On global special solutions for Navier-Stokes equations with boundary slip conditions in a cylindrical domain. Existence.

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Abstract. Global existence and uniqueness of solutions to Navier-Stokes equations in a cylinder and with boundary slip conditions is proved under the assumption that the initial motion is sufficiently close to the axially symmetric motion. Then the solution remains close to the axially symmetric solution for all time. Moreover the solution has some properties of the axially symmetric solution. Therefore the existence is proved in the Sobolev weighted spaces where the weight is a power function of the distance to the axis of symmetry. The existence is proved in such classes that velocity belongs to H^3 locally.
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1. Introduction

We consider the motion of a viscous incompressible fluid in a bounded domain $\Omega \subset \mathbb{R}^3$ under boundary slip conditions

$$
\begin{aligned}
&v_{,t} + v \cdot \nabla v - \mathrm{div}\mathbb{T}(v,p) = f && \text{in } \ \Omega^T = \Omega \times (0,T),\\
&\mathrm{div} v = 0 && \text{in } \ \Omega^T,\\
&v \cdot \bar{n} = 0 && \text{on } \ S^T = S \times (0,T),\\
&\bar{n} \cdot \mathbb{T}(v,p) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } \ S^T,\\
&v\big|_{t=0} = v(0) && \text{in } \ \Omega,
\end{aligned}
\tag{1.1}
$$

where $v = v(x,t) = (v_1(x,t), v_2(x,t), \ v_3(x,t)) \in \mathbb{R}^3$ is the velocity field, $p = p(x,t) \in \mathbb{R}$ the pressure, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ the external force field, $\bar{n}$ is the unit outward vector normal to $S = \partial\Omega$, $\bar{\tau}_1, \bar{\tau}_2$ are the unit vectors tangent to S and $\mathbb{T}(v,p)$ is the stress tensor of the form

$$
\mathbb{T}(v,p) = \{\nu(v_{j,x_i} + v_{i,x_j}) - p\delta_{ij}\}_{i,j=1,2,3} \equiv \mathbb{D}(v) - pI,
\tag{1.2}
$$

where ν is the constant positive viscosity coefficient, $\mathbb{D}(v)$ is the dilatation tensor and I is the unit matrix.

We assume that the domain Ω is a cylinder with the axis x_3 as the axis of symmetry.

Our aim is to prove global existence of solutions which are close to the axially symmetric solutions. Let us introduce the cylindrical coordinates by the relations $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$. Then we assume that the domain Ω is a cylinder with the axis z as the axis of symmetry which is determined by relations $r < R$, $-a < z < a$, where a and R are given positive numbers. Let us introduce vectors $\bar{e}_r = (\cos\varphi, \sin\varphi, 0)$, $\bar{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$, $\bar{e}_z = (0,0,1)$. Then any vector u has the following cylindrical coordinates $u_r = u \cdot \bar{e}_r$, $u_\varphi = u \cdot \bar{e}_\varphi$, $u_z = u \cdot \bar{e}_z$.

Definition 1.1. By the axially symmetric solutions to (1.1) we mean such solutions that $v(0)_\varphi = v_\varphi = 0$, $f_\varphi = 0$, $v_{r,\varphi} = 0$, $v_{z,\varphi} = 0$, $p_{,\varphi} = 0$, $f_{r,\varphi} = 0$, $f_{z,\varphi} = 0$ and $v(0)_{r,\varphi} = 0$, $v(0)_{z,\varphi} = 0$.

Since we are looking for a solution which is close to the axially symmetric solution, so v_φ, f_φ, $v(0)_\varphi$, $v_{r,\varphi}$, $v_{z,\varphi}$, $p_{,\varphi}$, $f_{r,\varphi}$, $f_{z,\varphi}$, $v(0)_{r,\varphi}$, $v(0)_{z,\varphi}$ are small quantities. Therefore we introduce the quantities h (see (1.3)), $q = p_{,\varphi}$, $\omega = v_\varphi$ which measure the distance to the axially symmetric solution.

In view of [12, 13] to show a long time existence of solutions to (1.1) we introduce

$$
h = v_{r,\varphi}\bar{e}_r + v_{\varphi,\varphi}\bar{e}_\varphi + v_{z,\varphi}\bar{e}_z, \qquad q = p_{,\varphi}
\tag{1.3}
$$

which are solutions to the problem (see [13])

$$
\begin{aligned}
&h_{,t} - \mathrm{div}\mathbb{D}(h) + \nabla q = -v \cdot \nabla h + h \cdot \nabla v + g \equiv G && \text{in } \ \Omega^T,\\
&\mathrm{div} h = 0 && \text{in } \ \Omega^T,\\
&h \cdot \bar{n} = 0 && \text{on } \ S^T,\\
&\bar{n} \cdot \mathbb{D}(h) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } \ S^T,\\
&h\big|_{t=0} = h(0) && \text{in } \ \Omega,
\end{aligned}
\tag{1.4}
$$

where $h(0) = v(0)_{r,\varphi}\bar{e}_r + v(0)_{\varphi,\varphi}\bar{e}_\varphi + v(0)_{z,\varphi}\bar{e}_z$ and $g = f_{r,\varphi}\bar{e}_r + f_{\varphi,\varphi}\bar{e}_\varphi + f_{z,\varphi}\bar{e}_z$.
In view of (1.3) we have

$$
v_{,\varphi} = h + v_r\bar{e}_\varphi - v_\varphi\bar{e}_r.
\tag{1.5}
$$

We recall the equation of continuity in the cylindrical coordinates

$$
v_{r,r} + \frac{1}{r}v_{\varphi,\varphi} + v_{z,z} + \frac{v_r}{r} = 0.
\tag{1.6}
$$

Moreover

$$
\begin{aligned}
&\omega_{,t} + v\cdot\nabla\omega + \frac{v_r}{r}\omega - \nu\Delta\omega + \nu\frac{\omega}{r^2} = \frac{1}{r}q + \frac{2\nu}{r^2}h_r + f_\varphi && \text{in } \ \Omega^T,\\
&\omega_{,r} = \frac{1}{R}\omega && \text{on } \ S_1^T,\\
&\omega_{,z} = 0 && \text{on } \ S_2^T,\\
&\omega\big|_{t=0} = \omega(0) \equiv v_\varphi(0) && \text{in } \ \Omega,
\end{aligned}
\tag{1.7}
$$

where v, q, h are treated as given functions and

$$
\begin{aligned}
&S_1 = \{x\in\mathbb{R}^3:\ r = R,\ -a < z < a,\ \varphi\in[0,2\pi\},\\
&S_2 = \{x\in\mathbb{R}^3:\ z = a \text{ and } z = -a,\ r < R,\ \varphi\in[0,2\pi]\}.
\end{aligned}
$$

Since in the proof of global existence of axially symmetric solutions the main role plays the equations for vorticity (see [3, 4]) we have to consider also a problem for vorticity. Let $\alpha = \mathrm{rot}v$. Then α is a solution to the problem

$$
\begin{aligned}
&\alpha_{,t} + v\cdot\nabla\alpha - \alpha\cdot\nabla v - \nu\Delta\alpha = F \equiv \mathrm{rot}f && \text{in } \ \Omega^T,\\
&(\alpha\cdot\bar n)_{,n} = -\frac{1}{R^2}h_z - \frac{1}{R}\omega_{,z} && \text{on } \ S_1^T,\\
&\alpha\cdot\bar\tau_1 = \frac{2}{R}\omega,\quad \alpha\cdot\bar\tau_2 = 0 && \text{on } \ S_1^T,\\
&(\alpha\cdot\bar n)_{,n} = 0\quad \alpha\cdot\bar\tau_1 = 0,\quad \alpha\cdot\bar\tau_2 = 0 && \text{on } \ S_2^T,\\
&\alpha\big|_{t=0} = \alpha_0 \equiv \mathrm{rot}v(0) && \text{in } \ \Omega,
\end{aligned}
\tag{1.8}
$$

where

$$
\begin{aligned}
&\bar n\big|_{S_1} = \bar e_r,\ \ \bar\tau_1\big|_{S_1} = \bar e_z,\ \ \bar\tau_2\big|_{S_1} = \bar e_\varphi,\\
&\bar n\big|_{S_2} = \bar e_z,\ \ \bar\tau_1\big|_{S_2} = \bar e_r,\ \ \bar\tau_2\big|_{S_2} = \bar e_\varphi.
\end{aligned}
$$

In our considerations the problem for vorticity plays a crucial role so v and p are calculated from the following elliptic problems

$$
\begin{aligned}
&\mathrm{rot}v = \alpha && \text{in } \ \Omega,\\
&\mathrm{div}v = 0 && \text{in } \ \Omega,\\
&v\cdot\bar n = 0 && \text{on } \ S,
\end{aligned}
\tag{1.9}
$$

and

$$
\begin{aligned}
&\Delta p = -\nabla v\cdot\nabla v + \mathrm{div}f && \text{in } \ \Omega,\\
&\frac{\partial p}{\partial n} = f\cdot\bar n - \frac{1}{R}v_\varphi^2 + \frac{2\nu}{R^2}v_{\varphi,\varphi} && \text{on } \ S_1\\
&\frac{\partial p}{\partial n} = f\cdot\bar n && \text{on } \ S_2,
\end{aligned}
\tag{1.10}
$$

(see Lemma 1.1 in [12]).

We have the following relations between cylindrical components of vorticity and velocity

$$
\begin{aligned}
&\alpha_1 = \alpha_r = \frac{1}{r}(v_{z,\varphi} - rv_{\varphi,z}),\\
&\alpha_2 = \alpha_\varphi = v_{r,z} - v_{z,r} \equiv \chi,\\
&\alpha_3 = \alpha_z = \frac{1}{r}[(rv_\varphi)_{,r} - v_{r,\varphi}].
\end{aligned}
\tag{1.11}
$$

From (1.11) we see that for solutions which are close to the axially symmetric solutions components α_1 and α_3 are small, because for the axially symmetric solutions only component α_2 is different from zero.
This is the main property used in [3, 4] to prove global existence of axially symmetric solutions because it implied a possibility to find a global estimate for α_2. For these reasons we distinguish the component of vorticity denoting it by χ.

Since different components of vorticity play different roles we have to replace problem (1.8) by a system of problems for the particular components of vorticity.
Then α_1 is a solutions of the problem

$$
\begin{aligned}
&\alpha_{1,t} + v\cdot\nabla\alpha_1 - \alpha_1 v_{r,r} - \frac{\alpha_2}{r}h_r - \alpha_3 v_{r,z} + \frac{2\nu}{r^2}(h_{r,z} - h_{z,r}) && \\
&+ \frac{\nu\alpha_1}{r^2} - \nu\Delta\alpha_1 = F_1 && \text{in } \Omega^T,\\
&\alpha_{1,r} = -\frac{1}{R^2}h_z - \frac{1}{R}\omega_{,z} && \text{on } S_1^T,\\
&\alpha_1 = 0 && \text{on } S_2^T,\\
&\alpha_1\big|_{t=0} = \alpha_{10} && \text{in } \Omega.
\end{aligned}
\tag{1.12}
$$

Next α_2 satisfies the problem

$$
\begin{aligned}
&\alpha_{2,t} + v\cdot\nabla\alpha_2 + \frac{v_\varphi}{r}\alpha_1 - \alpha\cdot\nabla v_\varphi - \frac{\alpha_2}{r}v_r - \frac{2\nu}{r^2}\left(\frac{1}{r}h_{z,\varphi} - h_{\varphi,z}\right) && \\
&+ \frac{\nu\alpha_2}{r^2} - \nu\Delta\alpha_2 = F_2 && \text{in } \Omega^T,\\
&\alpha_2 = 0 && \text{on } S^T,\\
&\alpha_2\big|_{t=0} = \chi(0) \equiv \alpha_{20} && \text{in } \Omega.
\end{aligned}
\tag{1.13}
$$

Finally α_3 is a solution to the problem

$$
\begin{aligned}
&\alpha_{3,t} + v\cdot\nabla\alpha_3 - (\alpha_1 v_{z,r} + \alpha_3 v_{z,z}) - \frac{\alpha_2}{r}h_z - \nu\Delta\alpha_3 = F_3 && \text{in } \Omega^T,\\
&\alpha_3 = \frac{2}{R}\omega && \text{on } S_1^T,\\
&\alpha_{3,z} = 0 && \text{on } S_2^T,\\
&\alpha_3\big|_{t=0} = \alpha_{30} && \text{in } \Omega.
\end{aligned}
\tag{1.14}
$$

The aim of this paper is to prove the existence of global solutions to problem (1.1). Since global existence of axially symmetric is known for a long time (see [3, 4]) we were restricted in our considerations to prove the existence of global solutions which remain close to the axially symmetric solutions. Since the axially symmetric solution is such that $v_\varphi = 0$, $\alpha_1 = \alpha_3 = 0$, $v_{r,\varphi} = v_{z,\varphi} = 0$, $\alpha_{2,\varphi} = 0$, $p_{,\varphi} = 0$ in our case the above quantities are small in corresponding norms. This plays a role of small parameter in our considerations. Since that the estimate for the vorticity α_2 is so important (see (5.5') in [13]) we replace problem (1.1) by the system of problems: (1.4), (1.7), (1.9), (1.10), (1.12)–(1.14). This suggests a method of successive approximations described by the system of problems (3.1)–(3.6), (3.5^2), (3.5^3). Then we prove the existence of local solutions to problem (1.1) (see Theorem 3.1) with large time of existence and with small parameters X_5, X_6 (see assumptions of Lemma 3.1), which measure a distance to the axially symmetric solutions. Finally the local existence is prolongated up to infinity step by step.

Now we formulate the main result of this paper.

Theorem 1. *Assume $\mu \in \left(1/2, 1/\sqrt{2}\right)$. Assume the decays*

$$
\begin{aligned}
&|g(t)|_{2,-1,\Omega} + |g_t(t)|_{2,\Omega} \le (|g(0)|_{2,-1,\Omega} + |g_t(0)|_{2,\Omega})e^{-\nu_0 t},\\
&|f(t)|_{2,\Omega} \le |f(0)|_{2,\Omega}e^{-\nu_1 t},\\
&|f_\varphi(t)|_{2,-\mu,\Omega} \le |f_\varphi(0)|_{2,-\mu,\Omega}e^{-\nu_2 t},\\
&|F'(t)|_{2,1-\mu,\Omega} \le |F'(0)|_{2,1-\mu,\Omega}e^{-\nu_3 t},\\
&|F_2(t)|_{2,1-\mu,\Omega} \le |F_2(0)|_{2,1-\mu,\Omega}e^{-\nu_3 t},
\end{aligned}
$$

where $\nu_0, \nu_1, \nu_2, \nu_3$ are positive numbers.

Assume that $h(0) \in H^1_{-1}(\Omega)$, $h_t(0) \in L_2(\Omega)$, $\omega(0) \in H^1_{-\mu}(\Omega)$, $\chi(0) \in L_{2,-1}(\Omega)$, $h_t(0) \in L_2(\Omega)$, $g(0) \in L_{2,-1}(\Omega)$, $g_t(0) \in L_2(\Omega)$, $\chi(0) \in H^1_{1-\mu}(\Omega)$, $v(0) \in L_2(\Omega)$.
Assume $g \in L_2(0,\infty; L_2(\Omega))$, $f \in L_2(0,\infty; L_2(\Omega))$, $F_2 \in L_2(0,\infty; L_{2,-1}(\Omega))$, $F' \in L_2(0,\infty; L_{2,1-\mu}(\Omega))$, $|\int_0^\infty \mathrm{d}t \int_\Omega f_\eta(x,t)\mathrm{d}x| < \infty$, where $F' = (F_1, F_3)$.
Then there exist finite numbers $X_1^*, X_2^*, Y_1^*, Y_2^*, D^*$ determined by (4.3), the numbers T and A such that (4.4) holds for X_1^*, X_2^* sufficiently small.
Next there exists a global solution to problem (1.1) such that $v \in A_{1-\mu}(\Omega \times (k, T(k+1)T)$ and $p \in L_2(kT, (k+1)T; W^2_{2,1-\mu}(\Omega))$, $k \in \mathbb{N}$.

Next we formulate the uniqueness result

Theorem 2. *Let the assumptions of Theorem 1 be satisfied. Then there exists only one solution to problem (1.1).*

In view of the decay assumptions in Theorem 1 we could expect a convergence of the considered solution to the axially symmetric solution. However, the problem is still open. Generally, we might expect two possibilities: either the decays are so strong that we have the convergence or they are strong enough to have global existence but not so strong to obtain the convergence. Hence we should have the relation $\nu_i = \nu_i(X_1^*, X_2^*, Y_1^*, Y_2^*, D^*)$, $i = 0, \dots, 3$, and the quantities ν_i^*, $i = 0,1,2,3$, such that for $\nu_i > \nu_i^*$, $i = 0, \dots, 3$, we have the converges but for $\nu_i < \nu_i^*$, $i = 0, \dots, 3$, we can show the global existence of solutions which are all time close to the axially symmetric solution.

2. Notation and spaces

To simplify considerations we introduce

$$\begin{aligned}
&|u|_{p,Q} = \|u\|_{L_p(Q)}, && Q \in \{\Omega, S, \Omega^T, S^T\}, && p \in [1,\infty],\\
&\|u\|_{s,Q} = \|u\|_{H^s(Q)}, && Q \in \{\Omega, S\}, && s \in \mathbb{Z}_+ \cup \{0\},\\
&\|u\|_{s,Q} = \|u\|_{W_2^{s,s/2}(Q)}, && Q \in \{\Omega^T, S^T\}, && s \in \mathbb{Z}_+ \cup \{0\},
\end{aligned}$$

where $\|u\|_{0,Q} = |u|_{2,Q}$.
We introduce weighted spaces with the weight equal to the distance to the axis of symmetry,

$$\|u\|_{L_{p,\mu}(Q)} = (\int_Q |u|^p r^{p\mu} \mathrm{d}Q)^{1/p}, \quad p \in [1,\infty), \quad \mu \in \mathbb{R}, \quad Q \in \{\Omega, S, \Omega^T, S^T\},$$

and for simplicity we define

$$|u|_{p,\mu,Q} = \|u\|_{L_{p,\mu}(Q)}.$$

Next

$$\|u\|_{H^s_\mu(Q)} = \left(\sum_{|\alpha|\le s} \int_Q |D_x^\alpha u|^2 r^{2(\mu - s + |\alpha|)} \mathrm{d}Q\right)^{1/2},$$

where $Q \in \{\Omega, S\}$, $s \in \mathbb{Z}_+ \cup \{0\}$, $\mu \in \mathbb{R}$, and

$$\|u\|_{H^{s,s/2}_\mu(Q)} = \left(\sum_{|\alpha|+2i\le s} \int_Q |D_x^\alpha \partial_t^i u|^2 r^{2(\mu - s + |\alpha|)} \mathrm{d}Q\right)^{1/2},$$

where $Q \in \{\Omega^T, S^T\}$, $s \in \mathbb{Z}_+ \cup \{0\}$, $\mu \in \mathbb{R}$. Moreover, for simplicity, we introduce

$$\|u\|_{s,\mu,Q} = \|u\|_{H^s_\mu(Q)}, \quad Q \in \{\Omega, S\}, \quad s \in \mathbb{Z}_+ \cup \{0\}, \quad \mu \in \mathbb{R},$$

and

$$\|u\|_{s,\mu,Q} = \|u\|_{H^{s,s/2}_\mu(Q)}, \quad Q \in \{\Omega^T, S^T\}, \quad s \in \mathbb{Z}_+ \cup \{0\}, \quad \mu \in \mathbb{R}.$$

We define

$$\begin{aligned} \||u\||_{l,r,Q} &= \|u\|_{W^{l,l/2}_r(Q)} \quad \text{for} \quad Q \in \{\Omega^T, S^T\}, \\ \||u\||_{l,r,Q} &= \|u\|_{W^l_r(Q)} \qquad \text{for} \quad Q \in \{\Omega, S\}. \end{aligned}$$

We introduce

$$\|u\|_{p_1,p_2,\Omega^T} = \|u\|_{L_{p_2}(0,T;L_{p_1}(\Omega))}, \quad \|u\|_{p_1,p_2,\mu,\Omega^T} = \|u\|_{L_{p_2}(0,T;L_{p_1,\mu}(\Omega))},$$

$$\|u\|_{W^s_{p,\mu}(\Omega)} = (\sum_{|\alpha|\le s} \int_\Omega |D^\alpha_x u|^p r^{p\mu} \mathrm{d}x)^{1/p},$$

and

$$\||u\||_{s,p,\mu,\Omega} = \|u\|_{W^s_{p,\mu}(\Omega)}, \quad \||u\||_{s,p,\mu,\Omega^T} = \|u\|_{W^{s,s/2}_{p,\mu}(\Omega^T)}.$$

Moreover

$$\|u\|_{L_{p_1,p_2}(\Omega^T)} = (\int_0^T (\int_\Omega |u|^{p_1} \mathrm{d}x)^{p_2/p_1})^{1/p_2}.$$

We introduce also the space $V^l_{p,\mu}(Q)$, $Q \in \{\Omega, S\}$ with the norm

$$\|u\|_{V^l_{p,\mu}(Q)} = (\sum_{|\alpha|\le l} \int_Q |D^\alpha_x u| r^{p(\mu-l+|\alpha|)} \mathrm{d}Q)^{1/p},$$

where $\mu \in \mathbb{R}$, $p \in [1,\infty)$.
We introduce also the notation

$$\|u\|_{l,p,\mu,Q^T} = \|u\|_{V^{l,l/2}_{p,\mu}(Q^T)},$$

$$\|u\|_{l,p,Q} = \|u\|_{V^l_{p,\mu}(Q)},$$

where $Q \in \{\Omega, S\}$, $l \in \mathbb{N} \cup \{0\}$, $p \in [1,\infty)$, $\mu \in \mathbb{R}$. Finally we introduce

$$\|u\|_{A_\mu(\Omega^T)} = \|u\|_{W^{2,1}_{2,\mu}(\Omega^T)} + \|u_{,x}\|_{W^{2,1}_{2,\mu}(\Omega^T)}.$$

For the reader convenience we introduce estimates and imbeddings which are used in this paper

$$\|u(t)\|_{s,\Omega} \le c(\|u\|_{s+1,\Omega^t} + \|u(0)\|_{s,\Omega}), \quad s \in \mathbb{Z}_+ \cup \{0\}, \tag{2.1}$$

$$\|u(t)\|_{s,\mu,\Omega} \le c(\|u\|_{s+1,\mu,\Omega^t} + \|u(0)\|_{s,\mu,\Omega}), \quad s \in \mathbb{Z}_+ \cup \{0\}, \quad \mu \in \mathbb{R}, \tag{2.2}$$

where c does not depend on t.
From [10, Ch.1, Sect. 2.15] we have the Hardy inequality

$$|f|_{q,-(\frac{1}{q}+\alpha),\mathbb{R}_+} \le \left(\frac{\mu}{|\alpha|}\right)^\mu |f_{,x}|_{p,1-(\frac{1}{p}+\alpha),\mathbb{R}_+}, \tag{2.3}$$

where $\alpha \ne 0$, $1 \le p \le q \le \infty$, $\mu = 1 - \frac{1}{p} + \frac{1}{q}$, and for $p = 1$, $q = \infty$ we assume that $\left(\frac{\mu}{|\alpha|}\right)^\mu = 1$.
More useful for us is the following Hardy inequality [10, Ch.1, Sect. 2.15]

$$|x^{-\alpha} f|_{p,\mathbb{R}_+} \le \frac{1}{|\alpha - 1/p|} |x^{-\alpha+1} f_{,x}|_{p,\mathbb{R}_+}, \quad \alpha \ne \frac{1}{p}. \tag{2.4}$$

Using (2.4) we obtain the Hardy inequality for the domain Ω. For this purpose we write (2.4) in the form

$$\int_0^\infty r^{-p\alpha}|f|^p\mathrm{d}r \le \frac{1}{\left|\alpha-\frac{1}{p}\right|^p}\int_0^\infty r^{-p(\alpha-1)}|f_{,r}|^p\mathrm{d}r \tag{2.5}$$

Using the cylindrical coordinates we obtain from (2.5) the inequality

$$\int_{-a}^{a}\mathrm{d}z\int_0^{2\pi}\mathrm{d}\varphi\int_0^\infty r^{-p\alpha-1}|f|^p r\mathrm{d}r \le \frac{1}{\left|\alpha-\frac{1}{p}\right|^p}\int_{-a}^{a}\mathrm{d}z\int_0^{2\pi}\mathrm{d}\varphi\int_0^\infty r^{-p(\alpha-1)-1}|f_{,r}|^p r\mathrm{d}r. \tag{2.6}$$

Hence we have

$$|f|_{p,-(\alpha+\frac{1}{p}),\Omega} \le \frac{1}{\left|\alpha-\frac{1}{p}\right|}|f_{,x}|_{p,1-(\alpha+\frac{1}{p}),\Omega}. \tag{2.7}$$

We need also the imbedding theorem for weighted Sobolev spaces (see Lemma 1.5 in [9])

$$\|u\|_{V^s_{q,\beta+s-l+\frac{n}{p}-\frac{n}{q}}(\Omega)} \le c\|u\|_{V^l_{p,\beta}(\Omega)}, \quad \Omega\subset\mathbb{R}^n, \tag{2.8}$$

and $s-l+\frac{n}{p}-\frac{n}{q}\le 0$.

3. Local existence

To prove local existence of solutions we use the following method of successive approximations. Let v_m be given. Then h_m and q_m are calculated from the problem

$$\begin{aligned}
&h_{m,t}-\mathrm{div}\mathbb{D}(h_m)+\nabla q_m+v_m\cdot\nabla h_m+h_m\cdot\nabla v_m=g &&\text{in } \Omega^T,\\
&\mathrm{div}h_m=0 &&\text{in } \Omega^T,\\
&h_m\cdot\bar n=0 &&\text{on } S^T,\\
&\bar n\cdot\mathbb{D}(h_m)\cdot\bar\tau_\alpha=0,\quad \alpha=1,2, &&\text{on } S^T,\\
&h_m\big|_{t=0}=h(0) &&\text{in } \Omega.
\end{aligned} \tag{3.1}$$

Let v_m,q_m,h_m be given. Then ω_m is a solutions of the problem

$$\begin{aligned}
&\omega_{m,t}+v_m\cdot\nabla\omega_m+\frac{v_{mr}}{r}\omega_m-\nu\Delta\omega_m+\nu\frac{\omega_m}{r^2}=\frac{q_m}{r}+\frac{2\nu}{r^2}h_{mr}\\
&+f_\varphi &&\text{in } \Omega^T,\\
&\omega_{m,r}=\frac{1}{R}\omega_m &&\text{on } S_1^T,\\
&\omega_{m,z}=0 &&\text{on } S_2^T,\\
&\omega_m\big|_{t=0}=\omega(0) &&\text{in } \Omega.
\end{aligned} \tag{3.2}$$

Let v_m,q_m,h_m,ω_m be given. Then $\alpha_{1m+1},\alpha_{2m+1},\alpha_{3m+1}$ are solutions of the problems

$$\begin{aligned}
&\alpha_{1m+1,t}+v_m\cdot\nabla\alpha_{1m+1}-\alpha_{1m+1}v_{mr,r}-\frac{\alpha_{2m+1}}{r}h_{mr}-\alpha_{3m+1}v_{mr,z}\\
&+\frac{2\nu}{r^2}(h_{mr,z}-h_{mz,r})+\frac{\nu\alpha_{1m+1}}{r^2}-\nu\Delta\alpha_{1m+1}=F_1 &&\text{in } \Omega^T,\\
&\alpha_{1m+1,r}=-\frac{1}{R^2}h_{mz}-\frac{1}{R}\omega_{m,z} &&\text{on } S_1^T,\\
&\alpha_{1m+1}=0 &&\text{on } S_2^T,\\
&\alpha_{1m+1}\big|_{t=0}=\alpha_{10} &&\text{in } \Omega
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\alpha_{2m+1,t}+v_m\cdot\nabla\alpha_{2m+1}+\frac{v_{m\varphi}}{r}\alpha_{1m+1}-\alpha_{m+1}\cdot\nabla v_{m\varphi}-\frac{\alpha_{2m+1}}{r}v_{mr}\\
&-\frac{2\nu}{r^2}\left(\frac{1}{r}h_{mz,\varphi}-h_{m\varphi,z}\right)+\frac{\nu\alpha_{2m+1}}{r^2}-\nu\Delta\alpha_{2m+1}=F_2 &&\text{in } \Omega^T,\\
&\alpha_{2m+1}=0 &&\text{on } S^T,\\
&\alpha_{2m+1}\big|_{t=0}=\chi(0)\equiv\alpha_{20} &&\text{in } \Omega,
\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}
&\alpha_{3m+1,t}+v_m\cdot\nabla\alpha_{3m+1}-(\alpha_{1m+1}v_{mz,r}+\alpha_{3m+1}v_{mz,z})\\
&-\frac{\alpha_{2m+1}}{r}h_{mz}-\nu\Delta\alpha_{3m+1}=F_3 &&\text{in } \Omega^T,\\
&\alpha_{3m+1}=\frac{2}{R}\omega_m &&\text{on } S_1^T,\\
&\alpha_{3,z}=0 &&\text{on } S_2^T,\\
&\alpha_{3m+1}\big|_{t=0}=\alpha_{30} &&\text{in } \Omega.
\end{aligned}\tag{3.5}$$

Collecting together problems (3.3)–(3.5) we know that the vector-valued function α_{m+1} is a solution to the system

$$\alpha_{m+1,t}+v_m\cdot\nabla\alpha_{m+1}-\alpha_{m+1}\cdot\nabla v_m-\nu\Delta\alpha_{m+1}=F,\tag{3.5^1}$$

with boundary conditions $(3.3)_{2,3}$, $(3.4)_2$, $(3.5)_{2,3}$.
Assuming that $\alpha_{m+1}=\mathrm{rot}v_m$ we know that v_m is a solution to the problem

$$\begin{aligned}
&v_{mt}+v_m\cdot\nabla v_m-\mathrm{div}\mathbb{D}(v_m)+\nabla p_m=f &&\text{in } \Omega^T,\\
&\mathrm{div}v_m=0 &&\text{in } \Omega^T,\\
&v_m\cdot\bar n=0 &&\text{on } S^T,\\
&\bar n\cdot\mathbb{D}(v_m)\cdot\bar\tau_\alpha=0,\quad \alpha=1,2, &&\text{on } S^T,\\
&v_m\big|_{t=0}=v_{(0)} &&\text{in } \Omega,
\end{aligned}\tag{3.5^2}$$

where p_m is a solution to the problem

$$\begin{aligned}
&\Delta p_m=-\nabla v_m\cdot\nabla v_m+\mathrm{div}f &&\text{in } \Omega^T,\\
&\frac{\partial p_m}{\partial n}=-\frac{1}{R}v_{m\varphi}^2+\frac{2\nu}{R^2}v_{m\varphi,\varphi}+f\cdot\bar n &&\text{on } S_1^T,\\
&\frac{\partial p_m}{\partial n}=f\cdot\bar n &&\text{on } S_2^T.
\end{aligned}\tag{3.5^3}$$

For the next step v_{m+1} is determined by the problem

$$\begin{aligned}
&\mathrm{rot}v_{m+1}=\alpha_{m+1} &&\text{in } \Omega,\\
&\mathrm{div}v_{m+1}=0 &&\text{in } \Omega,\\
&v_{m+1}\cdot\bar n=0 &&\text{on } S.
\end{aligned}\tag{3.6}$$

The problem (3.6) determines a new quantity v_{m+1}, so equation $(3.6)_1$ is not any definition of α_{m+1}, because α_{m+1} is already defined by the equation $\alpha_{m+1}=\mathrm{rot}v_m$.

By the zero approximation v_0 we assume an extension of the initial data $v(0)$ in such a way that the compatibility conditions are satisfied.

In [12] we proved the existence of local solutions to problem (3.1)–(3.6) on sufficiently small time interval $[0,T_0]$. Moreover it was shown that there exists a constant A such that the solution satisfies

$$\|v\|_{A_{1-\mu}(\Omega^{T_0})}\le A.\tag{3.7}$$

Our aim is to prolongate the local solution for some data sufficiently small. For this purpose we apply again the above method of successive approximations to obtain an uniform bound on an interval $[0,T]$, where T is reciprocally proportional to the sum of norms which measure the distance between the considered solution and the axially symmetric solution. Next step by step on the interval $[0,T_0]$, where T_0 is sufficiently small, we show convergence of the constructed sequence. Now we obtain an uniform bound

Lemma 3.1. *Let*

$$
\begin{aligned}
&X_5=|g|_{2,\Omega^t}+|f_\varphi|_{2,-\mu,\Omega^t}+\|h(0)\|_{1,-1,\Omega}+|\omega_{,x}(0)|_{2,-\mu,\Omega}+|\omega(0)|_{2,-(1+\mu),\Omega}\\
&X_6=X_5+|F'|_{2,1-\mu,\Omega^t}+|\alpha_0'|_{2,1-\mu,\Omega},\\
&D(t)=|f|_{2,\Omega^t}+|\int\limits_\Omega v_\eta(0)\mathrm{d}x|+\sup_t|\int\limits_0^t\mathrm{d}\tau\int\limits_\Omega f_\eta(x,\tau)\mathrm{d}x|+|v(0)|_{2,\Omega},\\
&Y_1=|\chi(0)|_{2,-1,\Omega}+|F_2|_{2,-1,\Omega^t},\quad Y_2=|F_2|_{2,1-\mu,\Omega^t}+\|\chi(0)\|_{1,1-\mu,\Omega},
\end{aligned}
$$

be bounded, where $F'=(F_1,F_3)$, $\alpha_0'=(\alpha_{10},\alpha_{30})$. *Assume that there exists a positive constant* A *such that*

$$
\begin{aligned}
&c(Y_1+\varphi(A)e^{ct}X_5)^2D(t)+c(Y_1+Y_2)\le A,\quad \|v_0\|_{A_{1-\mu}(\Omega^t)}\le A\\
&\varphi(A)(e^{ct}X_5+X_6)+\varphi(A)X_5Y_1\le A,
\end{aligned}
\tag{3.8}
$$

which hold for X_5 *and* X_6 *sufficiently small, where* φ *is an increasing positive function and* v_0 *is an extension of initial data* $v(0)$.

Then for $t\le T=\frac{1}{c}\ln\frac{A}{c\varphi(A)X_5D(\infty)}$ the estimate holds

$$
\|v_m\|_{A_{1-\mu}(\Omega^t)}\le A,\qquad m\in\mathbb{N},\quad m>0.
\tag{3.9}
$$

Proof. From Lemma 5.3 [13] we have

$$
|h_m|_{2,\Omega}+\nu(\int\limits_0^t\|h_m\|_{1,\Omega}^2\mathrm{d}\tau)^{1/2}\le\varphi(K_m(t))X_1,
\tag{3.10}
$$

where

$$
X_1=|g|_{2,\Omega^t}+|h(0)|_{2,\Omega},\qquad K_m(t)=\|v_m\|_{A_{1-\mu}(\Omega^t)}
\tag{3.11}
$$

and φ will always denote an increasing positive function.
From Lemma 5.4 [13] we have

$$
\|h_m\|_{2,-1,\Omega^t}+\left(\int\limits_0^t\|q_m\|_{1,-1,\Omega}^2\mathrm{d}\tau\right)^{1/2}\le\varphi(K_m(t))X_2,
\tag{3.12}
$$

where

$$
X_2=X_1+\|h(0)\|_{1,-1,\Omega}.
\tag{3.13}
$$

Assume that δ is sufficiently small and

$$
\sup_t\left|\frac{h_{m\varphi}}{r}\right|_{\frac{3}{2},\Omega}\le\delta.
\tag{3.14}
$$

Then Lemma 5.1 [13] yields

$$
\begin{aligned}
&|\chi_{m+1}|_{2,-1,\Omega}+\nu\left(\int\limits_0^t\left\|\frac{\chi_{m+1}}{r}\right\|_{1,\Omega}^2\mathrm{d}\tau\right)^{1/2}\le cY_1+\varphi(K_m(t))(\|h_m\|_{2,-1,\Omega^t}\\
&+\sup_t\|\omega_m\|_{1,-\mu,\Omega}),
\end{aligned}
\tag{3.15}
$$

where

$$Y_1 = |\chi(0)|_{2,-1,\Omega} + |F_2|_{2,-1,\Omega^t}. \tag{3.16}$$

Lemma 5.6 [13] implies

$$|\omega_m|_{2,-\mu,\Omega} + |\omega_{m,x}|_{2,-\mu,\Omega^t} + |\omega_m|_{2,-(1+\mu),\Omega^t} \le \varphi(K_m(t))e^{ct}X_3, \tag{3.17}$$

where

$$X_3 = X_2 + |f_\varphi|_{2,-\mu,\Omega^t} + |\omega(0)|_{2,-\mu,\Omega}. \tag{3.18}$$

Lemma 5.5 [13] gives

$$\|\omega_m\|_{2,1-\mu,\Omega^t} \le \varphi(K_m(t))e^{ct}X_4, \tag{3.19}$$

where

$$X_4 = X_3 + \|\omega(0)\|_{1,1-\mu,\Omega}. \tag{3.20}$$

From [14, (5.13)] we have

$$\begin{aligned} &|\omega_{m,x}(t)|_{2,-\mu,\Omega} + |\omega_m(t)|_{2,-(1+\mu),\Omega} \le \varphi(K_m(t))[\sup_t |\omega_m(t)|_{2,1,\Omega} \\ &+ \left(\int_0^t |\omega_m(\tau)|^2_{2,-\mu,\Omega} \mathrm{d}\tau\right)^{1/2} + X_5], \end{aligned} \tag{3.21}$$

where

$$X_5 = X_4 + |\omega_{,x}(0)|_{2,-\mu,\Omega} + |\omega(0)|_{2,-(1+\mu),\Omega}. \tag{3.22}$$

From [14, (5.34)] we obtain

$$|\omega_m(t)|_{2,1,\Omega} \le |\omega(0)|_{2,1,\Omega} + t^{1/2}X_3. \tag{3.23}$$

Using (3.17) and (3.23) in (3.21) yields

$$|\omega_{m,x}(t)|_{2,-\mu,\Omega} + |\omega_m(t)|_{2,-(1+\mu),\Omega} \le \varphi(K_m(t))e^{ct}X_5. \tag{3.24}$$

Using (3.12) and (3.24) in (3.15) gives

$$|\chi_{m+1}(t)|_{2,-1,\Omega} + \nu\left(\int_0^t \left\|\frac{\chi_{m+1}}{r}\right\|^2_{1,\Omega} \mathrm{d}\tau\right)^{1/2} \le cY_1 + \varphi(K_m(t))e^{ct}X_5. \tag{3.25}$$

From [15, (5.21)] we have

$$|||\alpha'_{m+1}|||_{2,2,1-\mu,\Omega^t} \le \varphi(K_m(t))[(Y_1 + e^{ct}X_5)X_2 + e^{ct}X_5 + X_6], \tag{3.26}$$

where

$$X_6 = X_5 + |F'|_{2,1-\mu,\Omega^t} + \|\alpha'_0\|_{1,1-\mu,\Omega}. \tag{3.27}$$

From Lemma 5.5 [15] we get

$$\begin{aligned}\|\alpha_{2m+1}\|_{2,1-\mu,\Omega^t} &\le (\varepsilon\|v_m\|_{A_{1-\mu}(\Omega^t)} + c(\varepsilon)\sup_t|v_m|_{2,\Omega} + |v(0)|_{2,\Omega}\\ &+ \varphi(K_m(t))X_5+1))(Y_1+\varphi(K_m(t))e^{ct}X_5)\\ &+ \varphi(K_m(t))e^{ct}X_5(1+X_5)+cY_2,\end{aligned}\tag{3.28}$$

where applying Lemma 5.2 from [13] to problem (3.5^2) gives

$$\sup_t|v_m|_{2,\Omega} \le e^{-\frac{\nu_0}{2}t}|v(0)|_{2,\Omega} + c(|f|_{2,\Omega^t} + |\int\limits_\Omega rv_\varphi(0)\mathrm{d}x| + \sup_t|\int\limits_{\Omega^t} rf_\varphi \mathrm{d}x\mathrm{d}t|,\ \nu_0<\nu$$

and

$$Y_2 = |F_2|_{2,1-\mu,\Omega^t} + \|\chi(0)\|_{1,1-\mu,\Omega}.\tag{3.29}$$

From (3.26) and (3.28) we obtain

$$\begin{aligned}&\||\alpha_{m+1}\||_{2,2,1-\mu,\Omega^t} \le [\varepsilon\|v_m\|_{A_{1-\mu}(\Omega^t)} + c(\varepsilon)D(t)]\cdot\\ &(Y_1+\varphi(K_m(t))e^{ct}X_5) + \varphi(K_m(t))[e^{ct}X_5(1+X_5)+X_6]\\ &+c(Y_2+Y_1)+\varphi(K_m(t))X_5Y_1,\end{aligned}\tag{3.30}$$

where

$$D(t) = |v(0)|_{2,\Omega} + |f|_{2,\Omega^t} + |\int\limits_\Omega rv_\varphi(0)\mathrm{d}x| + \sup_t|\int\limits_{\Omega^t} rf_\varphi\mathrm{d}x\mathrm{d}t|$$

Therefore from (3.6) (see [6]) we have

$$\begin{aligned}&K_{m+1}(t) \le [\varepsilon K_m(t) + c(\varepsilon)D(t)](Y_1+\varphi(K_m(t))e^{ct}X_5)\\ &+\varphi(K_m(t))(1+X_5)(e^{ct}X_5+X_6)+c(Y_1+Y_2)+\varphi(K_m(t))X_5Y_1.\end{aligned}\tag{3.31}$$

Let us assume for simplicity that $c(\varepsilon)=c\varepsilon^{-1}$. Then (3.31) takes the form

$$\begin{aligned}&K_{m+1}(t) \le \varepsilon K_m(t) + c\varepsilon^{-1}(Y_1+\varphi(K_m(t))e^{ct}X_5)^2D(t)\\ &+\varphi(K_m(t))(1+X_5)(e^{ct}X_5+X_6)+c_1(Y_2+Y_1)+\varphi(K_m(t))X_5Y_1.\end{aligned}\tag{3.32}$$

Let

$$K_m(t)\le A.\tag{3.33}$$

Take $\varepsilon=\frac{1}{3}$. Assume that $T<\infty$ be given. Assume that A is so large and X_5, X_6 are so small that for $t\le T$ we have

$$\begin{aligned}&c[3(Y_1+\varphi(A)e^{ct}X_5)]^2D(t)+c_1(Y_2+Y_1)\le\frac{1}{3}A,\\ &\varphi(A)(1+X_5)(e^{ct}X_5+X_6)+\varepsilon(A)X_5Y_1\le\frac{1}{3}A.\end{aligned}\tag{3.34}$$

then from (3.32) we have that

$$K_{m+1}(t)\le A\quad\text{for}\quad t\le T,\quad m\in\mathbb{N}.\tag{3.35}$$

From (3.34) we can also see that

$$T=\frac{1}{c}\ln\frac{A}{c\varphi(A)D(\infty)X_5}.\tag{3.36}$$

This concludes the proof.

To show convergence of the considered sequence we formulate problems on differences. Then for $t\le T_0$, where T_0 is sufficiently small, we show convergence. Since we have the bound (3.35) for $t\le T$ we can show existence step by step up to $t\le T$. Considering convergence in the interval $[T_0,2T_0]$ we have that $X_5(T_0)=\varphi(A)X_5(0)$, where $\varphi(A)>1$. Since $X_5(0)$ is sufficiently small and (3.35) holds for $[T_0,2T_0]$ the length of the convergence interval can be kept the same. Generally such procedure can be prolongated up to $t\le T$, where T is determined by (3.36).
Now we show convergence

Lemma 3.2. *Let the assumptions of Lemma 3.1 hold.*

Then

$$\|V_{m+1}\|_{A_{1-\mu}(\Omega^t)} \le \varphi(A)(t^a + X_7)\|V_m\|_{A_{1-\mu}(\Omega^t)}, \quad a > 0, \tag{3.37}$$

where V_m is defined below and X_7 by (3.63).
Hence the sequence $\{v_m\}$, $m = 1, \dots$, determined by problems (3.1)–(3.6) converges to a solution of problem (1.1).

Proof. Let us introduce the differences

$$H_m = h_m - h_{m-1}, \quad Q_m = q_m - q_{m-1}, \quad W_m = \omega_m - \omega_{m-1}, \quad P_m = p_m - p_{m-1},$$

$$A_m = \alpha_m - \alpha_{m-1}, \quad V_m = v_m - v_{m-1}.$$

From problem (3.1) we obtain

$$\begin{aligned}
&H_{m,t} - \mathrm{div}\mathbb{D}(H_m) + \nabla Q_m = -V_m \cdot \nabla h_m - v_{m-1} \cdot \nabla H_m - H_m \cdot \nabla v_m \\
&- h_{m-1} \cdot \nabla v_m && \text{in } \ \Omega^T, \\
&\mathrm{div} H_m = 0 && \text{in } \ \Omega^t, \\
&\bar{n} \cdot h_m = 0 && \text{on } \ S^T, \\
&\bar{n} \cdot \mathbb{D}(H_m) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } \ S^T, \\
&H_m\big|_{t=0} = 0 && \text{in } \ \Omega.
\end{aligned} \tag{3.38}$$

From (3.2) we have

$$\begin{aligned}
&W_{m,t} + v_m \cdot \nabla W_m + \frac{v_{mr}}{r} W_m - \nu \Delta W_m + \nu \frac{W_m}{r^2} \\
&= -V_m \cdot \nabla \omega_{m-1} - \frac{V_{mr}}{r}\omega_{m-1} + \frac{Q_m}{r} + \frac{2\nu}{r^2} H_{mr} && \text{in } \ \Omega^T, \\
&W_{m,r} = \frac{1}{R} W_m && \text{on } \ S_1^T, \\
&W_{m,z} = 0 && \text{on } \ S_2^T, \\
&W_m\big|_{t=0} = 0 && \text{in } \ \Omega.
\end{aligned} \tag{3.39}$$

Problem (3.3) implies

$$\begin{aligned}
&A_{1m+1,t} + v_m \cdot \nabla A_{1m+1} - A_{1m+1} v_{mr,r} - \frac{A_{2m+1}}{r} h_{mr} \\
&- A_{3m+1} v_{mr,z} + \frac{\nu A_{1m+1}}{r^2} - \nu \Delta A_{1m+1} \\
&= -V_m \nabla \alpha_{1m} + \alpha_{1m} V_{mr,r} + \frac{\alpha_{2m}}{r} H_{mr} + \alpha_{3m} V_{mr,z} - \frac{2\nu}{r^2}(H_{mr,z} - H_{mz,r}) && \text{in } \ \Omega^T, \\
&A_{1m+1,r} = -\frac{1}{R^2} H_{mz} - \frac{1}{R} W_{m,z} && \text{on } \ S_1^T, \\
&A_{1m+1} = 0 && \text{on } \ S_2^T, \\
&A_{1m+1}\big|_{t=0} = 0 && \text{in } \ \Omega.
\end{aligned} \tag{3.40}$$

Problem (3.4) gives

$$
\begin{aligned}
&A_{2m+1,t}+v_m\cdot\nabla A_{2m+1}+\frac{v_{m\varphi}}{r}A_{1m+1}-A_{m+1}\cdot\nabla v_{m\varphi}\\
&-\frac{A_{2m+1}}{r}v_{mr}+\frac{\nu}{r^2}A_{2m+1}-\nu\Delta A_{2m+1}\\
&=-V_m\cdot\nabla\alpha_{2m}-\frac{V_{m\varphi}}{r}\alpha_{1m}+\alpha_m\cdot\nabla V_{m\varphi}-\alpha_{2m}\frac{V_{mr}}{r}\\
&+\frac{2\nu}{r^2}\left(\frac{1}{r}H_{mz,\varphi}-H_{m\varphi,z}\right) && \text{in } \Omega^T,\\
&A_{2m+1}=0 && \text{on } S^T,\\
&A_{2m+1}\big|_{t=0}=0 && \text{in } \Omega.
\end{aligned}
\tag{3.41}
$$

Next problem (3.5) implies

$$
\begin{aligned}
&A_{3m+1,t}+v_m\cdot\nabla A_{3m+1}-(A_{1m+1}v_{mz,r}+A_{3m+1}v_{mz,z})\\
&-\frac{A_{2m+1}}{r}h_{mz}-\nu\Delta A_{3m+1}=-V_m\nabla\alpha_{3m}+\alpha_{1m}V_{mz,r}\\
&+\alpha_{3m}V_{mz,z}+\frac{\alpha_{2m}}{r}H_{mz} && \text{in } \Omega^T,\\
&A_{3m+1}=\frac{2}{R}W_{3m} && \text{on } S_1^T,\\
&A_{3m+1,z}=0 && \text{on } S_2^T,\\
&A_{3m+1}\big|_{t=0}=0 && \text{in } \Omega.
\end{aligned}
\tag{3.42}
$$

Next problem (3.5^2) implies

$$
\begin{aligned}
&V_{mt}+v_m\cdot\nabla V_m+V_m\cdot\nabla v_{m-1}-\mathrm{div}\mathbb{D}(V_m)+\nabla P_m=0 && \text{in } \Omega^T,\\
&\mathrm{div}V_m=0 && \text{in } \Omega^T,\\
&V_m\cdot\bar n=0 && \text{on } S^T,\\
&\bar n\cdot\mathbb{D}(V_m)\cdot\bar\tau_\alpha=0,\quad \alpha=1,2, && \text{on } S^T,\\
&V_m\big|_{t=0}=0 && \text{in } \Omega.
\end{aligned}
\tag{3.42^1}
$$

and problem (3.5^3) gives

$$
\begin{aligned}
&\Delta P_m=-\nabla v_m\cdot\nabla V_m-\nabla V_m\cdot\nabla v_{m-1} && \text{in } \Omega^T,\\
&\frac{\partial P_m}{\partial n}=-\frac{1}{R}V_{m\varphi}^2+\frac{2\nu}{R^2}V_{m\varphi,\varphi} && \text{on } S_1^T,\\
&\frac{\partial P_m}{\partial n}=0 && \text{on } S_2^T.
\end{aligned}
\tag{3.42^2}
$$

Finally from (3.6) we obtain

$$
\begin{aligned}
&\mathrm{rot}V_{m+1}=A_{m+1} && \text{in } \Omega,\\
&\mathrm{div}V_{m+1}=0 && \text{in } \Omega,\\
&V_{m+1}\cdot\bar n=0 && \text{on } S.
\end{aligned}
\tag{3.43}
$$

Now we obtain estimates for solutions of problems (3.38)–(3.43). For solutions of (3.40) we have

$$
\begin{aligned}
&\||A_{1m+1}\||_{2,2,1-\mu,\Omega^t}\le c[|v_m\cdot\nabla A_{1m+1}|_{2,1-\mu,\Omega^t}+|A_{1m+1}v_{mr,r}+A_{3m+1}v_{mr,z}|_{2,1-\mu,\Omega^t}\\
&+\left|\frac{A_{2m+1}}{r}h_{mr}\right|_{2,1-\mu,\Omega^t}+|A_{1m+1}|_{2,-(1+\mu),\Omega^t}+|V_m\cdot\nabla\alpha_{1m}|_{2,1-\mu,\Omega^t}\\
&+|\alpha_{1m}V_{mr,r}+\alpha_{3m}V_{mr,z}|_{2,1-\mu,\Omega^t}+\left|\frac{\alpha_{2m}}{r}H_{mr}\right|_{2,1-\mu,\Omega^t}+|H_{mr,z}|_{2,-(1+\mu),\Omega^t}\\
&+|H_{mz,r}|_{2,-(1+\mu),\Omega^t}+\|H_{mz}\|_{1/2,1-\mu,S_1^t}+\|W_m\|_{3/2,1-\mu,S_1^t}].
\end{aligned}
\tag{3.44}
$$

Now we estimate the terms from the r.h.s. of (3.44). Since in this proof time is small estimates must be done in a different way than in the proof of Lemma 3.1. To simplify considerations we omit the indices connected with m. The first term we estimate in the way

$$|v\cdot A_{1,x}|_{2,1-\mu,\Omega^t}=(\int_0^t|vA_{1,x}|^2_{2,1-\mu,\Omega}\mathrm{d}\tau)^{1/2}\le(\int_0^t\mathrm{d}\tau|v|^2_{2p_1,\Omega}|A_{1,x}|^2_{2p_2,1-\mu,\Omega})^{1/2}\equiv I_1,$$

where $1/p_1+1/p_2=1$.

Using the interpolation inequality (see [10, Ch. 3, Sect. 15])

$$|v|_{2p_1,\Omega}\le c|v|^{1-\theta_1}_{2,\Omega}\||v\||^{\theta_1}_{2,r,\Omega}$$

where $\theta_1=\frac{3}{2p_2(\frac{7}{2}-\frac{3}{r})}$ and r is such that inequality (2.8) can be applied in the way

$$\||v\||_{2,r,\Omega}\le c\||v\||_{3,2,1-\mu,\Omega}, \tag{3.45}$$

so $\frac{1}{2}-\frac{\mu}{3}\le\frac{1}{r}$. Hence $r\le\frac{6}{3-2\mu}$ with $\mu\in(0,1/\sqrt{2})$.

From [11] we have the interpolation inequality

$$|A_{1,x}|_{2p_2,1-\mu,\Omega}\le c|A_1|^{1-\theta_2}_{2,1-\mu,\Omega}\|A_1\|^{\theta_2}_{2,1-\mu,\Omega},$$

where $\theta_2=\frac{5}{4}-\frac{3}{4p_2}<1$ which implies that $p_2<3$.

Taking r such that $\frac{1}{r}=\frac{1}{2}-\frac{\mu}{3}$ we obtain $\theta_1=\frac{3}{2p_2(2+\mu)}$. Choosing μ and p_2 in such a way that $\theta_1+\theta_2<1$, so $p_2<\frac{3\mu}{2+\mu}$, we obtain

$$\begin{aligned}I_1&\le c\sup_t|v|^{1-\theta_1}_{2,\Omega}\sup_t|A_1|^{1-\theta_2}_{2,1-\mu,\Omega}\|A_1\|^{\theta_2}_{2,1-\mu,\Omega^t}\|v\|^{\theta_1}_{A_{1-\mu}(\Omega^t)}\\&\le\varepsilon\|A_1\|_{2,1-\mu,\Omega^t}+c(\varepsilon)t^{a_1}\sup_t|v|^{\frac{1-\theta_1}{\theta_1}}_{2,\Omega}\sup_t|A_1|_{2,1-\mu,\Omega}\|v\|_{A_{1-\mu}(\Omega^t)},\end{aligned}$$

where $a_1>0$.

In the bound for I_1 we can have in reality the norm $\||A_1\||_{2,2,1-\mu,\Omega}$ instead of $\|A_1\|_{2,1-\mu,\Omega}$. Since $\alpha_1\big|_{r=0}=0$, so $A_1\big|_{r=0}=0$ and then $\||A_1\||_{2,2,1-\mu,\Omega}\le c\|A_1\|_{2,1-\mu,\Omega}$, so the last norm appeared in the bound for I_1.

The second term on the r.h.s. of (3.44) we estimate by

$$|A'v_{,x}|_{2,1-\mu,\Omega^t}=(\int_0^t|A'v_{,x}|^2_{2,1-\mu,\Omega}\mathrm{d}\tau)^{1/2}\le(\int_0^t|A'|^2_{2p_1,1-\mu,\Omega}|v_{,x}|^2_{2p_2,\Omega}\mathrm{d}\tau)^{1/2}\equiv I_2,$$

where $1/p_1+1/p_2=1$ and $A'=(A_1,A_3)$.

Using (see [11])

$$|A'|_{2p_1,1-\mu,\Omega}\le c|A'|^{1-\theta_1}_{2,1-\mu,\Omega}\||A'\||^{\theta_1}_{2,2,1-\mu,\Omega},$$

with $\theta_1=\frac{3}{4p_2}$, and (see [10, Ch. 3, Sect 15])

$$|v_{,x}|_{2p_2,\Omega}\le c|v|^{1-\theta_2}_{2,\Omega}\||v\||^{\theta_2}_{2,r,\Omega},$$

where $\theta_2=\frac{\frac{5}{2}-\frac{3}{2p_2}}{\frac{7}{2}-\frac{3}{r}}$ and r is such that (3.45) holds, so $\frac{1}{2}-\frac{\mu}{3}\le\frac{1}{r}$. Choosing r such that $\frac{1}{2}-\frac{\mu}{3}=\frac{1}{r}$ we obtain $\theta_2=\frac{\frac{5}{2}-\frac{3}{2p_2}}{2+\mu}$. Imposing $\theta_1+\theta_1<1$ we have that $p_2<\frac{3\mu}{2(2\mu-1)}$, where $\mu\in\left(\frac{1}{2},\frac{1}{\sqrt{2}}\right)$.

Then

$$\begin{aligned} I_2 &\le c\sup_t |v|_{2,\Omega}^{1-\theta_2} \sup_t |A'|_{2,1-\mu,\Omega}^{1-\theta_1} \|v\|_{A_{1-\mu}(\Omega^t)}^{\theta_2} |||A'|||_{2,2,1-\mu,\Omega^t}^{\theta_1} \\ &\le \varepsilon |||A'|||_{2,2,1-\mu,\Omega^t} + c(\varepsilon) t^{a_2} \sup_t |v|_{2,\Omega}^{\frac{1-\theta_2}{\theta_2}} \sup_t |A'|_{2,1-\mu,\Omega} \|v\|_{A_{1-\mu}(\Omega^t)}, \end{aligned}$$

where $a_2 > 0$.
The third term on the r.h.s. of (3.44) we estimate by

$$\left(\int_0^t \left| \frac{A_2}{r} \right|_{2p_1,\Omega}^2 |h|_{2p_2,1-\mu,\Omega}^2 \mathrm{d}\tau \right)^{1/2} \equiv I_3,$$

where $1/p_1 + 1/p_2 = 1$.
Using the interpolation inequality

$$\left| \frac{A_2}{r} \right|_{2p_1,\Omega} \le c \left| \frac{A_2}{r} \right|_{2,\Omega}^{1-\theta_1} \left\| \frac{A_2}{r} \right\|_{1,\Omega}^{\theta_1}$$

with $\theta_1 = \frac{3}{2p_2}$ and

$$|h|_{2p_2,1-\mu,\Omega} \le c|h|_{2,1-\mu,\Omega}^{1-\theta_2} \|h\|_{2,1-\mu,\Omega}^{\theta_2}$$

with $\theta_2 = \frac{3}{4p_1}$ we obtain

$$I_3 \le c\sup_t |h|_{2,1-\mu,\Omega}^{1-\theta_2} \sup_t \left| \frac{A_2}{r} \right|_{2,\Omega}^{1-\theta_1} \left(\int_0^t \left\| \frac{A_2}{r} \right\|_{1,\Omega}^{2\theta_1} \|h\|_{2,1-\mu,\Omega}^{2\theta_2} \mathrm{d}\tau \right)^{1/2} \equiv I_4.$$

Assuming $\theta_1 + \theta_2 = 1$, which holds for $p_1 = \frac{3}{2}$, $p_2 = 3$, we obtain that

$$I_4 \le c\sup_t |h|_{2,1-\mu,\Omega} \left(\int_0^t \left\| \frac{A_2}{r} \right\|_{1,\Omega}^2 \mathrm{d}\tau \right)^{1/2} + c\|h\|_{2,1-\mu,\Omega^t} \sup_t \left| \frac{A_2}{r} \right|_{2,\Omega}.$$

The fifth term on the r.h.s. of (5.44) we estimate in the same way as the first term. Therefore it is bounded by

$$c\sup_t |V|_{2,\Omega} \|\alpha_1\|_{2,1-\mu,\Omega^t} + c\|V\|_{A_{1-\mu}(\Omega^t)} \sup_t |\alpha_1|_{2,1-\mu,\Omega},$$

where $\|\alpha_1\|_{2,1-\mu,\Omega^t}$ can be replaced by $|||\alpha_1|||_{2,2,1-\mu,\Omega^t}$ (see remarks below the bound for I_1).
The sixth term on the r.h.s. of (5.44) we estimate in the same way as the second term. Therefore it is estimated by

$$c\sup_t |V|_{2,1-\mu,\Omega} |||\alpha'|||_{2,2,1-\mu,\Omega^t} + c|||V|||_{2,2,1-\mu,\Omega^t} \sup_t |\alpha'|_{2,1-\mu,\Omega}.$$

The seventh term we examine in the same way as the third. Therefore it is bounded by

$$c\sup_t |H|_{2,1-\mu,\Omega} \left(\int_0^t \left\| \frac{\alpha_2}{r} \right\|_{1,\Omega}^2 \mathrm{d}\tau \right)^{1/2} + c\|H\|_{2,1-\mu,\Omega^t} \sup_t \left| \frac{\alpha_2}{r} \right|_{2,\Omega}.$$

Summarizing the above estimates we obtain from (3.44) the inequality

$$
\begin{aligned}
&\||A_{1m+1}\||_{2,2,1-\mu,\Omega^t} \le \varepsilon\||A_{3m+1}\||_{2,2,1-\mu,\Omega^t}\\
&+ c(\varepsilon)t^a\varphi(\|v_m\|_{A_{1-\mu}(\Omega^t)})\sup_t|A'_{m+1}|_{2,1-\mu,\Omega}\\
&+ c\sup_t|h_m|_{2,1-\mu,\Omega}\left(\int_0^t\left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2 \mathrm{d}\tau\right)^{1/2} + c\|h_m\|_{2,1-\mu,\Omega^t}\sup_t\left|\frac{A_{2m+1}}{r}\right|_{2,\Omega}\\
&+ c\||\alpha'_m\||_{2,2,1-\mu,\Omega^t}\sup_t|V_m|_{2,\Omega} + c\sup_t|\alpha'_m|_{2,1-\mu,\Omega}\|V_m\|_{A_{1-\mu}(\Omega^t)}\\
&+ c\sup_t|H_m|_{2,1-\mu,\Omega}\left(\int_0^t\left\|\frac{\alpha_{2m}}{r}\right\|_{1,\Omega}^2 \mathrm{d}\tau\right)^{1/2} + c\|H_m\|_{2,1-\mu,\Omega^t}\sup_t\left|\frac{\alpha_{2m}}{r}\right|_{2,\Omega}\\
&+ c(\|H_m\|_{2,1-\mu,\Omega^t} + \|W_m\|_{2,1-\mu,\Omega^t}),
\end{aligned} \tag{3.45}
$$

where $a > 0$.

Similarly for solutions of (3.42) we have

$$
\begin{aligned}
&\||A_{3m+1}\||_{2,2,1-\mu,\Omega^t} \le \varepsilon\||A_{1m+1}\||_{2,2,1-\mu,\Omega^t}\\
&+ c(\varepsilon)t^a\varphi(\|v_m\|_{A_{1-\mu}(\Omega^t)})\sup_t|A'_{m+1}|_{2,1-\mu,\Omega}\\
&+ c\sup_t|h_m|_{2,1-\mu,\Omega}\left(\int_0^t\left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2 \mathrm{d}\tau\right)^{1/2} + c\|h_m\|_{2,1-\mu,\Omega^t}\sup_t\left|\frac{A_{2m+1}}{r}\right|_{2,\Omega}\\
&+ c\||\alpha'_m\||_{2,2,1-\mu,\Omega^t}\sup_t|V_m|_{2,\Omega} + c\sup_t|\alpha'_m|_{2,1-\mu,\Omega}\|V_m\|_{A_{1-\mu}(\Omega^t)}\\
&+ c\sup_t|H_m|_{2,1-\mu,\Omega}\left(\int_0^t\left\|\frac{\alpha_{2m}}{r}\right\|_{1,\Omega}^2 \mathrm{d}\tau\right)^{1/2} + c\|H_m\|_{2,1-\mu\Omega^t}\sup_t\left|\frac{\alpha_{2m}}{r}\right|_{2,\Omega}\\
&+ c\|W_m\|_{2,1-\mu,\Omega^t}.
\end{aligned} \tag{3.46}
$$

From (3.45) and (3.46) we obtain

$$
\begin{aligned}
&\||A'_{m+1}\||_{2,2,1-\mu,\Omega^t} \le ct^a\varphi(\|v_m\|_{A_{1-\mu}(\Omega^t)})\sup_t|A'_{m+1}|_{2,1-\mu,\Omega}\\
&+ c\sup_t|h_m|_{2,1-\mu,\Omega}\left(\int_0^t\left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2 \mathrm{d}\tau\right)^{1/2} + c\|h_m\|_{2,1-\mu,\Omega^t}\sup_t\left|\frac{A_{2m+1}}{r}\right|_{2,\Omega}\\
&+ c\||\alpha'_m\||_{2,2,1-\mu,\Omega^t}\sup_t|V_m|_{2,\Omega} + c\sup_t|\alpha'_m|_{2,1-\mu,\Omega}\|V_m\|_{A_{1-\mu}(\Omega^t)}\\
&+ c\sup_t|H_m|_{2,1-\mu,\Omega}\left(\int_0^t\left\|\frac{\alpha_{2m}}{r}\right\|_{1,\Omega}^2 \mathrm{d}\tau\right)^{1/2} + c\|H_m\|_{2,1-\mu,\Omega^t}\sup_t\left|\frac{\alpha_{2m}}{r}\right|_{2,\Omega}\\
&+ c(\|H_m\|_{2,1-\mu,\Omega^t} + \|W_m\|_{2,1-\mu,\Omega^t}).
\end{aligned} \tag{3.47}
$$

Using (3.10), (3.12), (3.25), (3.26) and (3.35) in (3.47) gives

$$
\begin{aligned}
&|||A'_{m+1}|||_{2,2,1-\mu,\Omega^t} \le \varphi(A)[t^a \sup_t |A'_{m+1}|_{2,1-\mu,\Omega} + |A_{1m+1}|_{2,-(1+\mu),\Omega^t} \\
&+ \left(\sup_t \left| \frac{A_{2m+1}}{r} \right|_{2,\Omega} + \left(\int_0^t \left\| \frac{A_{2m+1}}{r} \right\|_{1,\Omega}^2 \mathrm{d}\tau \right)^{1/2} \right) X_2 \\
&+ (\sup_t |V_m|_{2,\Omega} + \|V_m\|_{A_{1-\mu}(\Omega^t)})[(Y_1 + e^{ct} X_5) X_2 + X_6] \\
&+ (\sup_t |H_m|_{2,1-\mu,\Omega} + \|H_m\|_{2,1-\mu,\Omega^t})(Y_1 + e^{ct} X_5)] \\
&+ c(\|H_m\|_{2,1-\mu,\Omega^t} + \|W_m\|_{2,1-\mu,\Omega^t}).
\end{aligned} \tag{3.48}
$$

Applying [15, (5.9)] to problems (3.40) and (3.42) we obtain

$$
\begin{aligned}
&|A'_{m+1}(t)|_{2,\Omega} \le \varphi(A)\Big[\sup_t \left| \frac{A_{2m+1}}{r} \right|_{2,\Omega} \|h_m\|_{2,-\mu,\Omega^t} \\
&+ \varepsilon_1 \left(\int_0^t |A_{3m+1,t'}|_{2,1-\mu,\Omega}^2 \mathrm{d}t' \right)^{1/2} + c(\varepsilon_1)\|W_m\|_{2,1-\mu,\Omega^t} \\
&+ |V_m \nabla \alpha'_m|_{2,1-\mu,\Omega^t} + |\alpha'_m \cdot \nabla V_m|_{2,1-\mu,\Omega^t} + \left| \frac{\alpha_{2m}}{r} H_m \right|_{2,1-\mu,\Omega^t} + \left| \frac{1}{r^2} H_{m,x} \right|_{2,1-\mu,\Omega^t} \Big],
\end{aligned} \tag{3.49}
$$

where the last three terms we estimate by

$$
\begin{aligned}
&c \sup_t |V_m|_{2,1-\mu,\Omega} |||\alpha'_m|||_{2,2,1-\mu,\Omega^t} + c\|V_m\|_{A_{1-\mu}(\Omega^t)} \sup_t |\alpha'_m|_{2,1-\mu,\Omega} \\
&+ c \sup_t |H_m|_{2,1-\mu,\Omega} \left(\int_0^t \left\| \frac{\alpha_{2m}}{r} \right\|_{1,\Omega}^2 \mathrm{d}\tau \right)^{1/2} + c\|H_m\|_{2,1-\mu,\Omega^t} \sup_t \left| \frac{\alpha_{2m}}{r} \right|_{2,\Omega} \\
&+ c\|H_m\|_{2,-\mu,\Omega^t}.
\end{aligned} \tag{3.50}
$$

Applying [15, (5.17)] yields

$$
\begin{aligned}
&|A'_{m+1}|_{2,\Omega}^2 + \nu \int_0^t \|A'_{m+1}\|_{1,\Omega}^2 \mathrm{d}\tau \le \varphi(A)\Big[\sup_t \left| \frac{A_{2m+1}}{r} \right|_{2,\Omega}^2 \|h_m\|_{2,-\mu,\Omega^t}^2 \\
&+ \varepsilon_1 \int_0^t |A_{3m+1,t'}|_{2,\Omega}^2 \mathrm{d}t' + c(\varepsilon_1)\|W_m\|_{2,1-\mu,\Omega^t}^2 \\
&+ c \sup_t |V_m|_{2,1-\mu,\Omega}^2 |||\alpha'_m|||_{2,2,1-\mu,\Omega^t}^2 + c\|V_m\|_{A_{1-\mu}(\Omega^t)}^2 \sup_t |\alpha'_m|_{2,1-\mu,\Omega}^2 \\
&+ c \sup_t |H_m|_{2,1-\mu,\Omega}^2 \int_0^t \left\| \frac{\alpha_{2m}}{r} \right\|_{1,\Omega}^2 \mathrm{d}\tau + c\|H_m\|_{2,1-\mu,\Omega^t}^2 \sup_t \left| \frac{\alpha_{2m}}{r} \right|_{2,\Omega}^2 \Big],
\end{aligned} \tag{3.51}
$$

where we used (3.50).

Applying [15, (5.18)] to (3.40) implies

$$
\begin{aligned}
&|A_{1m+1}|^2_{2,-\mu,\Omega}+\nu|A_{1m+1,x}|^2_{2,-\mu,\Omega^t}+\nu(1-2\mu^2)|A_{1m+1}|^2_{2,-(1+\mu),\Omega^t}\\
&\le cA^2\sup_t|A'_{m+1}|^2_{2,\Omega}+c|A_{1m+1}|^2_{2,\Omega}+c\left|\frac{A_{2m+1}}{r}\right|^2_{\frac{10}{3},\Omega^t}\|h_m\|^2_{2,-\mu,\Omega^t}\\
&+c\sup_t|V_m|^2_{2,1-\mu,\Omega}|||\alpha'_m|||^2_{2,2,1-\mu,\Omega^t}+c\|V_m\|^2_{A_{1-\mu}(\Omega^t)}\sup_t|\alpha'_m|^2_{2,1-\mu,\Omega}\\
&+c\sup_t|H_m|^2_{2,1-\mu,\Omega}\int_0^t\left\|\frac{\alpha_{2m}}{r}\right\|^2_{1,\Omega}\mathrm{d}\tau+c\|H_m\|^2_{2,1-\mu,\Omega^t}\sup_t\left|\frac{\alpha_{2m}}{r}\right|^2_{2,\Omega}\\
&+c(\|H_m\|^2_{2,-\mu,\Omega^t}+\|W_m\|^2_{2,1-\mu,\Omega^t}).
\end{aligned}
\tag{3.52}
$$

Now we obtain estimates for H_m. Multiplying (3.38) by H_m integrating over Ω yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|H_m|^2_{2,\Omega}+\nu\|H_m\|^2_{1,\Omega}\\
&\le c|\nabla v_m|^2_{3,\Omega}|H_m|^2_{2,\Omega}+c|\nabla h_m|^2_{3,\Omega}|V_m|^2_{2,\Omega}+c|h_{m-1}|^2_{3,\Omega}|\nabla v_m|^2_{2,\Omega}.
\end{aligned}
\tag{3.53}
$$

Integrating (3.53) with respect to time and using the estimates

$$
|V_m(t)|^2_{2,\Omega}\le t\int_0^t|V_{m,t'}|^2_{2,\Omega}\mathrm{d}t',\qquad |\nabla V_m(t)|^2_{2,\Omega}\le t\int_0^t|\nabla V_{m,t'}|^2_{2,\Omega}\mathrm{d}t',
$$

we obtain

$$
\begin{aligned}
&|H_m|^2_{2,\Omega}+\nu\int_0^t\|H_m\|^2_{1,\Omega}\mathrm{d}\tau\le t\varphi(A)(\|h_m\|^2_{2,\Omega^t}+\|h_{m-1}\|^2_{2,\Omega^t})\|V_m\|_{A_{1-\mu}(\Omega^t)}\\
&\le\varphi(A)X_2\|V_m\|_{A_{1-\mu}(\Omega^t)}).
\end{aligned}
\tag{3.54}
$$

From (3.38) we have

$$
\begin{aligned}
&\|H_m\|_{2,-1,\Omega^t}+(\int_0^t\|Q_m\|^2_{1,-1,\Omega}\mathrm{d}\tau)^{1/2}\le c(|V_m\nabla h_m|_{2,-1,\Omega^t}+|h_{m-1}\cdot\nabla V_m|_{2,-1,\Omega^t}\\
&+|v_m\cdot\nabla H_m|_{2,-1,\Omega^t}+|H_m\cdot\nabla v_m|_{2,-1,\Omega^t}),
\end{aligned}
\tag{3.55}
$$

where the first two terms on the r.h.s. of (3.55) we estimate by

$$
c(\|h_m\|_{2,-1,\Omega^t}+\|h_{m-1}\|_{2,-1,\Omega^t})\|V_m\|_{A_{1-\mu}(\Omega^t)},
$$

and the second two by

$$
\varepsilon\|H_m\|_{2,-1,\Omega^t}+c(\varepsilon)\varphi(A)\sup_t|H_m|_{2,-1,\Omega},
$$

where

$$
\sup_t|H_m|_{2,-1,\Omega}\le t^{1/2}\|H_m\|_{2,-1,\Omega^t}.
$$

Taking $\varepsilon=1/2$ and assuming that t is so small that

$$
\varphi(A)t^{1/2}\le 1/4,
\tag{3.56}
$$

we obtain from (3.55) the inequality

$$\|H_m\|_{2,-1,\Omega^t}+\Big(\int_0^t\|Q_m\|_{1,-1,\Omega}^2\mathrm{d}\tau\Big)^{1/2}\le c(\|h_m\|_{2,-1,\Omega^t}+\|h_{m-1}\|_{2,-1,\Omega^t})\|V_m\|_{A_{1-\mu}(\Omega^t)}\le\varphi(A)X_2\|V_m\|_{A_{1-\mu}(\Omega^t)}. \tag{3.57}$$

Applying [13, (5.15)] to (3.39) yields

$$\begin{aligned}\|W_m\|_{2,1-\mu,\Omega^t}&\le c|W_m|_{2,-(1+\mu),\Omega^t}+\varphi(A)[|W_m|_{2,1-\mu,\Omega^t}\\&+X_2\|V_m\|_{A_{1-\mu}(\Omega^t)}]+c\left(|V_m\cdot\nabla\omega_{m-1}|_{2,1-\mu,\Omega^t}+\left|\frac{V_m}{r}\omega_{m-1}\right|_{2,1-\mu,\Omega^t}\right),\end{aligned} \tag{3.58}$$

where (3.54) and (3.57) were used.

The last two terms on the r.h.s. of (3.58) we estimate by

$$\begin{aligned}&c\Big[\Big(\int_0^t|V_m|_{2p_1,\Omega}^2|\nabla\omega_{m-1}|_{2p_2,1-\mu,\Omega}^2\mathrm{d}t'\Big)^{1/2}+\Big(\int_0^t|V_m|_{2p_1,\Omega}^2|\omega_{m-1}|_{2p_2,-\mu,\Omega}^2\mathrm{d}t'\Big)^{1/2}\Big]\\&\le c\Big(\int_0^t|V_m|_{2p_1,\Omega}^2|\omega_{m-1,x}|_{2p_2,1-\mu,\Omega}^2\mathrm{d}t'\Big)^{1/2}\equiv I,\end{aligned}$$

where $1/p_1+1/p_2=1$ and the Hardy inequality was used.
Using the imbeddings $\nabla_xW_{2,1-\mu}^2(\Omega)\subset L_{2p_2,1-\mu}(\Omega)$ with $\frac{3}{2}-\frac{3}{2p_2}\le1$ (see [11]) and $V_{2,1-\mu}^1(\Omega)\subset L_{2p_1}(\Omega)$ with $1-\mu-1+\frac{3}{2}-\frac{3}{2p_1}\le0$ (see (2.8)), which hold together because $\frac{3}{2}<1+\mu$, $\mu\in\left(\frac{1}{2},\frac{1}{\sqrt{2}}\right)$, we obtain

$$\begin{aligned}I&\le c\sup_t|V_m|_{2p_1,\Omega}\Big(\int_0^t|\omega_{m-1,x}|_{2p_2,1-\mu,\Omega}^2\mathrm{d}t'\Big)^{1/2}\\&\le c\sup_t\|V_m\|_{1,1-\mu,\Omega}\Big(\int_0^t\||\omega_{m-1}\||_{2,2,1-\mu,\Omega}^2\mathrm{d}t'\Big)^{1/2}\\&\le ct^{1/2}\Big(\int_0^t\|V_{m,x}\|_{1,1-\mu,\Omega}^2\mathrm{d}t'\Big)^{1/2}\|\omega_{m-1}\|_{2,1-\mu,\Omega^t}\\&\le ct^{1/2}\|V_m\|_{A_{1-\mu}(\Omega^t)}\|\omega_{m-1}\|_{2,1-\mu,\Omega^t},\end{aligned}$$

where we exploited that $\omega_{m-1}\big|_{r=0}=0$.
Using (3.19) we obtain

$$\begin{aligned}\|W_m\|_{2,1-\mu,\Omega^t}&\le c|W_m|_{2,-(1+\mu),\Omega^t}+\varphi(A)[|W_m|_{2,1-\mu,\Omega^t}\\&+(X_2+t^{1/2}e^{ct}X_4)\|V_m\|_{A_{1-\mu}(\Omega^t)}].\end{aligned} \tag{3.59}$$

Applying [13, (5.17)] to (3.39) implies

$$|W_m|_{2,-\mu,\Omega}^2+|W_{m,x}|_{2,-\mu,\Omega^t}^2+|W_m|_{2,-(1+\mu),\Omega^t}^2\le\varphi(A)e^{ct}[X_2+t^{1/2}e^{ct}X_4]\|V_m\|_{A_{1-\mu}(\Omega^t)}. \tag{3.60}$$

From (3.59) and (3.60) we have

$$\|W_m\|_{2,1-\mu,\Omega^t}\le\varphi(A)e^{ct}[X_2+t^{1/2}e^{ct}X_4]\|V_m\|_{A_{1-\mu}(\Omega^t)}. \tag{3.61}$$

In view of (3.54), (3.57), (3.61) and (3.25) we obtain from (3.48) the inequality

$$
\begin{aligned}
&|||A'_{m+1}|||_{2,2,1-\mu,\Omega^t} \le \varphi(A)\Big[t^a \sup_t |A'_{m+1}|_{2,1-\mu,\Omega} + |A_{1m+1}|_{2,-(1+\mu),\Omega^t} \\
&+ \left(\sup_t \left|\frac{A_{2m+1}}{r}\right|_{2,\Omega} + \left(\int_0^t \left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2 \mathrm{d}\tau\right)\right)^{1/2} X_2 + \|V_m\|_{A_{1-\mu}(\Omega^t)} X_7\Big],
\end{aligned}
\tag{3.62}
$$

where

$$
X_7 = (Y_1 + e^{ct} X_5 + 1) X_2 + e^{ct} X_2 + t^{1/2} e^{ct} X_4 + X_6
\tag{3.63}
$$

Similarly (3.49) implies

$$
\begin{aligned}
&|A'_{m+1}(t)|_{2,\Omega} \le \varphi(A)\Big[\sup_t \left|\frac{A_{2m+1}}{r}\right|_{2,\Omega} X_7 \\
&+ \varepsilon\Big(\int_0^t |A_{3m+1,t'}|_{2,1-\mu,\Omega}^2 \mathrm{d}t'\Big)^{1/2} + c(\varepsilon)\|V_m\|_{A_{1-\mu}(\Omega^t)} X_7\Big].
\end{aligned}
\tag{3.64}
$$

Continuing (3.51) takes the form

$$
\begin{aligned}
&|A'_{m+1}(t)|_{2,\Omega}^2 + \nu \int_0^t \|A'_{m+1}(\tau)\|_{1,\Omega}^2 \mathrm{d}\tau \le \varepsilon \int_0^t |A_{3m+1,t'}(t')|_{2,1-\mu,\Omega}^2 \mathrm{d}t' \\
&+ c(\varepsilon)\|V_m\|_{A_{1-\mu}(\Omega^t)}^2 X_7^2.
\end{aligned}
\tag{3.65}
$$

Finally (3.52) equals to

$$
\begin{aligned}
&|A_{1m+1}|_{2,-\mu,\Omega}^2 + \nu|A_{1m+1,x}|_{2,-\mu,\Omega^t}^2 + \nu|A_{1m+1}|_{2,-(1+\mu),\Omega^t}^2 \\
&\le \varphi(A)\Big[\sup_t |A'_{m+1}|_{2,\Omega}^2 + |A_{1m+1}|_{2,\Omega^t}^2 + \|V_m\|_{A_{1-\mu}(\Omega^t)}^2 X_7^2 \\
&+ \left|\frac{A_{2m+1}}{r}\right|_{10/3,\Omega^t}^2 X_7^2\Big].
\end{aligned}
\tag{3.66}
$$

Using (3.65)–(3.66) in (3.62) yields

$$
\begin{aligned}
&|||A'_{m+1}|||_{2,2,1-\mu,\Omega^t} \le \varphi(A)\Big[\sup_t \left|\frac{A_{2m+1}}{r}\right|_{2,\Omega} + \Big(\int_0^t \left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2 \mathrm{d}\tau\Big)^{1/2} \\
&+ \|V_m\|_{A_{1-\mu}(\Omega^t)}\Big] X_7.
\end{aligned}
\tag{3.67}
$$

To apply Lemma 5.1 from [13] to (3.41) we write (3.41) in the form

$$
\begin{aligned}
&A_{2m+1,t} + v_{mr} A_{2m+1,r} + \frac{v_{m\varphi}}{r} A_{2m+1,\varphi} + v_{mz} A_{2m+1,z} \\
&+ (v_{mr,r} + v_{mz,z}) A_{2m+1} \\
&= \nu\left[\left(r\left(\frac{A_{2m+1}}{r}\right)_{,r}\right)_{,r} + \frac{1}{r^2} A_{2m+1,\varphi\varphi} + A_{2m+1,zz} + 2\left(\frac{A_{2m+1}}{r}\right)_{,r}\right] \\
&- V_m \nabla \alpha_{2m} - (V_{mr,r} + V_{mz,z})\alpha_{2m} + \frac{2\nu}{r^2}\left(-H_{m\varphi,z} + \frac{1}{r} H_{mz,\varphi}\right) \\
&- \frac{1}{r}\left(\omega_{m,z} H_{mr} - \omega_{mr} H_{mz} + \frac{\omega_m}{r} H_{mz}\right) + \frac{2}{r}\omega_m V_{m\varphi,z} \\
&- \frac{1}{r}\left(W_{m,z} h_{m-1r} - W_{m,r} h_{m-1z} + \frac{W_m}{r} h_{m-1z}\right) + \frac{2}{r} W_m v_{m-1\varphi,z}, \\
&A_{2m+1}\big|_S = 0, \\
&A_{2m+1}\big|_{t=0} = 0.
\end{aligned}
\tag{3.68}
$$

Multiplying (3.68) by $\frac{A_{2m+1}}{r^2}$ and integrating over Ω^t yields

$$
\begin{aligned}
&\left|\frac{A_{2m+1}}{r}\right|_{2,\Omega}^2+\nu\int_0^t\left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2\mathrm{d}\tau\le c\sup_t\|h_{m\varphi}\|_{1,-1,\Omega}^2\int_0^t\left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2\mathrm{d}\tau\\
&+\left|\int_{\Omega^t}V_m\nabla\alpha_{2m}\frac{A_{2m+1}}{r^2}\mathrm{d}x\right|+\left|\int_{\Omega^t}(V_{mr,r}+V_{mz,z})\alpha_{2m}\frac{A_{2m+1}}{r^2}\mathrm{d}x\right|\\
&+2\nu\left|\int_{\Omega^t}\frac{1}{r^2}\left(-H_{m\varphi,z}+\frac{1}{r}H_{mz,\varphi}\right)\frac{A_{2m+1}}{r^2}\mathrm{d}x\right|\\
&+\left|\int_{\Omega^t}\frac{1}{r}\left(\omega_{m,z}H_{mr}-\omega_{m,r}H_{mz}+\frac{\omega_m}{r}H_{mz}\right)\frac{A_{2m+1}}{r^2}\mathrm{d}x\right|\\
&+\left|\int_{\Omega^t}\frac{2}{r}\omega_mV_{m\varphi,z}\frac{A_{2m+1}}{r^2}\mathrm{d}x\right|+\left|\int_{\Omega^t}\frac{1}{r}\left(W_{m,z}h_{m-1r}-W_{m,r}h_{m-1z}+\frac{W_m}{r}h_{m-1z}\right)\frac{A_{2m+1}}{r^2}\mathrm{d}x\right|\\
&+\left|\int_{\Omega^t}\frac{2}{r}W_mv_{m-1\varphi,z}\frac{A_{2m+1}}{r^2}\mathrm{d}x\right|.
\end{aligned}
\tag{3.69}
$$

Now we have to estimate the particular terms on the r.h.s. of (3.69). The second term on the r.h.s. of (3.69) we estimate by

$$
\varepsilon\int_0^t\left|\frac{A_{2m+1}}{r}\right|_{6,\Omega}^2\mathrm{d}t+c(\varepsilon)\int_0^t\left|V_m\left(\nabla\frac{\alpha_{2m}}{r}-\frac{\alpha_{2m}}{r^2}\nabla r\right)\right|_{\frac{6}{5},\Omega}^2\mathrm{d}t\equiv I_1,
$$

where the second term in I_1 we estimate by

$$
\begin{aligned}
&\int_0^t\left(\left|V_m\nabla\frac{\alpha_{2m}}{r}\right|_{\frac{6}{5},\Omega}^2+\left|V_m\frac{\alpha_{2m}}{r^2}\right|_{\frac{6}{5},\Omega}^2\right)\mathrm{d}t\\
&\le\int_0^t|V_m|_{3,\Omega}^2\left|\nabla\frac{\alpha_{2m}}{r}\right|_{2,\Omega}^2\mathrm{d}t+\int_0^t|V_m|_{3,-1+\sigma,\Omega}^2|\alpha_{2m}|_{2,-(1+\sigma),\Omega}^2\mathrm{d}t\equiv I_2,
\end{aligned}
$$

where σ is close to 1.
Using the Hardy inequality we obtain

$$
I_2\le c\sup_t|V_m|_{3,-1+\sigma,\Omega}^2\int_0^t\left|\nabla\frac{\alpha_{2m}}{r}\right|_{2,\Omega}^2\mathrm{d}t\equiv I_3.
$$

Finally using (2.8) with $1-\mu-1+\frac{3}{2}-\frac{3}{3}\le\sigma-1$ which implies that $\mu\ge\frac{3}{2}-\sigma$ so $\mu>\frac{1}{2}$ we have

$$
I_3\le\varphi(A)\sup_t\|V_m\|_{1,1-\mu,\Omega}^2\le\varphi(A)t\|V_m\|_{A_{1-\mu}(\Omega^t)}^2.
$$

The third term on the r.h.s. of (3.69) we estimate by

$$\begin{aligned}&\int\limits_0^t |V_{m,x}|_{\frac{3}{2},\Omega}\left|\frac{\alpha_{2m}}{r}\right|_{6,\Omega}\left|\frac{A_{2m+1}}{r}\right|_{6,\Omega}\mathrm{d}t\\&\le\varepsilon\int\limits_0^t\left|\frac{A_{2m+1}}{r}\right|_{6,\Omega}^2\mathrm{d}t+c(\varepsilon)\sup_t|V_{m,x}|_{3/2,\Omega}^2\int\limits_0^t\left|\frac{\alpha_{2m}}{r}\right|_{6,\Omega}^2\mathrm{d}t\\&\le\varepsilon\int\limits_0^t\left|\frac{A_{2m+1}}{r}\right|_{6,\Omega}^2\mathrm{d}t+c(\varepsilon)\varphi(A)t\|V_m\|_{A_{1-\mu}(\Omega^t)}^2.\end{aligned}$$

Repeating the proof of Lemma 5.1 [13] we estimate the fourth term on the r.h.s. of (3.69) by

$$\varepsilon\int\limits_{\Omega^t}\left|\frac{A_{2m+1,\varphi}}{r^2}\right|^2\mathrm{d}x+c(\varepsilon)\|H_m\|_{2,-1,\Omega^t}^2.$$

The fifth term on the r.h.s. of (3.69) we estimate by

$$\begin{aligned}&c\int\limits_{\Omega^t}\left(|\omega_{m,x}|+\left|\frac{\omega_m}{r}\right|\right)\left|\frac{H_m}{r^2}\right|\left|\frac{A_{2m+1}}{r}\right|\mathrm{d}x\\&\le\varepsilon\int\limits_0^t\left|\frac{A_{2m+1}}{r}\right|_{6,\Omega}^2\mathrm{d}t+c(\varepsilon)\sup_t\left(|\omega_{m,x}|_{2,\Omega}^2+\left|\frac{\omega_m}{r}\right|_{2,\Omega}^2\right)\int\limits_0^t\left|\frac{H_m}{r^2}\right|_{3,\Omega}^2\mathrm{d}t\\&\le\varepsilon\int\limits_0^t\left|\frac{A_{2m+1}}{r}\right|_{6,\Omega}^2\mathrm{d}t+c(\varepsilon)\varphi(A)e^{ct}X_5\|H_m\|_{2,-1,\Omega^t}^2,\end{aligned}$$

where we used (3.24).

The sixth term on the r.h.s. of (3.69) we estimate by

$$\begin{aligned}&\varepsilon\int\limits_0^t\left|\frac{A_{2m+1}}{r}\right|_{6,\Omega}^2\mathrm{d}t+c(\varepsilon)\int\limits_0^t\left|\frac{\omega_m}{r^{1+\mu}}\right|_{2,\Omega}^2\left|\frac{V_{m\varphi,z}}{r^{1-\mu}}\right|_{3,\Omega}^2\mathrm{d}t\\&\le\varepsilon\int\limits_0^t\left|\frac{A_{2m+1}}{r}\right|_{6,\Omega}^2\mathrm{d}t+c(\varepsilon)\sup_t|\omega_m|_{2,-(1+\mu),\Omega}^2\|V_m\|_{A_{1-\mu}(\Omega^t)}^2\\&\le\varepsilon\int\limits_0^t\left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2\mathrm{d}t+c(\varepsilon)\varphi(A)e^{ct}X_5\|V_m\|_{A_{1-\mu}(\Omega^t)}^2.\end{aligned}$$

The seventh term on the r.h.s. of (3.69) we estimate by

$$\begin{aligned}&\varepsilon\int\limits_0^t\left|\frac{A_{2m+1}}{r}\right|_{6,\Omega}^2\mathrm{d}t+c(\varepsilon)\int\limits_0^t(|W_{m,x}|_{2,1-\mu,\Omega}^2+|W_m|_{2,-\mu,\Omega}^2)\cdot\cdot|h_m|_{3,-3+\mu,\Omega}^2\mathrm{d}t\\&\le\varepsilon\int\limits_0^t\left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2\mathrm{d}t+c(\varepsilon)\sup_t(|W_{m,x}|_{2,1-\mu,\Omega}^2+|W_m|_{2,-\mu,\Omega}^2)\|h_m\|_{2,-1,\Omega^t}^2\\&\le\varepsilon\int\limits_0^t\left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2\mathrm{d}t+c(\varepsilon)\varphi(A)X_1^2X_7^2\|V_m\|_{A_{1-\mu}(\Omega^t)}^2,\end{aligned}$$

where we used that

$$\sup_t |W_{m,x}|_{2,1-\mu,\Omega} \le c\|W_m\|_{2,1-\mu,\Omega^t}$$

and (3.10), (3.60), (3.61).
Finally the last term on the r.h.s. of (3.69) we estimate by

$$\begin{aligned}
&\varepsilon \int_0^t \left|\frac{A_{2m+1}}{r}\right|_{6,\Omega}^2 \mathrm{d}t + c(\varepsilon) \int_0^t \left|\frac{W_m}{r^{1+\mu}}\right|_{3,\Omega}^2 \left|\frac{v_{m-1\varphi,z}}{r^{1-\mu}}\right|_{3,\Omega}^2 \mathrm{d}t \\
&\le \varepsilon \int_0^t \left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2 \mathrm{d}t + c(\varepsilon)\sup_t |W_m|_{2,-(1+\mu),\Omega}^2 |||v_{m-1}|||_{3,2,1-\mu,\Omega^t}^2,
\end{aligned}$$

where to estimate the first factor of the second term we apply the proof of (5.13) from [14] to (3.39). Hence we have

$$\begin{aligned}
&|W_{m,x}(t)|_{2,-\mu,\Omega}^2 + |W_m(t)|_{2,-(1+\mu),\Omega}^2 \le \varphi(A)\Big[\int_0^t \|Q_m\|_{1,-1,\Omega}^2 \mathrm{d}t \\
&+ \|H_m\|_{2,-1,\Omega^t}^2 + \int_0^t |V_m|_{2,-\mu,\Omega}^2 \mathrm{d}t + \int_0^t |V_m \nabla\omega_{m-1}|_{2,-\mu,\Omega}^2 \mathrm{d}t \\
&+ \int_0^t \left|V_m \frac{\omega_{m-1}}{r}\right|_{2,-\mu,\Omega}^2 \mathrm{d}t\Big] \le \varphi(A)(t^a + X_7^2)\|V_m\|_{A_{1-\mu}(\Omega^t)}^2,
\end{aligned} \tag{3.70}$$

where the last two terms in the middle expression we estimate by

$$\sup_t \left(|\nabla\omega_{m-1}|_{2,-\mu,\Omega}^2 + \left|\frac{\omega_{m-1}}{r}\right|_{2,-\mu,\Omega}^2\right) \int_0^t |V_m|_{\infty,\Omega}^2 \mathrm{d}t \le \varphi(A)X_7^2\|V_m\|_{A_{1-\mu}(\Omega^t)}^2,$$

and we exploited (3.57).
Using the above estimates in (3.69) and assuming that ε and $\sup_t \|h_{m\varphi}\|_{1,-1,\Omega}$ are sufficiently small we obtain

$$\begin{aligned}
&\left|\frac{A_{2m+1}}{r}\right|_{2,\Omega}^2 + \nu \int_0^t \left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2 \mathrm{d}t \le \varphi(A)\Big[\int_0^t \|Q_m\|_{1,-1,\Omega}^2 \mathrm{d}t \\
&+ \|H_m\|_{2,-1,\Omega^t}^2 + (X_7^2 + t^a)\|V_m\|_{A_{1-\mu}(\Omega^t)}^2\Big] \le \varphi(A)(X_7^2 + t^a)\|V_m\|_{A_{1-\mu}(\Omega^t)}^2,
\end{aligned} \tag{3.71}$$

where $a > 0$.
To obtain further estimates for A_{2m+1} we write (3.41) in the form

$$\begin{aligned}
&A_{2m+1,t} + v_m \cdot \nabla A_{2m+1} + (v_{mr,r} + v_{mz,z})A_{2m+1} + \nu\frac{A_{2m+1}}{r^2} \\
&- \nu\Delta A_{2m+1} = -V_m \cdot \nabla\alpha_{2m} - (V_{mr,r} + V_{mz,z})\alpha_{2m} \\
&+ \frac{2\nu}{r^2}\left(\frac{1}{r}H_{mz,\varphi} - H_{m\varphi,z}\right) - \frac{1}{r}\Big(W_{m,z}h_{mr} + \omega_{m-1,z}H_{mr} \\
&- W_{m,r}h_{mz} - \omega_{m-1,r}H_{mz} + \frac{W_m}{r}h_{mz} + \frac{\omega_{m-1}}{r}H_{mz}\Big) + \frac{2}{r}W_m v_{m\varphi,z} + \frac{2}{r}\omega_{m-1}V_{m\varphi,z}, \\
&A_{2m+1}\big|_S = 0, \\
&A_{2m+1}\big|_{t=0} = 0.
\end{aligned} \tag{3.72}$$

Applying the proof of Lemma 5.1 from [15] to (3.72) yields

$$
\begin{aligned}
&\|A_{2m+1}\|_{2,1-\mu,\Omega^t} \le c(\|v_m\|_{A_{1-\mu}(\Omega^t} + \|h_m\|_{2,1-\mu,\Omega^t} + 1)\cdot \\
&\cdot\left(|A_{2m+1}|_{2,-1,\Omega} + \left(\int\limits_0^t \left\|\frac{A_{2m+1}}{r}\right\|_{1,\Omega}^2 \mathrm{d}\tau\right)^{1/2}\right) \\
&+ c(\|\omega_{m-1}\|_{2,1-\mu,\Omega^t} + 1)\|H_m\|_{2,-1,\Omega^t} + c(\|h_m\|_{2,-1,\Omega^t}\|W_m\|_{2,1-\mu,\Omega^t} \\
&+ \sup_t \|\omega_{m-1}\|_{1,-\mu,\Omega}\|V_m\|_{A_{1-\mu}(\Omega^t)} + \|v_m\|_{A_{1-\mu}(\Omega^t)} \sup_t \|W_m\|_{1,-\mu,\Omega}) \\
&+ c(|\nabla\alpha_{2m}V_m|_{2,1-\mu,\Omega^t} + |\alpha_{2m}V_{m,x}|_{2,1-\mu,\Omega^t}),
\end{aligned}
\tag{3.73}
$$

where the last two terms we estimate by (see [12])

$$ct^a\|\alpha_{2m}\|_{2,1-\mu,\Omega^t}\|V_m\|_{A_{1-\mu}(\Omega^t)},$$

where $a > 0$.

Summarizing we have

$$
\begin{aligned}
&\|A_{2m+1}\|_{2,1-\mu,\Omega^t} \le \varphi(A)[(\int\limits_0^t \|Q_m\|_{1,-1,\Omega}^2 \mathrm{d}t)^{1/2} + \|H_m\|_{2,-1,\Omega^t} \\
&+ (t^a + X_7)\|V_m\|_{A_{1-\mu}(\Omega^t)} + \|W_m\|_{2,1-\mu,\Omega^t} + \sup_t \|W_m\|_{1,-\mu,\Omega}] \\
&\le \varphi(A)(t^a + X_7)\|V_m\|_{A_{1-\mu}(\Omega^t)},
\end{aligned}
\tag{3.74}
$$

where to show the second inequlaity we used (3.57), (3.61), (3.70).

From (3.43), (3.67), (3.71), (3.74) we obtain (3.37). This concludes the proof.

From Lemmas 3.1, 3.2 we have

Theorem 3.3. *Let the assumptions of Lemmas 3.1, 3.2 hold. Then there exists a solution of (1.1) such that $v \in A_{1-\mu}(\Omega^T)$, $p \in L_2(0,T); W_{2,1-\mu}^2(\Omega))$ and $\|v\|_{A_{1-\mu}(\Omega^T)} \le A$.*

4. Global existence

To prove global existence we have to prolongate the local solution from Section 3. To prolongate the local solution we have to show (3.9) in the form

$$\|v_m\|_{A_{1-\mu}(\Omega\times(T_0,t))} \le A, \tag{4.1}$$

where $t \le T_0 + T$, and T_0 is an arbitrary number. Then by Theorem 3.3 we show existence of solutions to (1.1) in the interval $(T_0, T_0 + T)$. For the inductive considerations we assume $T_0 = kT$, where $k \in \mathbb{N}$.

To show (4.1) we have to satisfy the assumptions of Lemma 3.1 at $t = T_0$. For this purpose we define:

$$
\begin{aligned}
&X_5(T_0) = |g|_{2,\Omega\times(T_0,t)} + |f_\varphi|_{2,-\mu,\Omega\times(T_0,t)} \\
&+ \|h(T_0)\|_{1,-1,\Omega} + |\omega_{,x}(T_0)|_{2,-\mu,\Omega} + |\omega(T_0)|_{2,-(1+\mu),\Omega}, \\
&X_6(T_0) = X_5(T_0) + |F'|_{2,1-\mu,\Omega\times(T_0,t)} + |\alpha'(T_0)|_{2,1-\mu,\Omega}, \\
&D(T_0,t) = |f|_{2,\Omega\times(T_0,t)} + |\int\limits_\Omega v_\eta(T_0)\mathrm{d}x| \\
&+ \sup_t |\int\limits_{T_0}^t \mathrm{d}\tau \int\limits_\Omega f_\eta(x,\tau)\mathrm{d}x| + |v(T_0)|_{2,\Omega},
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
&Y_1(T_0) = |\chi(T_0)|_{2,-1,\Omega} + |F_2|_{2,-1,\Omega\times(T_0,t)},\\
&Y_2(T_0) = |F_2|_{2,1-\mu,\Omega\times(T_0,t)} + \|\chi(T_0)\|_{1,1-\mu,\Omega},
\end{aligned}
$$

and $t \in (T_0, T_0 + T)$.
Let us introduce the quantities

$$
\begin{aligned}
&X_5^* = \max_{T_0} X_5(T_0), \qquad X_6^* = \max_{T_0} X_6(T_0),\\
&D^* = \max_{T_0} D(T_0,t), \qquad Y_1^* = \max_{T_0} Y_1(T_0), \qquad Y_2^* = \max_{T_0} Y_2(T_0)
\end{aligned}
\tag{4.3}
$$

and let us formulate formally the inequalities

$$
\begin{aligned}
&c(Y_1^* + \varphi(A)e^{cT}X_5^*)^2 D^* + c(Y_1^* + Y_2^*) \le A,\\
&\varphi(A)(e^{cT}X_5^* + X_6^*) + \varphi(A)X_5^* Y_1^* \le A.
\end{aligned}
\tag{4.4}
$$

If we show that Y_1^*, Y_2^*, D^* are bounded and if X_5^* and X_6^* are sufficiently small then (4.4) can be satisfied by finding an appropriate number A. Hence we can apply Lemma 3.1 and then Theorem 3.3 step by step up to infinity.

To simplify notation we introduce

$$
A(t_1,t_2) = \|v\|_{A_{1-\mu}(\Omega\times(t_1,t_2))}. \tag{4.5}
$$

We assume the decay

$$
\gamma(t) \equiv |g(t)|_{2,-1,\Omega} + |g_t(t)|_{2,\Omega} \le \gamma(0)e^{-\nu_0 t}. \tag{4.6}
$$

First we have

Lemma 4.1. *Assume that $v \in A_{1-\mu}(\Omega^t)$ and h is a solution of (1.2). Assume that $0 < t_1 < t \le T$. Assume (4.6). Then*

$$
\|h(t)\|_{1,-1,\Omega} \le \varphi(A(0,t))e^{-\nu_0 t_1}(\|h(0)\|_{1,-1,\Omega} + \gamma(0)), \tag{4.7}
$$

where φ is an increasing positive function and (4.5) is used.

Proof. Let $\zeta = \zeta(t)$ be a smooth function such that $\zeta(t) = 0$ for $t \le t_1$ and $\zeta(t) = 1$ for $t \ge t_2$, where $t_2 < T$ and T is the time of local solution. Multiplying (1.2) by $\zeta(t)$ and introducing new quantities $\tilde h = h\zeta$, $\tilde q = q\zeta$, $\tilde G = G\zeta$, problem (1.2) takes the form

$$
\begin{aligned}
&\tilde h_t - \mathrm{div}\mathbb{D}(\tilde h) + \nabla\tilde q = \tilde G + \dot\zeta h,\\
&\mathrm{div}\tilde h = 0,\\
&\tilde h\cdot\bar n = 0,\\
&\bar n\cdot\mathbb{D}(\tilde h)\cdot\bar\tau_\alpha = 0, \quad \alpha = 1,2,\\
&\tilde h\big|_{t=0} = 0.
\end{aligned}
\tag{4.8}
$$

Considering (4.8) in the interval (t_1, t) we obtain the estimate

$$
\begin{aligned}
&\|\tilde h\|_{2,-1,\Omega\times(t_1,t)} + \|\nabla\tilde q\|_{0,-1,\Omega\times(t_1,t)}\\
&\le c_1\Big(\|\tilde G\|_{0,-1,\Omega\times(t_1,t)} + \frac{1}{t_2 - t_1}\|h\|_{0,-1,\Omega\times(t_1,t)}\Big),
\end{aligned}
\tag{4.9}
$$

where c_1 does not depend on t (see [16]).

Now we examine the last term on the r.h.s. of (4.9). Applying the Hardy inequality and using that the domain Ω is bounded we obtain

$$\begin{aligned}\|h\|_{0,-1,\Omega\times(t_1,t)} &\le c\|h\|_{0,-1-\varepsilon,\Omega\times(t_1,t)} \le c\|h_{,x}\|_{0,-\varepsilon,\Omega\times(t_1,t)}\\ &\le c\|h_{,xx}\|_{0,1-\varepsilon,\Omega\times(t_1,t)} \le c\|h_{,xx}\|_{0,\Omega\times(t_1,t)},\end{aligned}\tag{4.10}$$

where $\varepsilon > 0$ and the constants c do not depend on t.

To estimate the r.h.s. of (4.10) we apply the energy method to problem (1.2). Multiplying $(1.2)_1$ by h_t and integrating over Ω yields

$$\int\limits_\Omega h_t^2 \mathrm{d}x + \int\limits_\Omega \mathbb{D}(h)\cdot\mathbb{D}(h_t)\mathrm{d}x = \int\limits_\Omega G\cdot h_t \mathrm{d}x$$

Continuing we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega |\mathbb{D}(h)|^2\mathrm{d}x + \int\limits_\Omega h_t^2\mathrm{d}x \le \int\limits_\Omega G^2\mathrm{d}x.\tag{4.11}$$

Integrating (4.11) with respect to t yields

$$\int\limits_\Omega |\mathbb{D}(h(t))|^2\mathrm{d}x + \int\limits_{\Omega^t} h_t^2\mathrm{d}x\mathrm{d}t \le \int\limits_{\Omega^t} G^2\mathrm{d}x\mathrm{d}t + \int\limits_\Omega |\mathbb{D}(h(0))|^2\mathrm{d}x.\tag{4.12}$$

In view of Lemma 3.5 and Remark 3.6 from [13] we have the Korn inequality

$$\|h\|_{1,\Omega}^2 \le c|\mathbb{D}(h)|_{2,\Omega}^2.\tag{4.13}$$

Using (4.13) in (4.12) implies

$$\|h(t)\|_{1,\Omega}^2 + |h_t|_{2,\Omega^t}^2 \le c(|G|_{2,\Omega^t}^2 + \|h(0)\|_{1,\Omega}^2),\tag{4.14}$$

where the constant c does not depend on t.

In view of (4.14) we have that $h_t \in L_2(\Omega^t)$, so instead of (1.2) we consider the elliptic problem

$$\begin{aligned}&-\mathrm{div}\mathbb{D}(h) + \nabla q = G - h_t,\\ &\mathrm{div}h = 0,\\ &h\cdot\bar{n} = 0,\\ &\bar{n}\cdot\mathbb{D}(h)\cdot\bar{\tau}_\alpha = 0, \quad \alpha = 1,2.\end{aligned}\tag{4.15}$$

For solutions of (4.15) we have the estimate

$$\|h\|_{2,\Omega}^2 + \|\nabla q\|_{0,\Omega}^2 \le c(\|G\|_{0,\Omega}^2 + \|h_t\|_{0,\Omega}^2),\tag{4.16}$$

where c does not depend on t.

Integrating (4.16) with respect to t and using (4.14) give

$$\|h\|_{2,\Omega^t}^2 + \|\nabla q\|_{0,\Omega^t}^2 \le c(\|G\|_{0,\Omega^t}^2 + \|h(0)\|_{1,\Omega}^2),\tag{4.17}$$

where c does not depend on t.

Taking $t_* < t_1$ and using the form of G we obtain from (4.9), (4.10) and (4.17) the estimate

$$\begin{aligned}\|h(t)\|_{1,-1,\Omega} \le c_1\Big[&\|v\|_{A_{1-\mu}(\Omega\times(t_1,t))}\sup_{t'\in(t_1,t)}\|h(t')\|_{1,\Omega}\\ &+\|g\|_{0,-1,\Omega\times(t_1,t)} + \frac{1}{t_2-t_1}\|h(t_*)\|_{1,\Omega}\Big],\end{aligned}\tag{4.18}$$

where $t_* < t_1$ and can be chosen as very close to t_1.
From (5.63) in [14] we have

$$\|h(t)\|_{1,\Omega} \le e^{-\nu_0 t + c\|v\|_{A_{1-\mu}(\Omega^t)}}[\|h(0)\|_{1,\Omega} + c(\int_0^t \|g(t')\|_{0,\Omega}^2 e^{\nu_0 t' - c\|v\|_{A_{1-\mu}(\Omega^{t'})}}\mathrm{d}t')^{1/2}], \tag{4.19}$$

where the constants c do not depend on t.

Using (4.6) and (4.19) in (4.18) yields

$$\begin{aligned} \|h(t)\|_{1,-1,\Omega} \le \Big[& A(t_1,t)\varphi(A(0,t_1))e^{-\nu_0 t_1}(\|h(0)\|_{1,-1,\Omega} + \gamma(0)) \\ & + \frac{1}{t_2 - t_1}\varphi(A(0,t_1))e^{-\nu_0 t_*}(\|h(0)\|_{1,-1,\Omega} + \gamma(0))\Big]. \end{aligned} \tag{4.20}$$

Using that t_2 is sufficiently greater than t_1 and t_* is sufficiently close to t_1 we obtain (4.7). This concludes the proof.

Let us introduce the quantity

$$Z_1(t) = |\omega_{,x}(t)|_{2,-\mu,\Omega} + |\omega(t)|_{2,-(1+\mu),\Omega}. \tag{4.21}$$

Lemma 4.2. *Assume that $v \in A_{1-\mu}(\Omega^t)$, $\omega \in L_\infty(0,t;H_1^1(\Omega))$, and the decay estimate (4.6) is satisfied.*

Then the inequality holds

$$Z_1(t) \le \varphi(A(0,t))[\sup_t |\omega|_{2,1,\Omega} + \gamma(0) + \|h(0)\|_{1,-1,\Omega} + |f_\varphi|_{2,-\mu,\Omega^t} + e^{-\nu_1 t}Z_1(0)]. \tag{4.22}$$

where φ is an increasing positive function and $|\omega(t)|_{2,1,\Omega}$ is estimated by (4.29).

Proof. From [14, (5.48)],

$$\begin{aligned} & |h_{,t}(t)|_{2,\Omega}^2 + \|h(t)\|_{1,\Omega}^2 \le e^{-\nu_0 t + c\|v\|_{A_{1-\mu}(\Omega^t)}}[|h_t(0)|_{2,\Omega}^2 \\ & + \|h(0)\|_{1,\Omega}^2 + c\int_0^t (\|g(t')\|_{0,\Omega}^2 + \|g_{t'}(t')\|_{0,\Omega}^2)e^{\nu_0 t' - c\|v\|_{A_{1-\mu}(\Omega^{t'})}}\mathrm{d}t'] \\ & \le \varphi(A(0,t))e^{-\nu_0 t}[|h_t(0)|_{2,\Omega}^2 + \|h(0)\|_{1,\Omega}^2 + \gamma^2(0)], \end{aligned} \tag{4.23}$$

where (4.6) was used in the second inequality and constants c do not depend on t.
From [14, (5.52)],

$$\begin{aligned} & \|q(t)\|_{0,\Omega}^2 \le c(\|h_t(t)\|_{0,\Omega}^2 + \|h(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2\|h(t)\|_{1,\Omega}^2 \\ & + \|g(t)\|_{0,\Omega}^2) \le c(1 + \|v\|_{A_{1-\mu}(\Omega^t)}^2)(\|h_t(t)\|_{0,\Omega}^2 + \|h(t)\|_{1,\Omega}^2) \\ & + c\|g(t)\|_{0,\Omega}^2 \le \varphi(A(0,t))e^{-\nu_0 t}[|h_t(0)|_{2,\Omega}^2 + \|h(0)\|_{1,\Omega}^2 + \gamma^2(0)]. \end{aligned} \tag{4.24}$$

From [13, (5.16)] we get

$$\begin{aligned} & \|h\|_{2,-1,\Omega^t} + (\int_0^t \|q(t')\|_{1,-1,\Omega}^2 \mathrm{d}t')^{1/2} \le \varphi(\|v\|_{A_{1-\mu}(\Omega^t)})(|g|_{2,-1,\Omega^t} \\ & + \|h(0)\|_{1,-1,\Omega}) \le \varphi(A(0,t))(\gamma(0) + \|h(0)\|_{1,-1,\Omega}). \end{aligned} \tag{4.25}$$

From [14, (5.13)] we have

$$
\begin{aligned}
Z_1(t) &\le \varphi(\|v\|_{A_{1-\mu}(\Omega^t)}[(e^{-2\nu_0 t}\int_0^t |\omega(t')|^2_{2,-\mu,\Omega}e^{2\nu_0 t'}\mathrm{d}t')^{1/2}\\
&+(\int_0^t |q|^2_{2,-(1+\mu),\Omega}+|h|^2_{2,-(2+\mu),\Omega}+|f_\varphi|^2_{2,-\mu,\Omega})\mathrm{d}t')^{1/2}]\\
&+ce^{-\nu_0 t+c\|v\|_{A_{1-\mu}(\Omega^t)}}Z_1(0)\\
&\le \varphi(A(0,t)[(e^{-2\nu_0 t}\int_0^t |\omega(t')|^2_{2,-\mu,\Omega}e^{2\nu_0 t'}\mathrm{d}t')^{1/2}+\gamma(0)+\|h(0)\|_{1,-1,\Omega}\\
&+|f_\varphi|_{2,-\mu,\Omega^t}+e^{-\nu_0 t}Z_1(0)],
\end{aligned}
\tag{4.26}
$$

where (4.25) was used.
From [14, (5.76)] we obtain

$$
\begin{aligned}
|\omega(t)|_{2,-\mu,\Omega} &\le \varphi(\|v\|_{A_{1-\mu}(\Omega^t)})[\sup_t |\omega|_{2,1,\Omega}\\
&+(\int_0^t (|q|^2_{2,-(1+\mu),\Omega}+|h|^2_{2,-(1+\mu),\Omega}+|f_\varphi|^2_{2,-\mu,\Omega})\mathrm{d}t')^{1/2}\\
&+ce^{-\nu_1 t+\|v\|_{A_{1-\mu}(\Omega^t)}}|\omega(0)|_{2,-\mu,\Omega}]\\
&\le \varphi(A(0,t))[\sup_t |\omega|_{2,1,\Omega}+\gamma(0)+\|h(0)\|_{1,-1,\Omega}+|f_\varphi|_{2,-\mu,\Omega^t}\\
&+e^{-\nu_1 t}|\omega(0)|_{2,-\mu,\Omega}],
\end{aligned}
\tag{4.27}
$$

where (4.25) was used in the second inequality and $\nu_1 \le \nu_0$.

To estimate the first term on the r.h.s. of (4.27) we use [14, (5.34)] in the form

$$
|\omega(t)|_{2,1,\Omega} \le |\omega(0)|_{2,1,\Omega}+c\int_0^t (|h_{,x}(t')|_{2,\Omega}+|q(t')|_{2,\Omega}+|f_\varphi(t')|_{2,1,\Omega})\mathrm{d}t'. \tag{4.28}
$$

Applying (4.23) and (4.24) in (4.28) yields

$$
|\omega(t)|_{2,1,\Omega} \le |\omega(0)|_{2,1,\Omega}+\varphi(A(0,t))(|h_t(0)|_{2,\Omega}+\|h(0)\|_{1,\Omega})+c\int_0^t |f_\varphi(t')|_{2,1,\Omega}\mathrm{d}t'. \tag{4.29}
$$

Using (4.27) in (4.26) yields (4.22). This concludes the proof.

Lemma 4.3. *Assume that $v \in A_{1-\mu}(\Omega^t)$, $\omega \in L_\infty(0,t;L_2(\Omega))$, $f_\varphi \in L_{2,-1}(\Omega^t)$, $F \in L_{2,-1}(\Omega^t)$, $h(0) \in H^1_{-1}(\Omega)$, $\chi(0) \in L_{2,-1}(\Omega)$, $\omega(0) \in H^1_{-\mu}(\Omega)$ and the decay estimate (4.6) holds.*

Then

$$
\begin{aligned}
|\chi(t)|_{2,-1,\Omega} &\le \varphi(A(0,t))[\sup_t |\omega|_{2,1,\Omega}+\gamma(0)+\|h(0)\|_{1,-1,\Omega}\\
&+|f_\varphi|_{2,-\mu,\Omega^t}]+c|F_\varphi|_{2,-1,\Omega^t}+e^{-\nu_1 t}Z_1(0)+e^{-\nu t}|\chi(0)|_{2,-1,\Omega}.
\end{aligned}
\tag{4.30}
$$

Proof. From [13, (5.7)] we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\chi|^2_{2,-1,\Omega}+\nu|\chi|^2_{2,-1,\Omega} &\le c(1+\|\omega\|^2_{1,0,\Omega})\|h\|^2_{2,-1,\Omega}\\
&+c\|\omega\|^2_{1,-\mu,\Omega}|||\omega|||^2_{2,2,1-\mu,\Omega}+c|F_2|^2_{2,-1,\Omega}.
\end{aligned}
\tag{4.31}
$$

Integrating (4.31) with respect to time yields

$$
\begin{aligned}
|\chi(t)|_{2,-1,\Omega}^2 &\le ce^{-\nu t}\int_0^t (1+\|\omega\|_{1,0,\Omega}^2)\|h\|_{2,-1,\Omega}^2 e^{+\nu t'}\mathrm{d}t' \\
&+ c\|v\|_{A_{1-\mu}(\Omega^t)}^2 \sup_t \|\omega\|_{1,-\mu,\Omega}^2 + ce^{-\nu t}\int_0^t |F_\varphi(t')|_{2,-1,\Omega}^2 e^{\nu t'}\mathrm{d}t' \\
&+ e^{-\nu t}|\chi(0)|_{2,-1,\Omega}^2.
\end{aligned}
\tag{4.32}
$$

Continuing,

$$
\begin{aligned}
|\chi(t)|_{2,-1,\Omega} &\le c(1+\sup_t \|\omega\|_{1,0,\Omega})\|h\|_{2,-1,\Omega^t} \\
&+ c\|v\|_{A_{1-\mu}(\Omega^t)} \sup_t \|\omega\|_{1,-\mu,\Omega^t} + c|F_\varphi|_{2,-1,\Omega^t} + e^{-\nu t}|\chi(0)|_{2,-1,\Omega}.
\end{aligned}
\tag{4.33}
$$

Using (4.22) and (4.25) yields (4.30). This concludes the proof.

Lemma 4.4. *Assume that $A(kT,(k+1)T)\le A$ for $\forall k\in\mathbb{N}$,*

$$
Z_2(t) = |h_t(t)|_{2,\Omega} + \|h(t)\|_{1,\Omega} + \gamma(t)
\tag{4.34}
$$

$\varphi(A)e^{-\nu_0 T}<1$, $\int_0^\infty |f_\varphi(t')|_{2,1,\Omega}\mathrm{d}t' < \varepsilon_1$, *where ε_1 is sufficiently small.*

Then

$$
|\omega(kT)|_{2,1,\Omega} \le |\omega(0)|_{2,1,\Omega} + \frac{\varphi(A)}{1-\varphi(A)e^{-\nu_0 T}} Z_2(0) + c\int_0^\infty |f_\varphi(t')|_{2,1,\Omega}\mathrm{d}t',
\tag{4.35}
$$

for any $k\in\mathbb{N}$.

Proof. From (4.29) and (4.23), (4.6) we obtain

$$
|\omega(kT)|_{2,1,\Omega} \le |\omega(0)|_{2,1,\Omega} + \sum_{i=0}^{k-1}(\varphi(A)e^{-\nu_0 T})^i\varphi(A)Z_2(0) + c\int_0^{kT}|f_\varphi(t')|_{2,1,\Omega}\mathrm{d}t'.
\tag{4.36}
$$

Passing with $k\to\infty$ on the r.h.s. of (4.36) and using the assumptions of the lemma we obtain (4.35). This concludes the proof.

Now we want to find X_5^*. From $(4.2)_1$ we have

$$
\begin{aligned}
X_5(\tau) &\le |g|_{2,\Omega\times(\tau,t)} + |f_\varphi|_{2,-\mu,\Omega\times(\tau,t)} + \varphi(A)e^{-\nu_0(t-\tau)}Z_3(\tau) \\
&+ \varphi(A)[\sup_{t\ge\tau}|\omega|_{2,1,\Omega} + \gamma(\tau) + Z_3(\tau) + |f_\varphi|_{2,-\mu,\Omega\times(\tau,t)} + e^{-\nu_1(t-\tau)}Z_1(\tau)],
\end{aligned}
$$

where

$$
Z_3(t) = \|h(t)\|_{1,-1,\Omega} + \gamma(t).
\tag{4.37}
$$

In (4.7) we have that $T>t_1>0$. Let us assume that t_1 is so large that

$$
\varphi(A)e^{-\nu t_1} \equiv a_1 < 1.
$$

Then (4.7) implies $Z_3(T)\le a_1 Z_3(0)$. Continuing $Z_3(kT)\le a_1^k Z_3(0)$. Since $|\omega(t)|_{2,1,\Omega}$ is bounded for all $t>0$ we obtain from (4.22) that

$$
Z_1(kT) \le a_2^k(Z_3(0)+Z_1(0)) + \varphi(A)(\sup_t |\omega|_{2,1,\Omega} + |f_\varphi|_{2,-\mu,\Omega^t}).
$$

If $a_2\equiv\varphi(A)e^{-\nu T}<1$ then we obtain the bound

$$
\begin{aligned}
X_5^* &= |g|_{2,\Omega\times(0,\infty)} + |f_\varphi|_{2,-\mu,\Omega\times(0,\infty)} + \int_0^\infty |f_\varphi(t')|_{2,1,\Omega}\mathrm{d}t' \\
&+ \varphi(A)(|\omega(0)|_{2,1,\Omega} + Z_1(0) + Z_2(0) + Z_3(0)).
\end{aligned}
\tag{4.38}
$$

To find X_6^* we need

Lemma 4.5. *Assume that* $v \in A_{1-\mu}(\Omega^t)$, $\omega(0) \in L_{2,1}(\Omega)$, $h_t(0) \in L_2(\Omega)$, $h(0) \in H^1_{-1}(\Omega)$, $g(0) \in L_{2,-1}(\Omega)$, $g_t(0) \in L_2(\Omega)$, $f_\varphi \in L_{2,-\mu}(\Omega^t)$, $f_\varphi \in L_1(0,t;L_{2,1}(\Omega))$, $\omega(0) \in H^1_{-\mu}(\Omega)$, $F_2 \in L_2(0,t;L_{2,-1}(\Omega))$, $F' \in L_2(0,t;L_{2,1-\mu}(\Omega))$.

Then

$$
\begin{aligned}
\|\alpha'(t)\|_{1,1-\mu,\Omega} &\le \varphi(A(0,t))[|\omega(0)|_{2,1,\Omega} + |h_t(0)|_{2,\Omega} + \|h(0)\|_{1,-1,\Omega} \\
&+ \gamma(0) + |f_\varphi|_{2,-\mu,\Omega^{t_0}} + \int_0^{t_0} |f_\varphi(t')|_{2,1,\Omega}\mathrm{d}t' + e^{-\nu_1 t_0} Z_1(0)],
\end{aligned}
\tag{4.39}
$$

where $t_0 < t$.

Proof. Multiplying problems (3.3), (3.5) by the function $\zeta(t)$ (see the proof of Lemma 4.1) and using [15, (5.21)], [13, (5.5′)] we have

$$
\begin{aligned}
\|\alpha'(t)\|_{1,1-\mu,\Omega} &\le \varphi(A(0,t))[\|h\|_{2,-1,\Omega\times(t_0,t)} + \sup_{t'\in(t_0,t)} \|\omega(t')\|_{1,-\mu,\Omega} \\
&+ (|F_2|_{2,-1,\Omega\times(t_0,t)} + |\chi(t_0)|_{2,-1,\Omega})\|h\|_{2,-\mu,\Omega\times(t_0,t)} \\
&+ \|\omega\|_{2,\Omega\times(t_0,t)} + |F'|_{2,1-\mu,\Omega\times(t_0,t)}] + \frac{c}{t_2 - t_1}\|\alpha'\|_{0,1-\mu,\Omega\times(t_0,t)},
\end{aligned}
\tag{4.40}
$$

where $0 < t_0 < t_1 < t_2 < t$.
Using the form of α' (see (1.11)) we have

$$
\begin{aligned}
&|\alpha'(t_0)|_{2,\Omega} \le c(\|h(t_0)\|_{0,-1,\Omega} + \|\omega(t_0)\|_{1,0,\Omega}) \\
&\le \varphi(A(0,t_0))e^{-\nu_1 t_1}(\|h(0)\|_{1,-1,\Omega} + \gamma(0)), \\
&\|\alpha'\|_{0,1-\mu,\Omega\times(t_0,t)} \le c(\|h\|_{0,-\mu,\Omega\times(t_0,t)} + \|\omega\|_{1,1-\mu,\Omega\times(t_0,t)}) \\
&\le \varphi(A(0,t_0))[|\omega(0)|_{2,1,\Omega} + |h_t(0)|_{2,\Omega} + \|h(0)\|_{1,-1,\Omega} + \gamma(0) \\
&+ |f_\varphi|_{2,-\mu,\Omega^{t_0}} + \int_0^{t_0} |f_\varphi(t')|_{2,1,\Omega}\mathrm{d}t' + e^{-\nu_1 t_0} Z_1(0)],
\end{aligned}
\tag{4.41}
$$

where $0 < t_1 < t_0$.

From (4.25) we have

$$
\begin{aligned}
&\|h\|_{2,-1,\Omega\times(t_0,t)} \le \varphi(A(t_0,t))[\|g\|_{0,\Omega\times(t_0,t)} + \|h(t_0)\|_{1,-1,\Omega}] \\
&\le \varphi(A(0,t))e^{-\nu_1 t_1}(\|h(0)\|_{1,-1,\Omega} + \gamma(0)),
\end{aligned}
\tag{4.42}
$$

where $0 < t_1 < t_0$.

Finally using (4.22) and (4.29) we have

$$
\begin{aligned}
\sup_t \|\omega(t)\|_{1,-\mu,\Omega} &\le \varphi(A(0,t))[|\omega(0)|_{2,1,\Omega} + |h_t(0)|_{2,\Omega} + \|h(0)\|_{1,-1,\Omega} \\
&+ \gamma(0) + |f_\varphi|_{2,-\mu,\Omega^t} + \int_0^t |f_\varphi(t')|_{2,1,\Omega}\mathrm{d}t' + e^{-\nu_1 t} Z_1(0)].
\end{aligned}
\tag{4.43}
$$

Using Lemma 5.7 from [13] and (4.41)–(4.43) in (4.40) yield (4.39). This concludes the proof.

Assuming any decay of $|F'(t)|_{2,1-\mu,\Omega}$ we obtain that there exists $\sup_{T_0} |F'|_{2,1-\mu,\Omega\times(T_0,t)}$.
Therefore to determine X_6^* we have to show that $\sup_t \|\alpha'(t)\|_{1,1-\mu,\Omega}$ could exist and remains small.
The inequality (4.39) is proved for $t \le T$, where T is the time of local existence. From (4.36) we see that $\sup_t |\omega(kT)|_{2,1,\Omega}$ remains small.

Moreover we have the decay

$$|h_t(t)|_{2,\Omega} + \|h(t)\|_{1,\Omega} + \gamma(t) \le e^{-\nu t}(|h_t(0)|_{2,\Omega} + \|h(0)\|_{1,\Omega} + \gamma(0)). \tag{4.44}$$

From (4.22) and (4.36) we see that $Z_1(t)$ remains small for all $t > 0$.
Assuming the decay

$$|f_\varphi(t)|_{2,-\mu,\Omega} + |f_\varphi(t)|_{2,1,\Omega} \le ce^{-\nu_2 t}(|f_\varphi(0)|_{2,-\mu,\Omega} + |f_\varphi(0)|_{2,1,\Omega}),$$

we see that the two norms with f_φ remain small for all $t > 0$.

In this way we have shown that there exists Y_6^* which could remain as small as we wish.

From [13, (3.30)] we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|v|_{2,\Omega} \le |f|_{2,\Omega},$$

so

$$|v(t)|_{2,\Omega} \le \int_0^t |f(t')|_{2,\Omega}\mathrm{d}t' + |v(0)|_{2,\Omega}. \tag{4.45}$$

Therefore we have a global estimate for $|v(t)|_{2,\Omega}$ if we assume an appropriate decay

$$|f(t)|_{2,\Omega} \le e^{-\nu t}|f(0)|_{2,\Omega}. \tag{4.46}$$

From [13, (3.28)] we have that

$$\Big|\int_\Omega v_\eta(t)\mathrm{d}x\Big| \le \Big|\int_\Omega v_\eta(0)\mathrm{d}x\Big| + \Big|\int_0^t \mathrm{d}t'\int_\Omega f_\eta(x,t')\mathrm{d}x\Big|, \tag{4.47}$$

so $\int_\Omega v_\eta(t)\mathrm{d}x$ remains bounded for all $t > 0$ if

$$\Big|\int_0^t \mathrm{d}t'\int_\Omega f_\eta(x,t')\mathrm{d}x\Big| < \infty \quad \text{for} \ \ t > 0. \tag{4.48}$$

In view of the above remarks we see that there exists $D^* < \infty$ such that

$$\sup_k D(kT,(k+1)T) \equiv D^*.$$

To find Y_1^* and Y_2^* it is enough to show that there exists $\sup_t \|\chi(t)\|_{1,-1,\Omega} < \infty$.
For this purpose we need.

Lemma 4.6. *Assume that* $v \in A_{1-\mu}(\Omega^T)$, $h \in H^{2,1}_{-1}(\Omega^T)$, $\omega \in L_\infty(0,T;H^1_{-\mu}(\Omega))$, $F_2 \in L_2(0,T;L_{2,-1}(\Omega))$, $\chi(0) \in L_{2,-1}(\Omega)$.

Then

$$\|\alpha_2(t)\|_{1,1-\mu,\Omega} \le \varphi(A(0,t))[\|h\|_{2,-1,\Omega^t} + \sup_t \|\omega\|_{1,-\mu,\Omega} + |F_2|_{2,-1,\Omega^t} + |\chi(0)|_{2,-1,\Omega}]. \tag{4.49}$$

Proof. From [15, Lemma 5.5] we have

$$\begin{aligned} \|\alpha_2(t)\|_{1,1-\mu,\Omega} &\le \varphi(A(0,t))\Big[|\alpha_2|_{2,-1,\Omega} + \bigg(\int_0^t \Big\|\frac{\alpha_2}{r}\Big\|^2_{1,\Omega}\mathrm{d}\tau\bigg)^{1/2} \\ &\quad + \|h\|_{2,-1,\Omega^t} + \sup_t \|\omega\|_{1,-\mu,\Omega} + |F_2|_{2,1-\mu,\Omega^t}\Big] + c|\alpha_2|_{2,1-\mu,\Omega\times(t_1,t_2)}, \end{aligned} \tag{4.50}$$

where $0 < t_1 < t_2 < t$.
Lemma 5.1 from [13] yields.

$$|\alpha_2|_{2,-1,\Omega} + \left(\int_0^t \left\|\frac{\alpha_2}{r}\right\|_{1,\Omega}^2 \mathrm{d}\tau\right)^{1/2} \le A(0,t)[\|h\|_{2,-1,\Omega^t} + \sup_t \|\omega\|_{1,-\mu,\Omega}] + c|F_2|_{2,-\mu,\Omega^t} + |\chi_0|_{2,-1,\Omega}. \tag{4.51}$$

Using (4.51) in (4.50) yields (4.49). This concludes the proof.

From (4.49) in view of (4.43), (4.25) and (4.30) we see that there exists $\sup_t \|\chi(t)\|_{1,1-\mu,\Omega}$. Therefore Y_1^* and Y_2^* exist.

Now we can prove Theorem 1. The proof will be done step by step. By Lemmas 4.1–4.6 we have the existence of numbers X_1^*, X_2^*, Y_1^*, Y_2^*, D^*. Assuming that X_1^*, X_2^* are sufficiently small we have by Theorem 3.3 the existence of solutions to problem (1.1) such that $v \in A_{1-\mu}(\Omega^T)$, $p \in L_2(0,T;W^2_{2,1-\mu}(\Omega))$ and $\|v\|_{A_{1-\mu}(\Omega^T)} \le A$.
Assume that we have proved the existence of solutions in the interval $(0,kT)$, where $k \in \mathbb{N}$ is fixed. Then in view of the decay estimates and Lemmas 4.1–4.6 we can repeat the proof of Theorem 3.3 in the interval $(kT,(k+1)T)$, so $\|v\|_{A_{1-\mu}(\Omega\times(kT;(k+1)T))} \le A$. Continuing we conclude the proof.

5. Uniqueness

To prove uniqueness of solutions to problem (1.1) we assume that we have two solutions of problems (1.4), (1.7), (1.8), (1.9) and (1.10): $h^i, q^i, \omega^i, \alpha^i, v^i, p^i$, $i = 1,2$.
Introducing

$$H = h^1 - h^2,\quad Q = q^1 - q^2,\quad W = \omega^1 - \omega^2,\quad A = \alpha^1 - \alpha^2,\quad V = v^1 - v^2,\quad P = p^1 - p^2, \tag{5.1}$$

we see that they are solutions to the problems (3.38)–(3.42²), where $H_m, V_m, \ldots$, are replaced by $H, V, \ldots$, respectively, $v_m, h_m, v_{m-1}, \alpha_m$ by v^1, h^1, v^2, α^2, respectively.

Proof (of Theorem 2). We shall show uniqueness step by step on sufficiently small subintervals. Instead of (3.67) we obtain

$$|||A'|||_{2,2,1-\mu,\Omega^t} \le \varphi(A)\left[\sup_t \left|\frac{A_2}{r}\right|_{2,\Omega} + \left(\int_0^t \left\|\frac{A_2}{r}\right\|_{1,\Omega}^2 \mathrm{d}t'\right)^{1/2} + \|V\|_{A_{1-\mu}(\Omega^t)}\right] X_7. \tag{5.2}$$

In view of (3.71) we get

$$\left|\frac{A_2}{r}\right|_{2,\Omega} + \left(\int_0^t \left\|\frac{A_2}{r}\right\|_{1,\Omega}^2 \mathrm{d}t'\right)^{1/2} \le \varphi(A)(t^a + X_7)\|V\|_{A_{1-\mu}(\Omega^t)}, \tag{5.3}$$

where $a > 0$.
Next, (3.74) implies

$$\|A_2\|_{2,1-\mu,\Omega^t} \le \varphi(A)(t^a + X_7)\|V\|_{A_{1-\mu}(\Omega^t)}. \tag{5.4}$$

From (5.2)–(5.4) we have

$$|||A|||_{2,2,1-\mu,\Omega^t} \le \varphi(A)(t^a + X_7)\|V\|_{A_{1-\mu}(\Omega^t)} \le \varphi(A)(t^a + X_7)|||A|||_{2,2,1-\mu,\Omega^t}. \tag{5.5}$$

Therefore for sufficiently small t and X_7 we have uniqueness on interval $[0,t]$. Continuing such considerations we can show uniqueness on the interval of the time of local existence $[0,T]$. Next using Theorem 1 we can repeat the considerations on any inteval $[kT,(k+1)T]$, $k \in \mathbb{N}$, $k \ge 0$. This concludes the proof.

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