On Tableau consistency in weak theories

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Abstract. We prove that the Gödel second incompleteness theorem holds for Tableau consistency and weak arithmetic. Exactly, we prove that

$$I\Delta_0 + \Omega_m \nvdash Tabcons(I\Delta_0 + \Omega_m),$$

for $m \geq 1$.

Also, we show that it is consistent with $I\Delta_0 + \Omega_m$ that a Tableau inconsistency proof of $I\Delta_0 + \Omega_m$ can be found in a cut which is "slightly" larger than the cut determined by the $m - 1$th iteration of log (theorem 1.1 and 1.2).

§0 Introduction

In this section we give the basic intuitions.

First let us explain what we mean by weak arithmetic and in particular by the theory $I\Delta_0 + \Omega_m$.

By $I\Delta_0$ we mean arithmetic with the induction scheme restricted to bounded formulas.

By $I\Delta_0 + exp$ we mean $I\Delta_0$ together with the axiom stating that exponentiation is total. Also we will consider the theory $I\Delta_0 + supexp$. The axiom $supexp$ is "$\forall x\, exp^x x$ exists", where $exp^0 x = x$, $exp^{n+1} x = 2^{exp^n x}$.

By E_n we mean here the subclass of the class Δ_0 consisting of formulas of restricted complexity – with n alternating blocks of bounded quantifiers starting with an existential block. In $I\Delta_0$ there is a Π_1 universal formula for E_n. By IE_n we mean arithmetic with the induction scheme restricted to E_n formulas.

The name "weak arithmetic" usually applies to all strict subtheories of $I\Delta_0 + exp$. The main such subtheories are the theories $I\Delta_0 + \Omega_m$ which are defined as follows. The function ω_m is defined inductively as

$$\omega_0(x) = x^2$$
$$\omega_{m+1}(x) = 2^{\omega_m(\log x)}$$

and Ω_m is the axiom stating the totality of the function ω_m.

The main aim of the study of the theories $I\Delta_0 + \Omega_m$ is finding how much power of the theory $I\Delta_0 + exp$ they have.

Now let us explain the notion of Tableau consistency of a theory T. Besides the ordinary Hilbert notion of a proof there are several other notions. The notion of a Tableau proof, introduced by Beth, is very close to Gentzen's cut free proof. If we define the relation $S \vdash \phi$ (the theory S proves the sentence ϕ) in a theory T, then for strong enough T, the Tableau provability of ϕ is equivalent to the Hilbert provability. If T is a fragment of Peano Arithmetic then "strong enough" means $T \supseteq I\Delta_0 + supexp$. This is due to the fact that a Tableau proof is in general super exponentially longer than an ordinary proof. However, if T is weak, Tableau provability and provability are different notions in T.

Of special importance is the question of Tableau provability or non-provability of the contradiction in a theory T, that is the Tableau inconsistency or consistency of T.

Why Tableau proofs are interesting in weak arithmetic?

First, the Tableau consistency of T gives a direct construction of a model of T – a henkinized completion of T. This is used in proofs of theorems on the existence of end extensions, e.g.

If K is a countable model of $I\Delta_0 + exp + B\Sigma_1$ then K has a proper end extension to a model of $I\Delta_0$,

see [WP$_2$], [A$_1$] ($B\Sigma_1$ is the collection scheme for Σ_1 formulas).

This theorem is connected with the following:

$$I\Delta_0 + exp \vdash Tabcons(I\Delta_0), \text{ see [WP}_1\text{]}$$

(although $I\Delta_0 + exp \nvdash Cons(I\Delta_0)$).

There are also negative results. For instance there is a model of $I\Delta_0 + exp$ with no proper end extension to a model of $I\Delta_0 + exp$, see [WP$_2$]. This theorem is connected with the following:

$$I\Delta_0 + exp \nvdash Tabcons(I\Delta_0 + exp).$$

Secondly, sentences of the form $TabconsT$ are candidates for separating the Π_1 consequences of the theories $I\Delta_0+\Omega_{m+1}$ and $I\Delta_0+\Omega_m$ (the Π_1 conservativeness of $I\Delta_0+\Omega_{m+1}$ over $I\Delta_0 + \Omega_m$ is a well known open problem), see e.g. [KT].

So, the main known results are the following:

Positive result:

$$(I) \qquad I\Delta_0 + exp \vdash Tabcons(I\Delta_0)$$

Also we have

$$(I'). \qquad I\Delta_0 + \Omega_2 \vdash Tabcons(IE_n + \Omega_1)$$

Also,

(I”) each of the theories $I\Delta_0 + \Omega_m$ does prove its own Tableau consistency relativised to a cut.

Negative result:

$$(II) \qquad I\Delta_0 + exp \nvdash Tabcons(I\Delta_0 + exp)$$

(I), (I') can be found in [WP_1] or derived from there, (I") can be found in [P], (II) can be derived from [WP_2]. The proof of our main theorem applied to the theory $I\Delta_0 + exp$ gives a direct proof of (II) (although in this case it can be simplified considerably).

The main motivation for the investigation of the problems mentioned above concerning Tableau provability and weak arithmetic is their close connection with the problem of the collapsing of the hierarchy of Δ_0 formulas.

A special case of the provability of the Tableau consistency of one theory in another is the provability of the Tableau consistency of a theory in itself.

For the ordinary consistency, the second Gödel theorem is true for all fragments of arithmetic extending Robinson's Arithmetic, in particular for $I\Delta_0$, $I\Delta_0 + \Omega_1$, $I\Delta_0 + exp$ etc. ([WP_1, [P]). That is, no of the above theories proves its own consistency. For Tableau consistency several difficulties arose. We discuss those difficulties below.

The main paper on Tableau consistency in connection with weak systems of arithmetic is [WP_1]. See also [CD], [P]. For a survey of problems around see [A_3].

Let us remind the notion of Tableau consistency. A strict formalization is given in §1 by the notion of a condition. First consider the following example.

Example

Let T be the theory consisting of the following two axioms:

$$\exists x \forall y (y \not< x \& x \not< y)$$

$$\forall u \exists v (v < u \vee u < v).$$

The following tree is a Tableau proof of a contradiction in T. Its root consists of the axioms, it splits at every node which contains a disjunction, every node which contains a formula beginning with a quantifier is followed by a substitution of a term in place of the quantifier and every conjunction is replaced by the set of its conjuncts.

$$\exists x \forall y (y \not< x \wedge x \not< y),\ \forall u \exists v (v < u \vee u < v)$$

$$|$$

$$\exists x \forall y (y \not< x \wedge x \not< y),\ \forall y (y \not< c_1 \wedge c_1 \not< y),\ \forall u \exists v (v < u \vee u < v)$$

$$|$$

$$\exists x \forall y (y \not< x \wedge x \not< y),\ \forall y (y \not< c_1 \wedge c_1 \not< y),\ \exists v (v < c_1 \vee c_1 < v),\ \forall u \exists v (v < u \vee u < v)$$

$$|$$

$$\exists x \forall y (y \not< x \wedge x \not< y),\ \forall y (y \not< c_1 \wedge c_1 \not< y),\ c_2 < c_1 \vee c_1 < c_2,\ \forall u \exists v (v < u \vee u < v)$$

$$|$$

$$\exists x \forall y (y \not< x \wedge x \not< y), \forall y (y \not< c_1 \wedge c_1 \not< y),\ c_2 \not< c_1 \wedge c_1 \not< c_2,\ c_2 < c_1 \vee c_1 < c_2,$$
$$\forall u \exists v (v < u \vee u < v)$$

$$|$$

$$\exists x \forall y (y \not< x \wedge x \not< y), \forall y (y \not< c_1 \wedge c_1 \not< y),\ c_2 \not< c_1,\ c_1 \not< c_2,\ c_2 < c_1 \vee c_1 < c_2,$$
$$\forall u \exists v (v < u \vee u < v)$$

$\exists x \forall y (y \not< x \wedge x \not< y)$, $\forall y (y \not< c_1 \wedge c_1 \not< y)$, $c_2 \not< c_1$, $c_1 \not< c_2$, $c_2 < c_1$, $\forall u \exists v (v < u \vee u < v)$	$\exists x \forall y (y \not< x \wedge x \not< y)$, $\forall y (y \not< c_1 \wedge c_1 \not< y)$, $c_2 \not< c_1$, $c_1 \not< c_2$, $c_1 < c_2$, $\forall u \exists v (v < u \vee u < v)$

The tree has two leaves. The atomic parts of the leaves are respectively

$$c_2 \not< c_1,\ c_1 \not< c_2,\ c_2 < c_1;$$

and

$$c_2 \not< c_1,\ c_1 \not< c_2,\ c_1 < c_2.$$

Here the atomic parts of all leaves contain an immediate contradiction – the tree is a Tableau proof of a contradiction in T.

Wilkie Paris [WP1], following Beth [B], give the following definition of a Tableau proof of contradiction from the theory T (what we give is a slight modification of their definition):

Def.0.1
A Tableau proof of contradiction from the theory T is a sequence $\Gamma_0, \ldots, \Gamma_s$ of sets of formulas such that

i) $X \in \Gamma_0$ implies that X is included in the set of axioms of T and of logic
ii) for each $X \in \Gamma_r$ for $r < s$ one of the following holds:
α) $X \in \Gamma_{r+1}$

β) $X \cup \{\theta\} \in \Gamma_{r+1}$ for some $\neg\neg\theta \in \Gamma_r$

γ) $X \cup \{\theta_1, \theta_2\} \in \Gamma_{r+1}$ for some $\theta_1 \& \theta_2 \in \Gamma_r$

δ) $X \cup \{\theta_1\}, X \cup \{\theta_2\} \in \Gamma_{r+1}$ for some $\theta_1 \vee \theta_2 \in \Gamma_r$

ϵ) $X \cup \{\theta(t)\} \in \Gamma_{r+1}$ for some $\forall x \theta(x) \in \Gamma_r$ and for some term t freely substitutable in θ

ζ) $X \cup \{\theta(c)\} \in \Gamma_{r+1}$ for some $\exists y \theta(y) \in \Gamma_r$, and c a constant not occurring in any formula in X

ϑ) $X \cup \{\theta(t, c)\} \in \Gamma_{r+1}$ for some $\forall x \exists y \theta(x, y) \in \Gamma_r$ or $\forall x \exists y \theta(x, y)$ an axiom, and for some term t freely substitutable in θ and c a constant not occurring in any formula in X nor in t.

iii) For each $Y \in \Gamma_{r+1}$ there is an $X \in \Gamma_r$ such that Y is obtained by one of the rules listed in (ii).

iv) Γ_s contains an atomic formula and its negation

In the rules $\zeta), \vartheta)$ we may treat the constant c as a Henkin constant and the assignment $t \longrightarrow c$ as a Skolem function for the formula $\forall x \exists y \theta(x, y)$.

In our example the Γ's can be taken as the levels of the tree, i.e. the sets

$$\Gamma_0 = \{\{\exists x \forall y (y \not< x \& x \not< y),\ \forall u \exists v (v < u \vee u < v)\}\}$$

(rule ζ:)

$$\Gamma_1 = \{\{\exists x \forall y (y \not< x \& x \not< y),\ \forall y (y \not< c_1 \& c_1 \not< y),\ \forall u \exists v (v < u \vee u < v)\}\}$$

(rule ϵ for $t = c_1$:)

$$\Gamma_2 = \{\{\exists x \forall y (y \not< x \& x \not< y),\ \forall y (y \not< c_1 \& c_1 \not< y),\ \exists v (v < c_1 \vee c_1 < v),\\ \forall u \exists v (v < u \vee u < v)\}\}$$

(rule ζ:)

$$\Gamma_3 = \{\{\exists x \forall y (y \not< x \& x \not< y),\ \forall y (y \not< c_1 \& c_1 \not< y),\ c_2 < c_1 \vee c_1 < c_2,\\ \forall u \exists v (v < u \vee u < v)\}\}$$

(rule ϵ for $t = c_2$:)

$$\Gamma_4 = \{\{\exists x \forall y (y \not< x \& x \not< y),\ \forall y (y \not< c_1 \& c_1 \not< y),\ c_2 \not< c_1 \& c_1 \not< c_2,\ c_2 < c_1 \vee c_1 < c_2,\\ \forall u \exists v (v < u \vee u < v)\}\}$$

(rule γ:)

$$\Gamma_5 = \{\{\exists x \forall y (y \not< x \& x \not< y),\ \forall y (y \not< c_1 \& c_1 \not< y),\ c_2 \not< c_1,\ c_1 \not< c_2,\ c_2 < c_1 \vee c_1 < c_2,\\ \forall u \exists v (v < u \vee u < v)\}\}$$

(rule δ:)

$$\Gamma_6 = \{\{\exists x \forall y(y \not< x \& x \not< y),\ \forall y(y \not< c_1 \& c_1 \not< y),\ c_2 \not< c_1,\ c_1 \not< c_2,\ c_2 < c_1,\ \forall u \exists v(v < u \vee u < v)\},$$
$$\{\exists x \forall y(y \not< x \& x \not< y),\ \forall y(y \not< c_1 \& c_1 \not< y),\ c_2 \not< c_1,\ c_1 \not< c_2,\ c_2 < c_1,\ \forall u \exists v(v < u \vee u < v)\}\}$$

Pudlak [P] following Kleene [K] considers the notion of a Herbrand variant of a formula ϕ:

$$(*) \qquad \exists x_1 \forall x_2 \dots \exists x_{2n-1} \forall x_{2n} \theta(x_1, \dots, x_{2n}),$$

where θ is open.

Namely it is a disjunction of sentences of the form

$$\theta(t_1, f_2(t_1), \dots, t_{2n-1}, f_{2n}(t_1, t_3, \dots, t_{2n-1})),$$

where t_{2i+1} is a term built of function symbols of the language and of some new function symbols $f_2, \dots, f_{2n}$.

Def.0.2

We say that ϕ is Herbrand provable if some of its Herbrand variants is provable in propositional calculus.

Consider our example and let ϕ be the disjunction of the negations of our axioms which can be presented in the form:

$$\forall x \exists u \forall v \exists y \big((y < x \vee x < y) \vee (v \not< u \& u \not< v)\big)$$

(we could add a redundant existential quantifier in the front to have the above formula in the form $(*)$). Then the formula

$$\bigvee_{t_u, t_y} \big(f_x < t_y(f_v(t_u(f_x))) \vee t_y(f_v(t_u(f_x))) < f_x\big)$$
$$\vee \big(f_v(t_u(f_x)) \not< t_u(f_x) \& t_u(f_x) \not< f_v(t_u(f_x))\big)$$

is a Herbrand variant of ϕ, where f_x, f_v are new function symbols (f_x is a 0-argument function symbol, i.e. a constant) and t_u, t_y run over terms. This is a propositional tautology – taking t_u, t_y the identity terms, we obtain an evident propositional tautology

$$\big(f_x < f_v(f_x) \vee f_v(f_x) < f_x\big) \vee \big(f_v(f_x) \not< f_x \& f_x \not< f_v(f_x)\big).$$

Here, f_x corresponds to our previous constant c_1 and f_v to the Skolem function $c_1 \longrightarrow c_2$ for the axiom $\forall u \exists v(v < u \vee u < v)$.

Thus we see that a Tableau proof of a contradiction from the axioms of T corresponds to a Herbrand proof of the disjunction of the negations of the axioms of T.

Our main theorem is the following:

Theorem 0.1
$I\Delta_0 + \Omega_1 \nvdash Tabcons(I\Delta_0 + \Omega_1)$.

First let us outline the proof of this theorem.

Consider the following presentation of the proof of the second Gödel incompleteness theorem.

Let τ be the diagonal sentence equivalent (in $I\Delta_0$) to $Cons(\mathrm{PA} + \neg\tau)$. Then τ says: *it is consistent with PA that I am false.* Thus τ is equivalent in PA to the Gödel sentence saying *I am unprovable.* Hence τ is true. Thus PA$+\neg\tau$ is consistent. Now work in PA$+\neg\tau$. We infer

$$\text{“}\neg\tau \text{ is provable in PA”}$$

since $\neg\tau$ is Σ_1 and

$$(*) \qquad \mathrm{PA} \vdash (\varphi \Rightarrow \text{“}\varphi \text{ is provable in PA”}),$$

for $\varphi \in \Sigma_1$ (Σ_1-completeness).

Suppose that PA $\vdash Cons(\mathrm{PA})$. Then we have $Cons(\mathrm{PA})$, and so, $Cons(\mathrm{PA} + \neg\tau)$, since $\neg\tau$ is provable in PA. Thus τ. Contradiction.

How could we reformulate this proof for the case of Tableau consistency and $I\Delta_0+\Omega_1$?

The first attempt can be the following: replace in the above reasoning PA by $I\Delta_0+\Omega_1$ and $Cons$ by $Tabcons$.

This is not correct. First, the validity of Σ_1 completeness in $I\Delta_0 + \Omega_1$ is an open problem, secondly, the inference

(**) *If* $Tabcons(I\Delta_0 + \Omega_1) + \neg\tau$*, then* $Tabcons(I\Delta_0 + \Omega_1 + \neg\tau)$

requires cut elimination, which is not in general available in $I\Delta_0 + \Omega_1$.

However, if the witness for a Σ_1 true sentence is small enough the above inference is available in $I\Delta_0 + \Omega_1$, see remarks following lemma 2.6 and corollary 2.1.

Pudlak in [P], and Hajek, Pudlak in [HP], worked with the usual second Gödel incompleteness theorem (for Hilbert type provability) in weak theories. They faced some of the above problems. To overcome the difficulty $(*)$ – the Σ_1 completeness – Hajek and Pudlak [HP] introduced two provability predicates, π_1 and π_2, which fulfilled the following derivability conditions:

$$i) \qquad T \vdash \varphi \text{ implies } T \vdash \pi_1(\varphi)$$

$$ii) \qquad T \vdash \pi_1(\bar{\varphi}) \Rightarrow \pi_2(\bar{\pi}_1(\bar{\varphi}))$$

iii) $$T \vdash \pi_1(\bar{\varphi})\&\pi_1(\overline{\varphi \Rightarrow \psi}) \Rightarrow \pi_2(\bar{\psi})$$

($\bar{x}$ is the numeral for x).

The condition (*ii*) is a version of Σ_1 completeness $(*)$. Note that the condition (*iii*), which is "modus ponens", yields, together with (*ii*), an analogue of $(**)$.

Our approach is a bit similar. We shall also consider two Tableau provabilty (for $T = I\Delta_0 + \Omega_1$) predicates π_1, π_2. They induce consistency predicates σ_1, σ_2, defined as

$$\sigma_j(\varphi) \leftrightarrow \neg\pi_j(\neg\varphi).$$

Then a version of $(**)$ takes the form

$$\sigma_2(0=0)\&\pi_1(\varphi) \Rightarrow \sigma_1(\pi_1(\varphi)).$$

But, in our case, we cannot derive it from (*ii*) with the help of (*iii*) since there is no chance for any (*iii*)-type condition to hold. The Hajek Pudlak reasoning does not apply here. Note that (*iii*), being "modus ponens", is a special instance of the "cut" rule, so it is not true for any Tableau provability predicate. So, we have to postulate $(**)$ directly. It follows from the new derivability conditions, below.

The predicates π_j can be presented in the form

$$\exists y \in J_j \pi_j^y(\varphi),$$

(as in [P], [HP]), where J_j is a definable cut and π_j^y is a bounded formula. Consequently, π_j^y induces σ_j^y.

We consider the following conditions:

i) "$\sigma_1(\varphi)$ is true" implies "φ is consistent with T";

the next are theorems of T:

ii) $$\sigma_2(0=0)\&\theta(b) \Rightarrow \sigma_2(\theta(\bar{b})),$$

provided $b \in J_1$ and θ is bounded (not necessarily explicitely), and in particular

$$\sigma_2(0=0)\&\pi_1^b(\varphi) \Rightarrow \sigma_2(\pi_1^{\bar{b}}(\varphi))$$

iii) $$\sigma_2(\theta(\bar{b})) \Rightarrow \sigma_1\big(\exists y \in J_1 \theta(y)\big),$$

and in particular

$$\sigma_2(\pi_1^{\bar{b}}(\varphi)) \Rightarrow \sigma_1(\pi_1(\varphi)).$$

The predicates π_j, σ_j are defined after lemma 2.3. Condition (i) is proved after corollary 2.4, condition (ii) is a consequence of lemma 2.6 (on absoluteness of bounded formulas with small witnesses), and (iii) is the corollary 2.4.

Now let us show how the above version of $(**)$ follows from the conditions (although we shall be using the conditions directly, we do not need $(**)$ any more). From $\sigma_2(0=0)\&\pi_1(\varphi)$ we derive $\sigma_2(0=0)\&\pi_1^b(\varphi)$, whence by (ii) and (iii) we derive $\sigma_1^b(\pi_1(\varphi))$, which, in view of an arbitrary choice of b, gives $\sigma_1(\pi_1(\varphi))$.

Now our proof of the theorem goes as follows:

We take the same diagonal sentence as Hajek Pudlak do (however induced by our provability predicate), namely a sentence τ equivalent to $\neg\pi_1(\tau)$ (i.e. to $\sigma_1(\neg\tau)$).

To be more precise here we have to go into the construction of a diagonal sentence. We shall define τ as a special diagonal sentence for the formula $\neg\pi_1(u)$.

Let $\psi(w,x)$ be a bounded formula expressing the meaning "w is the formula $x(\bar{x})$". – under a suitable coding of formulas ψ can be taken bounded.

Let $H(x)$ be the formula

$$\neg\exists y\in J_1\exists w\big\{\psi(w,x)\&\pi_1^y(w)\big\}.$$

How to understand the formula $y\in J_1$ is described in corollary 2.2.

Note (taking into account the form of π_1), that this formula is logically equivalent to

$$\forall w\big\{\psi(w,x)\Rightarrow\neg\pi_1(w)\big\}.$$

Let n_0 be (the Gödel number of) the formula $H(x)$.

Let τ be the sentence

$$\neg\exists y\in J_1\exists w\big\{\psi(w,\bar{n}_0)\&\pi_1^y(w)\big\}.$$

We easily show that τ is diagonal for the formula $\neg\pi_1$.

Let the formula

$$\exists w\big\{\psi(w,\bar{n}_0)\&\pi_1^y(w)\big\}.$$

be denoted by $\neg\tau'(y)$. Thus $\neg\tau$ is

$$\neg\neg\exists y\in J_1\neg\tau'(y)$$

and it has the same normal form as

$$\exists y\in J_1\neg\tau'(y).$$

As we shall see, the consistency predicate σ_1 refers to a normal form of a sentence. What normal form of the above sentence should be taken is described in corollary 2.2. We can assume that this description is included in the formula π_1 and consequently in σ_1. Thus $\sigma_1(\neg\tau)$ is equivalent to

$$\sigma_1(\exists y\in J_1\neg\tau'(y)).$$

Since τ is true in $\mathbb{N}$ and, by (i), our consistency predicate σ_1 implies true consistency, we infer that the theory

$$I\Delta_0 + \Omega_1 + \neg\tau$$

is consistent. We work in this theory. We have (the enumeration 1)-5) corresponds to the enumeration 1')-5') in the actual proof of the theorem in §2):

$$\neg\sigma_1(\neg\tau)$$

whence

1) $$\neg\sigma_1^b(\neg\tau),$$

for a b in J_1.

On the other hand, directly from $\neg\tau$ we infer $\neg\tau'(b')$ for a b' in J_1. Note that from the form of $\neg\tau'$ it follows that $\neg\tau'(x)$ implies $\neg\tau'(y)$ whenever $x \leq y$. Thus we may assume that $b' = b$: if $b' < b$ then take b instead of b' and if $b < b'$ then take b' instead of b.

Supposing "ad absurdum" $I\Delta_0 + \Omega_1 \Vdash Tabcons(I\Delta_0 + \Omega_1)$, we infer

2) $$\sigma_2(0 = 0)$$

whence by (ii)

3) $$\sigma_2(\neg\tau'(\bar{b}))$$

whence

4) $$\sigma_2^{b'}(\neg\tau'(\bar{b}))$$

for a certain, specially chosen b' whence by (iii)

5) $$\sigma_1^b(\neg\tau)$$

which contradicts 1).

§1 Basic notions

Here we shall define in $I\Delta_0 + \Omega_1$ the notion of a Tableau proof of the contradiction or Herbrand proof of the contradiction.

Assume that T is a theory in the language of arithmetic whose axioms are of the form

$$\forall x_1 \exists x_2 \ldots \forall x_n \exists x_{2n} \theta(x_1, \ldots, x_{2n}),$$

where θ is open, is in a disjunctive conjunctive form and all atomic formulas in it are of the form $x_k + x_l = x_m$, $x_k \cdot x_l = x_m$ or $x_k = x_l$. When dealing with an open formula later we shall mean an open formula of the above form. Plus and times may be treated as functions or as relations. The case where $+$ and $\cdot$ are functions differs from the case where they are relations by the fact that the axioms concerning the totality of $+$ and $\cdot$ are not included. In the axioms, however, no non-simple terms are allowed.

Let us define the numbers $L^n(i)$. The numbers $< L^n(i)$ are meant to be identified with (to enumerate) Henkin constants introduced by the rule ϑ into a canonical Tableau proof of the contradiction from T. Such a Tableau proof is divided into stages numbered by i where the ith stage corresponds to i nested iterations of the application of the rule ϑ. Then, up to the ith stage we need $L^n(i)$ constants. This will be clearly seen from the discussion below.

Def.1.1
Let $L^n(0) = 2$. Assume that 0 represents the constant Λ_0 for 0.
For $i \geq 2$ let

$$L^n(i+1) = L^n(i) + iL^n(i) + (i-1)L^n(i)L^n(i-1) + (i-2)L^n(i)L^n(i-1)L^n(i-2)+ \\ \ldots + (i-n+1)L^n(i)L^n(i-1) \cdot \ldots \cdot L^n(i-n+1).$$

We include in the above sum only those terms which are nonnegative.

The first term here $L^n(i)$ reflects the fact that all constants of the i-th stage are included into the $i+1$-th.

The second term $iL^n(i)$ reflects the fact that to every axiom of the i first axioms of T (if the number of axioms x is less than i then we let the $x+j$th axiom be the jth) and to its first existential quantifier possibly preceded by a universal quantifier and to every constant of the i-th stage substituted for the variable in the universal quantifier there is a Skolem constant corresponding to the existential quantifier. Some Henkin constants are repeated here.

The third term $(i-1)L^n(i)L^n(i-1)$ corresponds to the fact that to every axiom of the $i-1$ first axioms of T and to its second existential quantifier possibly preceded by a prefix $\forall\exists\forall$ and to every constant of the $i-1$-th stage substituted for the variable in the first universal quantifier and to every constant of the i-th stage substituted for the variable in the second universal quantifier there is a Skolem constant corresponding to the second existential quantifier while the suitable constant from the i-th stage is substituted for the variable in the first universal quantifier. If $+$ and $\cdot$ are functions then we take $i-3$ instead of $i-1$ first axioms of T and the remaining two products $L^n(i)L^n(i-1)$ reflect the fact that to every constant of the $i-1$-th stage and to every constant of the i-th stage we include a constant for their sum and for their product.

And so on.

The graph of the function L^n is definable by an arithmetical formula and in $I\Delta_0+\Omega_1$ this definition defines a function which is total on a cut. To formulate this definition one may use coding described in [WP$_1$] or in [HP].

Def.1.2

Now we define the functions $g_s(i,j,h_1,\ldots,h_s)$.

The function $g_s(i,j,h_1,\ldots,h_s)$ for $h_1 < L^n(i+1-s), h_2 < L^n(i+1-(s-1)),\ldots,h_s < L^n(i), 1 \le s \le n, j \le n$ is defined as follows:

$$\begin{aligned}
&g_s(i,j,h_1,\ldots,h_s) = L^n(i)+\\
&+\sum_{1\le t\le s-1}(i-t)\cdot L^n(i-t)\cdot\ldots\cdot L^n(i-1)\cdot L^n(i)+\\
&+(j-1)L^n(i+1-s)\cdot L^n(i+1-(s-1))\cdot\ldots\cdot L^n(i-1)\cdot L^n(i)+\\
&+<h_1,\ldots,h_s>_i
\end{aligned}$$

where $<h_1,\ldots,h_s>_i$ is the number of the sequence $<h_1,\ldots,h_s>$ in the lexicographical ordering of

$$L^n(i+1-s)\cdot L^n(i+1-(s-1))\cdot\ldots\cdot L^n(i-1)\cdot L^n(i).$$

If $h_1,\ldots,h_s,i,j,s$ satisfy the above conditions then

$$g_s(i,j,h_1,\ldots,h_s) < L^n(i+1)$$

and conversely, if $x < L^n(i+1)$ then either $x < L^n(i)$ or there are $h_1,\ldots,h_s,j,s$ satisfying the above conditions such that $x = g_s(i,j,h_1,\ldots,h_s)$.

The family of the graphs of the functions $g_s(i,j,h_1,\ldots,h_s)$ is uniformly definable by an arithmetical formula of $s,i,j,\langle h_1,\ldots,h_s\rangle$ which defines a family of partial functions in $I\Delta_0+\Omega_1$. To formulate this definition one may use coding described in [WP$_1$] or in [HP].

Now we shall give a precise definition in $I\Delta_0+\Omega_1$ of Tableau consistency and inconsistency of a theory – def.1.4. Later we shall discuss its relationship to def. 0.1 and 0.2 and we shall justify the choice of our definition.

The next notion is the notion of a *condition* widely exploited in [A$_1$], [A$_2$] (called "precondition" there).

Def.1.3

An i-condition is a function p whose domain consists of all atomic and negated atomic sentences φ, without non-simple terms, with numbers $< L^n(i)$ treated as constants, p : dom $p \longrightarrow 2$ and the following holds:

1) If we are given a sequence $i_1,\ldots,i_s,h_1,\ldots,h_s$ such that $i_1 < i_2 < \ldots < i_s \le i$, $h_1 < L^n(i_1),\ldots,h_s < L^n(i_s)$, the jth axiom of T is of the form

$$\forall x_1\exists x_2\ldots\forall x_s\exists x_{2s}\theta(x_1,\ldots,x_{2s}),$$

where θ is open, then p decides positively

$$\theta(h_1, g_1(i_1, j, h_1), \ldots, h_s, g_s(i_s, j, h_1, \ldots, h_s)).$$

Here, by deciding positively an atomic or negated atomic sentence we mean taking the value 0 at it, deciding positively a conjunction as deciding positively both and a disjunction as deciding positively at least one.

2) $p(\varphi) = 0$ iff $p(\neg\varphi) = 1$;

3) If $p(g_1 = g_2) = 0$ then p takes the same value on sentences in which g_1 is replaced by g_2;

4) $p(g = g) = 0$.

The notion "p is an i condition" is arithmetically definable. To formulate this definition one may use coding described in [WP$_1$] or in [HP].

Def.1.4

Assume that T is either finite or is arithmetically definable. Let $T \restriction n$ denote the set of the axioms of T with at most n existential quantifiers. By $TabconsT$ we mean the sentence

$$\forall i \forall n \big(2^{L^n(i)} \text{ exists} \Rightarrow \exists p(p \text{ is an } i\text{-condition w.r.t. } T \restriction n)\big),$$

i.e.

$$\forall y \forall i \forall n \big(y = 2^{L^n(i)} \Rightarrow \exists p(p \text{ is an } i\text{-condition w.r.t. } T \restriction n)\big).$$

Of course, here the function $L^n(i)$ and the notion of an i-condition have to be replaced by their arithmetical definitions.

Consequently, by $\neg TabconsT$ we mean

$$\exists i \exists n \big(2^{L^n(i)} \text{ exists} \& \neg \exists p(p \text{ is an } i\text{-condition w.r.t. } T \restriction n)\big).$$

Our main theorem is now formulated as:

Theorem 1.1

It is consistent with $I\Delta_0 + \Omega_1$ that

$$\exists i \exists n \big(2^{L^n(i)} \text{ exists} \& \neg \exists p(p \text{ is an } i\text{-condition w.r.t. } IE_n + \Omega_1)\big).$$

The same proof will also yield

Theorem 1.2

It is consistent with $I\Delta_0 + \Omega_m$ that

$$\exists i \exists n \big(\exp^m L^n(i) \text{ exists} \& \neg \exists p(p \text{ is an } i\text{-condition w.r.t. } IE_n + \Omega_m)\big).$$

Discussion

Here we give a somewhat informal discussion of the relationship of our definition 1.4 of Tableau consitency to classical definitions def.0.1 and def.0.2. Later on we shall rely on definition 1.4, so we do not have to formalize the other approaches, however it is not difficult to make them formal in $I\Delta_0 + \Omega_1$.

The numbers $< L^n(i)$ can have double role – on one hand they can be treated as constants, on another hand they can be treated as numbers enumerating sentences and terms in the following way: if $h_1, \dots, h_s$ satisfy $h_1 < L^n(i+1-s), h_2 < L^n(i+1-(s-1)), \dots, h_s < L^n(i)$ then $g_s(i, j, h_1, \dots, h_s)$ (for $s \neq 2$ and $j \neq i-1, i-2$) can be treated as the sentence number

$$\varphi_j^{(s)}(h_1, g_1(i, j, h_1) \dots, h_s, g_s(i, j, h_1, \dots, h_s)),$$

where φ_j is the jth axiom of T, $\varphi_j^{(s)}$ is the subformula of φ_j consisting of dropping the first s pairs of quantifiers $\forall\exists$, and $g_2(i, i-2, h_1, h_2)$ can be treated as the term number $h_1 + h_2$ and $g_2(i, i-1, h_1, h_2)$ as the term number $h_1 \cdot h_2$ (in this way the number of a term of the above form is not determined uniquely).

Assume for a moment that $+$ and $\cdot$ are relations and assume that $T = \{\varphi_1, \dots, \varphi_m\}$ and the axioms φ_1, φ_2 are numbered by $0, 1$ respectively. Under this enumeration of sentences, consider the sequence of sets of sentences defined as follows:

$$\begin{aligned} \Gamma_0 = \Gamma_1 = \Gamma_{L^n(0)} = \Gamma_{L^n(1)} &= \{< L^n(1)\} \\ &= \{\{0, 1\}\} \\ &= \{\{\varphi_1, \varphi_2\}\}; \end{aligned}$$

we apply rule ϑ) to the sentence φ_1 and the term 0 (the new constant is the number 2):

$$\Gamma_{L^n(1)+1} = \{\{0, 1, 2\}\} = \{\{\varphi_1, \varphi_2, \varphi_1^{(1)}(0, 2)\}\};$$

we apply rule ϑ) to the sentence φ_1 and the term 1 (the new constant is the number 3):

$$\Gamma_{L^n(2)} = \{< L^n(2)\} = \{\{0, 1, 2, 3\}\} = \{\{\varphi_1, \varphi_2, \varphi_1^{(1)}(0, 2), \varphi_1^{(1)}(1, 3)\}\};$$

we apply rule ϑ) to the sentence φ_1 and again to the term 0 (the new constant is the number 4):

$$\Gamma_{L^n(2)+1} = \{\{0, 1, 2, 3, 4\}\} = \{\{\varphi_1, \varphi_2, \varphi_1^{(1)}(0, 2), \varphi_1^{(1)}(1, 3), \varphi_1^{(1)}(0, 4)\}\};$$

we apply consecutively rule ϑ) to the sentence φ_1 and the terms 1,2,3, to the sentence φ_2 and the terms 0,1,2,3, to the sentences $\varphi_1^{(1)}(c, d)$ and the pairs $\langle c, d\rangle$ where $c \in \{0, 1\}$ ($c < L^n(1)$) and $d \in \{0, 1, 2, 3\}$, $d = g_1(1, 1, c)$ ($d < L^n(2)$, the new constants are consecutively the numbers $5, \dots, 19$). We get consecutively the set $\Gamma_{L^n(2)+2}, \dots, \Gamma_{L^n(3)}$, where $\Gamma_{L^n(3)}$ is:

$$\Gamma_{L^n(3)} = \{< L^n(3)\} = \{\{0,\ldots,19\}\} =$$
$$= \{\{\varphi_1, \varphi_2, \varphi_1^{(1)}(0,2), \varphi_1^{(1)}(1,3),$$
$$\varphi_1^{(1)}(0,4), \varphi_1^{(1)}(1,5), \varphi_1^{(1)}(2,6), \varphi_1^{(1)}(3,7), \varphi_2^{(1)}(0,8), \varphi_2^{(1)}(1,9), \varphi_2^{(1)}(2,10), \varphi_2^{(1)}(3,11),$$
$$\varphi_1^{(2)}(0,2,0,12), \varphi_1^{(2)}(0,2,1,13), \varphi_1^{(2)}(0,2,2,14), \varphi_1^{(2)}(0,2,3,15), \varphi_1^{(2)}(1,3,0,16),$$
$$\varphi_1^{(2)}(1,3,1,17), \varphi_1^{(2)}(1,3,2,18), \varphi_1^{(2)}(1,3,3,19)\}\}$$

$$\Gamma_r = \{\text{the set of numbers} < r\}$$

Then the sequence Γ_r satisfies the requirements of def.0.1.

If there are terms $+$, $\cdot$ in the language then some numbers $< L^n(i)$ are not sentence-numbers, so we drop them in the sets X in the Γs, however we do substitute them for universal quantifiers in the φ_js. So, in this case $\Gamma_r = \{X_r\}$, where X_r is a certain subset of $< r$.

In def.0.1 every set X in a Γ_r is a part of a Hintikka set – it would be a Hintikka set if the tree was infinite (see Smullyan [S]). The sets Γ_r here have a special property with respect to the general definition 0.1 – they are obtained in a canonical way. Such a canonical procedure is also described in Smullyan [S] for the sake of the proof of the completeness theorem.

Assume that $\neg TabconsT$ holds in a model M of $I\Delta_0 + \Omega_1$, where $T \in M$. As we have remarked above the singletons of the sets " $< r$" for $r < L^n(i)$ can be treated as consecutive families Γ_r in a Tableau proof of contradiction in the sense of def.0.1. Also, there is no problem with coding those sets by numbers $\leq 2^{L^n(i)}$, so by elements of M. What the sequence misses to be a proof of contradiction from T in the sense of def.0.1, are the leaves, each of which should contain an immediate contradiction. To get the leaves we should now define further sets Γ_r by splitting open sentences in the single set X in $\Gamma_{L^n(i)}$, which are in the disjunctive conjunctive form, into their disjuncts, one by one, and finally to replace the conjunctions being result of the splittings by sets of their atomic and negated atomic conjuncts. Then finally we would access leaves. So, the atomic and negated atomic part of every leaf would be determined by a function f from the set of open sentences in the single element X of $\Gamma_{L^n(i)}$ assigning to every such open sentence one of its disjuncts. Since every such function can be coded roughly as a number $\leq 2^{L^n(i)}$, their set may be difficult to be coded by a single element of M.

Note that this is only the matter of size. We can express in M the meaning that Γ_0,...,$\Gamma_{L^n(i)}$ generates a proof of contradiction from T – namely we can say that for every function f from the set of open sentences in the single element X in $\Gamma_{L^n(i)}$ assigning to every such open sentence one of its disjuncts, the leaf determined by f contains an immediate contradiction.

So, we may treat the sequence Γ_0,...,$\Gamma_{L^n(i)}$ as a proof of a contradiction from T.

Now let us consider a definition which yields an alternative approach to the one given by def. 1.4:

Def.1.5

By a proof of a contradiction from T we mean a sequence $\Gamma_0,\ldots,\Gamma_{L^n(i)}$, built in the canonical way given by the application of the function L^n and the family of functions $g_s(i,j,h_1,\ldots,h_s)$, such that for every function f from the set of open sentences in the single element X of $\Gamma_{L^n(i)}$ assigning to every such open sentence one of its disjuncts, the leaf determined by f contains an immediate contradiction. To avoid triviality, let us put an additional requirement on $\Gamma_0,\ldots,\Gamma_{L^n(i)}$, namely that $L^n(i)$ is of logarithmic size (otherwise there may be no or too few functions f with the required property and a sequence could be trivially a proof).

We leave to the reader the proof that the existence of a Tableau proof of a contradiction according to this definition is equivalent in $I\Delta_0+\Omega_1$ to definition 1.4 (every f as above yields a condition and conversely).

Note also, that already in the case $m=2$, theorem 1.2 gives a proof $\Gamma_0,\ldots,\Gamma_{L^n(i)}$ of a contradiction from $I\Delta_0+\Omega_2$ in a model of $I\Delta_0+\Omega_2$ with the property that $2^{2^{L^n(i)}}$ exists in the model. So, in that model there is also a code for a full proof $\Gamma_0,\ldots,\Gamma_s$ in the sense of def.0.1. Also, in $I\Delta_0+\Omega_2$ we can be less strict with enumerating the sentences

$$\varphi_j^{(s)}(h_1,g_1(i,j,h_1),\ldots,h_s,g_s(i,j,h_1,\ldots,h_s))$$

and we do not have to identify them with the numbers $g_s(i,j,h_1,\ldots,h_s)$ – the sentences

$$\varphi_j^{(s)}(h_1,g_1(i,j,h_1),\ldots,h_s,g_s(i,j,h_1,\ldots,h_s))$$

can be enumerated conventionally as sequences

$$\langle\varphi_j^{(s)},h_1,g_1(i,j,h_1),\ldots,h_s,g_s(i,j,h_1,\ldots,h_s)\rangle,$$

where $\varphi_j^{(s)}$ is a conventional formula-number.

A condition can be treated as a truth assignment on the set X in $\Gamma_{L^n(i)}$ (a part of a Hintikka set). What is needed is a truth assignment on atomic and negated atomic sentences in X, i.e. just a condition – the rest is determined then. Note that a Hintikka set determines a Henkin model of T, and so a condition determines a fragment of a model of T. The function $g_s(i,j,h_1,\ldots,h_s)$ determines in this model a Skolem function for the sentence $\exists y\varphi_j^{(s)}(h_1,g_1(i,j,h_1),\ldots,h_s,y)$.

Let us refer now to Pudlak's approach. Still on the other hand we may treat the numbers $g_s(i,j,h_1,\ldots,h_s)$, for $s\neq 2$ and $j\neq i-1,i-2$, as terms obtained by applying an s-argument function symbol $g_s(i,j,\cdot,\ldots,\cdot)$ to $h_1,\ldots,h_s$ and treat $g_2(i,i-2,h_1,h_2)$ as the term number h_1+h_2 and $g_2(i,i-1,h_1,h_2)$ as the term number $h_1\cdot h_2$. Under this interpretation, the negation of the disjunction of the negations of those sentences

$$\varphi_j^{(s)}(h_1,g_1(i,j,h_1)\ldots,h_s,g_s(i,j,h_1,\ldots,h_s)),$$

for $h_1, \ldots, h_s$ satisfying $h_1 < L^n(i+1-s), h_2 < L^n(i+1-(s-1)), \ldots, h_s < L^n(i)$, which are open, is a Herbrand variant of the disjunction of the negations of axioms of T.

Consider Pudlak's definition 0.2 again. He defines a Herbrand variant H of a formula ϕ to be a propositional tautology, if there is no truth assignment to the atomic and negated atomic formulas occurring in H falsifying H. However again, this definition, when considered in $I\Delta_0 + \Omega_1$, has a weakness – there may trivially be no such truth assignment since the right truth assignments would have be too large. So, for the sake of $I\Delta_0 + \Omega_1$, Pudlak's definition could be refined as follows

Def.1.6

We say that T is Herbrand inconsistent if there is an i such that the negation of the disjunction of the negations of those sentences

$$\varphi_j^{(s)}(h_1, g_1(i,j,h_1) \ldots, h_s, g_s(i,j,h_1,\ldots,h_s)),$$

for $h_1, \ldots, h_s$ satisfying $h_1 < L^n(i+1-s), h_2 < L^n(i+1-(s-1)), \ldots, h_s < L^n(i)$, which are open, is a propositional tautology and if (additionally) $L^n(i)$ is of logarithmic size.

We leave to the reader the proof that this definition is equivalent in $I\Delta_0 + \Omega_1$ to definition 1.4 (conditions are truth assignments). So, T is Tableau consistent iff it is Herbrand consistent.

Note also, that our definitions 1.4, 1.5, 1.6 differ from the general ones 0.1, 0.2 by the restriction to canonical Tableau proofs or Herbrand variants. However this strengthens our main result, for, if the existence of a canonical contradiction proof is consistent then so is the existence of a proof. The only restriction is the assumption that the axioms of T are of a certain special form.

Finally let us note that theorems I, I', I", II of the introduction hold according to definition 1.4. This can also be considered as an argument that this definition is an adequate one.

For a brief illustration of the known methods let us outline the proof of (I).

Remark 1.1

So, let M be a model of $I\Delta_0 + exp$. Let $i \in M$. We shall construct an i-condition w.r.t $I\Delta_0$ in M – a part of a Henkin model of $I\Delta_0$ in M. We construct this Henkin model as a submodel of M. Let a be any element of M. We make within M an i-th stage the Skolem hull of a under the Skolem functions for axioms of $I\Delta_0$ (the stage consisting of i steps of the closing procedure). We do it in the sense of M, thus, when dealing with the axioms φ of $I\Delta_0$ different than the axioms $\forall x \exists y\ y = x + \Lambda_1$, $\forall x \exists y\ y = x + x$, $\forall x \exists y\ y = x^2$ we replace them by $Sat(\varphi)$, where Sat is a Π_1 satisfaction formula for Π_1 formulas (note that all the axioms except of the axioms $\forall x \exists y\ y = x + \Lambda_1$, $\forall x \exists y\ y = x + x$, $\forall x \exists y\ y = x^2$ are Π_1). We are able to make this i-th stage since the i-th iteration of the square function at a i.e. a^{2^i} exists in M – since M satisfies exp. (Note that the i-th iteration of the exponential function is not necessarily available in M. Thus, if we want to run the argument for $I\Delta_0 + exp$ instead of $I\Delta_0$ here, we may fail. That is, we would fail in proving $I\Delta_0 + exp \vdash Tabcons(I\Delta_0 + exp)$ in this way.) Using exp we can bound the i-th stage of the hull. Using Δ_0 induction with this bound as a parameter we can proof the existence of the i-th stage of the hull – we prove the existence of the j-th stage of the hull for $j \leq i$ by induction on j. A similar proof works for (I') and (I").

The theorems (I), (I'), (I") are of the form

$$S \vdash TabconsT,$$

where S and T are certain theories. Two conditions on S and T are essential for the above argument – the existence of a formula Sat for the axioms of T in S and the "room" in any model M of S for the i-th iteration at a of the provably total functions of T. However, the proof is somewhat round way. We get the required Henkin model of a certain special form – as a submodel of M. The construction goes via a Skolem hull. The following problem concerns the question weather it has to be like that.

PROBLEM

Assume that T is "$I\Delta_0 + f$ is total", where f is a Δ_0 definable function. Assume $S \supseteq T$ and $S \vdash TabconsT$. Does it follow that there exists a satisfaction formula for the axioms of T in S? Is it true that

$$S \vdash \forall i (2^{L^n(i)} \text{ exists} \Rightarrow f^i \text{ exists})?$$

§2 The proof

We may define the relation of forcing a sentence of our Henkin language by a condition.

DEF.2.1

Assume $i_1 < i_2 < \ldots < i_s \leq i$ and assume that $\exp^3 i$ exists. Then the numbers $L^n(i_j)$ satisfy

$$L^n(i_1) < L^n(i_2) < \ldots < L^n(i_s) \leq L^n(i).$$

Let p be an i condition. Then we say that p forces the formula

$$\forall x_1 \exists x_2 \ldots \forall x_{s-1} \exists x_s \theta(\bar{c}, x_1, \ldots, x_s),$$

where θ is open and $\bar{c}$ is a sequence of our Henkin constants, if for any i_1, i_2 such that $i_1 < i$, $i_2 \in (i_1, i]$ (where $(a, b]$ denotes $\{x : \ a < x \leq b\}$) and every term of $\bar{c}$ is $< L^n(i_1)$

$$\begin{aligned}&\forall c_1 < L^n(i_1) \exists c_2 < L^n(i_2) \forall i_3 \in (i_2, i], i_4 \in (i_3, i] \forall c_3 < L^n(i_3) \exists c_4 < L^n(i_4)\\ &\ldots \forall i_{s-1} \in (i_{s-2}, i], i_s \in (i_{s-1}, i] \forall c_{s-1} < L^n(i_{s-1}) \exists c_s < L^n(i_s) (p \Vdash \theta(\bar{c}, c_1, \ldots, c_s))\end{aligned}$$

where forcing an open formula is defined as follows:

p forces an atomic formula if it takes the value 0 at the formula, forces its negation if it takes the value 1 at the formula, forces a conjunction as if forces both and forces a disjunction if forces at least one.

This idea has much in common with Wilkie [W] and Ratajczyk [R]. See also [A_2].

Note that every axiom of T is forced by every condition. Assume that $2^{L^n(i)}$ exists. Then, in a model of $I\Delta_0+\Omega_1$ the quantifiers in the definition of forcing are sharply bounded (i.e. bounded by the logarithm of a parameter), and thus the formula defining the relation $p \Vdash \varphi(\bar{c})$ can be made Δ_1 with a fixed (independent of φ) number of bounded quantifiers. Later when we write the forcing relation we shall mean that it is defined by this formula.

If we write $p \Vdash \varphi(\bar{c})$ we deal with an inner language – the language in which the formula φ is formulated – and an outer language – the language in which the forcing relation is formulated. Sometimes we shall assume that the inner language has some additional constants C. If we write $p \Vdash \varphi(\bar{c}, C)$ then we mean that in the outer language the constant C is represented by the number (the forcing constant) 1 and if there is more than one constant then they are represented by consecutive numbers $1, 2 \ldots$, unless we explicitly define them as some other forcing constants. The other forcing constants $g_s(i, j, h_1, \ldots, h_s)$ respectively preserve their inductive meaning – see [A_1], [A_2].

In the case of $i = 0$ and $i = 1$ we let $L^n(1)$ to be a number between $L^n(0)$ and $2^{2^1} = 4$ and $L^n(2)$ a number between $L^n(1) + 1L^n(1)$ and $2^{2^2} = 16$. We want Λ_1, the constant for 1, to be $< L^n(1)$ and also we want to have constants representing 4 and $\omega_1(4)$ to be $< L^n(2)$ (this is for the sake of the proof of lemma 2.7). The constants $< L^n(1)$ which are $< L^n(0)$ and those $< L^n(2)$ which are $< L^n(1) + 1L^n(1)$ have the meaning described in definition 1.1.

Note that if $\bar{c}$ is an arbitrary string of forcing constants then in the definition of the forcing relation we require that i_1 is suitably large – such that $\bar{c} < L^n(i_1)$. So it is interesting to speak about forcing a formula with $\bar{c}$ by a condition p^i only if $\bar{c} < L^n(i)$. Although the relation "p forces φ is very close to "p is w.r.t. φ" it is useful to introduce it.

Consider the following lemma (it will be needed in the proof of lemma 2.7):

Lemma 2.1

If φ is of the form $\forall x_1, \ldots, x_k \exists y_1, \ldots, y_l \theta$, where θ is open, and $T \vdash \varphi$ then there is a fixed standard number m such that whenever $M \models I\Delta_0$ and $p \in M$ and and p is an i condition w.r.t. T and $j < i - m$ then for any $c_1, \ldots, c_k < L^n(j)$ there exist $d_1, \ldots, d_l < L^n(j + m)$ such that $p \Vdash \theta(c_1, \ldots, c_k, d_1, \ldots, d_l)$.

Proof.

Suppose the converse, i.e. for every $m \in \mathbb{N}$ there is an $M \models I\Delta_0$ an i condition p in M, a $j < i - m$ and $c_1, \ldots, c_k < L^n(j)$ such that for no $d_1, \ldots, d_l < L^n(j + m)$, $p \Vdash \theta(c_1, \ldots, c_k, d_1, \ldots, d_l)$. Let T_m be the following theory in the language with additional constants $P, I, J, C_1, \ldots, C_k$:

$$I\Delta_0 + P \text{ is an } I\text{-condition w.r.t. T} + J < I - m + C_1, \ldots, C_k < L^n(J) +$$
$$+ \forall d_1, \ldots, d_l < L^n(J + m)\ P \nVdash \theta(C_1, \ldots, C_k, d_1, \ldots, d_l).$$

Since every T_m is consistent, the following theory is consistent:

$I\Delta_0 + P$ is an I-condition w.r.t. $\mathrm{T} + C_1, \ldots, C_k < L^n(J) +$
$+ \{J < I - m + \forall d_1, \ldots, d_l < L^n(J+m)\ P \not\Vdash \theta(C_1, \ldots, C_k, d_1, \ldots, d_l) :\ m \in \mathbb{N}\}.$

Let M be a model of this theory, $p, i, j, c_1, \ldots, c_k \in M$ interpret appropriate constants. Let N be the following structure determined by p: the universe of N consists of equivalence classes $[c]$ of constants $< L^n(j+m)$ for $m \in \mathbb{N}$, where c is equivalent to c' if $p \Vdash c = c'$. Let $N \models \eta(\overline{[c]})$ if $p \Vdash \eta(\bar{c})$ for an open η. We shall write $N = p(j + \mathbb{N})$. Then N is a model of $T + \neg\varphi$. Contradiction. $\square$.

We consider the usual axiom systems for $I\Delta_0$, $I\Delta_0 + exp$, $I\Delta_0 + \Omega_1$, $I\Delta_0 + \Omega_2$ with the restriction on the form of the axioms formulated at the beginning of §1. They are formulated in terms of the relational symbols $+, \cdot, <$ and two constants Λ_0, Λ_1 for 0 and 1. We treat $+, \cdot$ either as functions or as relations. The axiom systems are the finite amount of algebraic axioms concerning $+, \cdot$ and $<$ (PA^-), the Δ_0 induction scheme presented in the Π_1 form, and, in the case where $+, \cdot$ are relations, the axioms concerning the totality of $+$ and $\cdot$ i.e.

$\forall x \exists y\ y = x + \Lambda_1,$
$\forall x \exists y\ y = x + x,$
$\forall x \exists y\ y = x^2$
and respectively the axioms
$\forall x \exists y\ y = 2^x,$
$\forall x \exists y\ y = \omega_1(x),$
$\forall x \exists y\ y = \omega_2(x)$, where ω_1, ω_2 are formulated in terms of the primitive notions $+, \cdot$ and $<$.

For $i \geq 2$ we have

$$
\begin{aligned}
&L^n(i+1) = L^n(i)\big(1 + i + (i-1)L^n(i-1) + (i-2)L^n(i-1)L^n(i-2) + \\
&\ldots + (i-n+1)L^n(i-1) \cdot \ldots \cdot L^n(i-n+1)\big) = \\
&= L^n(i)\big(1 + i + L^n(i) - L^n(i-1) - (i-1-n+1)L^n(i-1) \cdot \ldots \cdot L^n(i-1-n+1)\big).
\end{aligned}
$$

Now we shall bound $L^n(i)$ more carefully than in [A_1].

Lemma 2.2

$$L^n(i) \leq 2^{2^i}.$$

Proof

We have $L^n(0) = 2 \leq 2^{2^0}$, $L^n(i) \leq 2^{2^i}$ for $i \leq 2$. We have also $L^n(2) \geq 1 + (2+1)$. Assume that the lemma is true for $L^n(i)$, $i \geq 2$, and assume also that for $j \leq i$, $L^n(j) \geq 1 + (j+1)$. We have

$$
\begin{aligned}
&L^n(i+1) = L^n(i)\big(1+i+L^n(i)-L^n(i-1)-(i-1-n+1)L^n(i-1)\cdot\ldots \\
&\cdot L^n(i-1-n+1\big) \le \\
&\le 2^{2^i}\big(1+i+2^{2^i}-L^n(i-1)\big) = \\
&(2^{2^i})^2 + 2^{2^i}\big(1+i-L^n(i-1)\big) \le \\
&\le (2^{2^i})^2 = 2^{2^{i+1}}.
\end{aligned}
$$

Also, we still have $L^n(i+1) \ge 1+(i+1)+1$. $\square$.

Now define the following function $\varepsilon_n(i)$. Let $\varepsilon_n(i)$ be the number ε_n such that

$$L^n(i) = 2^{2^{i-\varepsilon_n}}.$$

Lemma 2.3

$\varepsilon_n(i)$ is an increasing function of i diverging to infinity.

Proof

First let us show that $\varepsilon_n(i+1) \ge \varepsilon_n(i)$. Indeed, if $L^n(i) = 2^{2^{i-\varepsilon_n(i)}}$ then

$$
\begin{aligned}
&L^n(i+1) = L^n(i)\big(1+i+L^n(i)-L^n(i-1)-(i-1-n+1)L^n(i-1)\cdot\ldots \\
&\cdot L^n(i-1-n+1)\big) \le \\
&\le (L^n(i))^2 + L^n(i)(1+i-L^n(i-1)) < (L^n(i))^2 \\
&\le (2^{2^{i-\varepsilon_n(i)}})^2 = 2^{2^{i+1-\varepsilon_n(i)}}.
\end{aligned}
$$

Thus we have shown that $\varepsilon_n(i)$ is increasing.

Now we have to show that it diverges to infinity.

By the above we have

$$L^n(i+1) \le (L^n(i))^2$$

for all i, whence

$$L^n(i) \le (L^n(i-n))^{2^n}$$

and so

$$L^n(i-n) \ge (L^n(i))^{\frac{1}{2^n}}.$$

Now,

$$
\begin{aligned}
&L^n(i+1) = \\
&= L^n(i) + iL^n(i) + (i-1)L^n(i)L^n(i-1) + (i-2)L^n(i)L^n(i-1)L^n(i-2) + \ldots \\
&+ (i-n+1)L^n(i)L^n(i-1)\cdot\ldots\cdot L^n(i-n+1) \le \\
&\le (n+2)(i-n+1)L^n(i)L^n(i-1)\cdot\ldots\cdot L^n(i-n+1) \le \\
&\le (L^n(i))^{\frac{1}{2^{n+1}}} L^n(i)L^n(i-1)\cdot\ldots\cdot L^n(i-n+1) \le \\
&\le (L^n(i))^{\frac{1}{2^{n+1}}} L^n(i)\cdot\frac{L^n(i)}{L^n(i-1-n+1)} \le (L^n(i))^{\frac{1}{2^{n+1}}+1+1-\frac{1}{2^n}} = (L^n(i))^{2-\frac{1}{2^{n+1}}}
\end{aligned}
$$

for all i such that
$$(n+2)(i-n+1) \leq (L^n(i))^{\frac{1}{2^{n+1}}}.$$
It can easily be proved that the last holds for $i \geq m$ for a certain $m \in \mathbb{N}$.

Let $2 - \frac{1}{2^{n+1}}$ be denoted by q_n.

Hence we easily prove by induction that
$$L^n(i) \leq 2^{2^m} \cdot 2^{q_n^{i-m}} = 2^{2^m} \cdot 2^{2^{(i-m)\log q_n}} = 2^{2^{i-(i(1-\log q_n)-m+m\log q_n)}}.$$

Thus $\varepsilon_n(i) \geq i(1-\log q_n) - m + m\log q_n$, hence diverges to infinity (when n is fixed and $i \to \infty$). $\square$.

From now on by $\varepsilon_n(i)$ we shall mean the integer part of $\varepsilon_n(i)$ defined above. So, with n fixed, this is a non-decreasing function of i diverging to infinity.

Note that the integer part of $\varepsilon_n(i)$ and $\varepsilon_n(i)$ differ at most by one and hence the existence of $2^{2^{2^{i-\varepsilon_n(i)}}}$ is not affected by the change from the number to its integer part and conversely.

So, by def.1.4, $Tabcons(IE_n + \Omega_1)$ is formulated as
$$\forall i\big(2^{2^{2^{i-\varepsilon_n(i)}}} \text{ exists} \Rightarrow \exists p^i(p^i \text{ is a condition w.r.t. } IE_n + \Omega_1)\big),$$
i.e.
$$\forall y \forall i\big(y = 2^{2^{2^{i-\varepsilon_n(i)}}} \Rightarrow \exists p^i(p^i \text{ is a condition w.r.t. } IE_n + \Omega_1)\big).$$

Note that the formula $y = 2^{2^{2^{i-\varepsilon_n(i)}}}$ is bounded by y in $I\Delta_0 + \Omega_1$. However, it is not clear wether the formula $y = \varepsilon_n(i)$ can be bounded by i. For, in the definition of $\varepsilon_n(i)$ we refer to 2^{2^i}. To avoid this difficulty we shall deal rather with the function $\varepsilon_n(\log^3 i)$ than with $\varepsilon_n(i)$ (we let $\log^1 x$ be $\log x$ and $\log^{m+1} x$ be $\log\log^m x$). So, we shall consider the following corollary from $Tabcons(IE_n + \Omega_1)$:
$$\forall i\big(2^{2^{2^{i-\varepsilon_n(\log^3 i)}}} \text{ exists} \Rightarrow \exists p^i(p^i \text{ is a condition w.r.t. } IE_n + \Omega_1)\big).$$

This gives below our consistency predicate σ_2 from the introduction.

The formula $y = \varepsilon_n(\log^3 i)$ can be bounded by i – we shall need this later – and also the formula $y = 2^{2^{2^{i-\varepsilon_n(\log^3 i)}}}$ is bounded.

Without loss of generality we may assume that the axioms of $IE_n + \Omega_1$ in the form given in definition 2.1 have at most n existential quantifiers.

Here we may introduce our provability and consistency predicates.

Consider the following definable cut J:
$$J = \{y :\ \exists y' \forall x \leq y\big(\varepsilon_x(\log^3 y) > 2x \Rightarrow \exp^3(y + 2x - \varepsilon_x(\log^3 y)) \leq y'\big)\}.$$

If $y \in J$ then let $x(y)$ denote the greatest x with the property $\varepsilon_x(\log^3 y) > 2x$.

Let σ_1 be the following consistency predicate:

$$\forall y \in J \exists p \leq y \big(p \text{ is an } x(y)\text{-condition w.r.t. } IE_{x(y)} + \Omega_1 \text{ and } p \Vdash \varphi\big),$$

Let $\sigma_1^b(\varphi)$, for a $b \in J$, denote

$$\exists p \leq b \big(p \text{ is an } x(b)\text{-condition w.r.t. } IE_{x(b)} + \Omega_1 \text{ and } p \Vdash \varphi\big).$$

Let $\sigma_2(\varphi)$ be:

$$\forall i \forall x \big(2^{2^{2^{i-\varepsilon_x(\log^3 i)}}} \text{ exists} \Rightarrow \exists p (p \text{ is a } i\text{-condition w.r.t. } IE_x + \Omega_1 \text{ and } p \Vdash \varphi)\big),$$

and consequently let $\sigma_2^{b,x}(\varphi)$ be:

$$\exists p (p \text{ is a } b\text{-condition w.r.t. } IE_x + \Omega_1 \text{ and } p \Vdash \varphi).$$

Of course, σ_j yields a provability predicate π_j defined as $\pi_j(\varphi) \Leftrightarrow \neg\sigma_j(\neg\varphi)$. Similarly, σ_j^b yields π_j^b.

The aim of next three lemmas is to show how from $Tabcons(IE_n + \Omega_1)$ we can infer $Tabcons(IE_n + \Omega_1 + \exists x \varphi(x))$ provided that $\exists x \varphi(x)$ is true and the witness for it is small. As a corollary we get (ii) of the introduction.

LEMMA 2.4 (on reenumeration of constants)

Let T be a theory, $L^n(i)$ a suitable function enumerating constants. Assume that a forcing constant d is given such that $d < L^n(i)$ and that φ is given. Let D be a new forcing constant. Assume that $p^{i+i'}$ is a condition w.r.t. T, $p^{i+i'} \Vdash \varphi(d)$.

Then there is a $p^{i'}$ which is a condition w.r.t. T such that $p^{i'} \Vdash \varphi(D)$.

Similarly if strings of constants $\bar{d}, \bar{D}$ are given.

Proof

The idea of the proof is the following:

Since $d < L^n(i)$, it is closed i' many times, by $p^{i+i'}$, under Skolem functions for the axioms of T and for φ. As a result we infer that $p^{i+i'}$ determines an i'-fold closure under Skolem functions of d w.r.t. $T + \varphi(d)$. This i'-fold closure is isomorphic with an i'-fold closure of D under Skolem functions w.r.t. $T + \varphi(D)$ and this determines the required condition $p^{i'}$.

Assume that the formula $\varphi(d)$ is of the form

$$\forall x_1 \exists x_2 \ldots \forall x_{k-1} \exists x_k \theta(d, x_1, \ldots, x_k),$$

where θ is open. Since $p^{i+i'} \Vdash \varphi(d)$, for any $i'' \leq i' - k$ we have

$$\begin{aligned}
&\forall c_1 < L^n(i + i'' + 1) \exists c_2 < L^n(i + i'' + 2) \ldots \\
&\forall c_{k-1} < L^n(i + i'' + k - 1) \exists c_k < L^n(i + i'' + k) \\
&\big(p^{i+i'} \Vdash \theta(d, c_1, \ldots, c_k)\big).
\end{aligned}$$

Later the c_2 corresponding for c_1 will be denoted by $c_{i'',2}(d, c_1)$, the c_4 corresponding for c_1, c_3 be denoted by $c_{i'',4}(d, c_1, c_3)$ and so on.

Now we shall construct an i'-condition $p^{i'}$ w.r.t. $T + \varphi(D)$.

Let $L^n(i + i')$ correspond to the enumeration of the Henkin constants for T and $\tilde{L}^n(i')$ for $T + \varphi(D)$. We define a function $f : \tilde{L}^n(i') \longrightarrow L^n(i + i')$. Let $f(D) = d$. Assume that $i'' < i'$, that f has been defined at $\tilde{L}^n(i'' + 2l - 1)$ and that $f : \tilde{L}^n(i'' + 2l - 1) \longrightarrow L^n(i + i'' + 2l - 1)$.

If $\tilde{c}_1, \tilde{c}_3, \ldots \tilde{c}_{2l-1}$ are such that $c_1 < \tilde{L}^n(i'' + 1)$,...,$c_{2l-1} < \tilde{L}^n(i'' + 2l - 1)$, $\tilde{c}_{i'',2l}$ is the Henkin constant for the quantifier $\exists c_{2l}$ in $\varphi(D)$ then let $f(\tilde{c}_{2l})$ be

$$c_{i'',2l}(d, f(\tilde{c}_1), f(\tilde{c}_3), \ldots f(\tilde{c}_{2l-1})).$$

If $\tilde{c}_j$ is the Henkin constant for a subformula of an axiom of T at constants $\tilde{c}_1, \ldots, \tilde{c}_{2l-1}$ then let $f(\tilde{c}_j)$ be the constant $c < L^n(i + i'' + 2l)$ which is the Henkin constant for that subformula of that axiom in the enumeration of constants $< L^n(i + i')$ at the constants $f(\tilde{c}_1), \ldots, f(\tilde{c}_{2l-1})$.

We define the values of $p^{i'}$ at the atomic and negated atomic formulas of the constants $\tilde{c} < \tilde{L}^n(i')$ as the values of $p^{i+i'}$ at those formulas of the corresponding constants $f(\tilde{c})$. Then, by the construction, $p^{i'}$ is a condition w.r.t. $T + \varphi(D)$. $\square$.

Lemma 2.5 (on end-extension)

Let p^x be a condition w.r.t. $IE_n + \Omega_1$. Then for every $x' \leq x$ there is a constant $c_{x'} < L^n(x')$ such that for any $c < L^n(x)$ if $p^x \Vdash c \leq c_{x'}$ then there is an $x'' \leq x'$ such that $p^x \Vdash c = c_{x''}$. Moreover, for $x', x'', x''' \leq x$,

$$p^x \Vdash c_{x'} + c_{x''} = c_{x'''} \text{ iff } x' + x'' = x'''$$
$$p^x \Vdash c_{x'} \cdot c_{x''} = c_{x'''} \text{ iff } x' \cdot x'' = x'''.$$

Proof

The idea is similar as in the proof that the model we get in the arithmetized completeness theorem is an end extension of the ground model.

Without loss of generality we may assume that the axiom

$$\forall x \exists y (y = x + \Lambda_1),$$

is the first axiom of $IE_n + \Omega_1$.

Define the following function f. We define $f(x')$ by induction on $x' \leq x$. Let $f(0)$ be the constant for 0, Λ_0. Assume that $f(x')$ has been defined and that it is a constant $c_{x'} < L^n(x')$. There is a constant $c_{x'+1} < L^n(x' + 1)$ which is forced by p^x to be $c_{x'} + \Lambda_1$, where Λ_1 is the constant for 1. Define $f(x' + 1)$ as $c_{x'+1}$. Then $f(x' + 1) < L^n(x' + 1)$.

If p^x forces $c \leq c_{x'}$ for a c and an x' then p^x forces

$$c \leq c_{x'-1} \vee c = c_{x'}$$

since

$$\forall z, u, w (w = u + 1 \& z \le w \Rightarrow z \le u \vee z = w)$$

is an axiom of IE_n.

Again by induction on x' we prove that whenever p^x forces $c \le c_{x'}$ then it forces $c = c_{x''}$ for an $x'' \le x'$. The “moreover” condition is proved by induction on x'' with x' fixed. $\Box$.

The constant $c_x < L^n(x)$ plays the role of the numeral for x. Later we shall denote this constant by $\underline{x}$. In our constructions the numbers x such that $2^{2^{2^{x-\varepsilon_n(x)}}}$ exists have numerals.

Lemma 2.6 (on absoluteness of bounded formulas with small witnesses).
Assume that $i = b$. Assume that ϕ is bounded and that $\phi(b)$ holds. Let $p^{i+i'}$ be a condition w.r.t. $IE_n + \Omega_1$. Then $p^{i+i'} \Vdash \phi(\underline{b})$.

Instead of assuming that ϕ is bounded we may assume that all the unbounded quantifiers of ϕ in its normal form are existential and that $\phi(b)$ holds with the additional requirement that all the the existential quantifiers in ϕ are witnessed by numbers $\le b$.

Proof
Assume that $\phi(b)$ is of the form

$$\forall x_1 \le b \exists x_2 \le b \ldots \forall x_{k-1} \le b \exists x_k \le b \theta(b, x_1, \ldots, x_k)$$

where θ is open.

We shall show that

$$(*) \qquad \begin{aligned} &\forall c_1 < L^n(i) \exists c_2 < L^n(i) \ldots \forall c_{k-1} < L^n(i) \exists c_k < L^n(i) \\ &\Big(p^{i+i'} \Vdash \big((c_1 \le \underline{b} \& c_3 \le \underline{b} \& \ldots c_{k-1} \le \underline{b}) \\ &\Rightarrow c_2 \le \underline{b} \& c_4 \le \underline{b} \& \ldots c_k \le \underline{b} \& \theta(\underline{b}, c_1, \ldots, c_k)\big)\Big). \end{aligned}$$

So let $c_1 < L^n(i'')$ be such that $p^{i+i'} \Vdash c_1 \le \underline{b}$. Thus, by the previous lemma, there is an x_1 such that $x_1 \le b$ and $p^i \Vdash c_1 = c_{x_1}$. Since $\phi(b)$ is true, there is an $x_2 = x_2(x_1) \le b$ such that $\phi'(b, x_1, x_2(x_1))$ holds, where ϕ' is the appropriate subformula of ϕ. So $c_{x_2} < L^n(x_2) < L^n(i)$ and $p^{i+i'} \Vdash c_{x_2} \le \underline{b}$. Let $c_2 = c_2(c_1) = c_{x_2}$. If $p^{i+i'} \not\Vdash c_1 \le \underline{b}$ then let $c_2 = c_2(c_1)$ be any constant $< L^n(i)$. In a similar way we find

$$c_4(c_1, c_3) < L^n(i), \ldots, c_k(c_1, c_3, \ldots, c_{k-1}) < L^n(i)$$

for any $c_3, \ldots c_{k-1} < L^n(i)$ such that $p^{i+i'} \Vdash c_3, \ldots c_{k-1} \le \underline{b}$. By the previous lemma (and def.2.1)

$$p^{i+i'} \Vdash \theta(\underline{b}, c_1, \ldots, c_k).$$

Thus we have proved $(*)$.

It follows that $p^{i+i'} \Vdash \phi(\underline{b})$.
$\Box$.

From the lemma it follows that from

$$\forall i \big(2^{2^{2^{i-\varepsilon_n(i)}}} \text{ exists} \Rightarrow \exists p^i (p^i \text{ is a condition w.r.t. } IE_n + \Omega_1) \big)$$

and

$$2^{2^{2^{b-\varepsilon_n(b)}}} \text{ exists}$$

and

$$\varphi(b)$$

we can infer

$$\forall i \geq b \big(2^{2^{2^{i-\varepsilon_n(i)}}} \text{ exists} \Rightarrow \exists p^i (p^i \text{ is a condition w.r.t. } IE_n + \Omega_1 \text{ and } p^i \Vdash \varphi(\underline{b})) \big)$$

and hence by a lemma 2.4

$$\forall i \big(2^{2^{2^{b+i-\varepsilon_n(b+i)}}} \text{ exists} \Rightarrow \exists p^i (p^i \text{ is a condition w.r.t. } IE_n + \Omega_1 \text{ and } p^i \Vdash \varphi(B)) \big).$$

In particular, we get the following corollary

Corollary 2.1
If b is so small that

$$\forall i (2^{2^{2^{i-\varepsilon_n(i)}}} \text{ exists} \Rightarrow 2^{2^{2^{b+i-\varepsilon_n(b+i)}}} \text{ exists}),$$

φ is bounded, $\varphi(b)$ holds and

$$\forall i \big(2^{2^{2^{i-\varepsilon_n(i)}}} \text{ exists} \Rightarrow \exists p^i (p^i \text{ is a condition w.r.t. } IE_n + \Omega_1) \big),$$

then

$$\forall i \big(2^{2^{2^{i-\varepsilon_n(i)}}} \text{ exists} \Rightarrow \exists p^i (p^i \text{ is a condition w.r.t. } IE_n + \Omega_1 \text{ and } p^i \Vdash \exists x \varphi(x) \big).$$

This is a version available in $I\Delta_0 + \Omega_1$ of the inferring
If $M \models Tabcons(IE_n + \Omega_1) + \exists x \varphi(x)$, *then* $M \models Tabcons(IE_n + \Omega_1 + \exists x \varphi(x))$.

Lemma 2.6 gives also (ii) of the introduction:

ii) $$\sigma_2(0 = 0) \& \theta(b) \Rightarrow \sigma_2(\theta(\underline{b})),$$

provided that $b \in J$ and θ satisfies the assumptions of lemma 2.6.

Choose the following formula defining the graph of the function $2^{2^{2^x}}$. Let $y = 2^{2^{2^{x-1}}}$ be defined by the following formula $\psi(x, y)$:

$$\begin{aligned}
&\exists k, l \forall u < x \forall t_1, \ldots, \forall t_m \exists q, r, q', r' \exists q'', q''' \\
&\big(k - r = q(1 + ul) \& r < 1 + ul \& \\
&\& k - r' = q'(1 + (u + 1)l) \& r' < 1 + (u + 1)l \& r' = \omega_1(r) \& \\
&\& k - 4 = q''(1 + l) \& 4 < 1 + l \& k - y = q'''(1 + xl) \& y < 1 + xl\big),
\end{aligned}$$

where m comes from lemma 2.1 for the sentence $\forall k, l, u \exists q, r(k - r = q(1 + ul) \& r < 1 + ul)$.

This is equivalent to

$$\begin{aligned}
&\exists k, l \forall u < x \exists r, r' \\
&\big(k \equiv r \operatorname{mod} 1 + ul \& r < 1 + ul \& \\
&\& k \equiv r' \operatorname{mod} 1 + (u + 1)l \& r' < 1 + (u + 1)l \& r' = \omega_1(r) \& \\
&\& k \equiv 4 \operatorname{mod} 1 + l \& 4 < 1 + l \& k \equiv y \operatorname{mod} 1 + xl \& y < 1 + xl\big).
\end{aligned}$$

Here the quantifiers $\forall t_1, \ldots, \forall t_m$ are redundant, they appear for technical reasons.

The sentence $\exp^3 B \leq A$ should be read as

$$(*) \qquad \exists u\big(B \leq u - 1 \& \psi(u, A)\big).$$

Lemma 2.7

Let p be an $i + i'$-condition w.r.t. $IE_n + \Omega_1$. Then there is a constant $c < L^n(i)$ such that $p \Vdash \exp^3(\underline{i}) \leq c$.

Proof.

We have: for every $z \leq i$ there is a constant $c^z < L^n(i)$ such that

$$p^{i+i'} \Vdash c^1 = 4 \& p^{i+i'} \Vdash c^2 = \omega_1(c^1) \& \ldots \& p^{i+i'} \Vdash c^z = \omega_1(c^{z-1})$$

– we prove it by induction on z (without loss of generality we may assume that $\forall x \exists y\, y = \omega_1(x)$ is the second axiom of $IE_n + \Omega_1$). Moreover, if we are in a model M of $I\Delta_0 + \Omega_1$ then the function $z \longrightarrow c^z$ is in M.

Now we are going to show that $p^{i+i'}$ "almost" forces $c^i = 2^{2^{2^{\underline{i}-1}}}$.

Claim 2.1

There are constants $K, L < L^n(i + m_0)$ for an $m_0 \in \mathbb{N}$ such that

$$p^{i+i'} \Vdash \psi'(\underline{i}, c^i, K, L),$$

where ψ' is the part of ψ following the quantifiers $\exists k, l$.

Prof of the claim.

First we shall show that there is an $m_0 \in \mathbb{N}$ and constants $K, L < L^n(i+m_0)$, $q^z < L^n(i+m_0+m)$ for $z \le i$ such that for every $z \le i$

$$p^{i+i'} \Vdash K - c^z = q^z(1+\underline{z}L) \& c^z < 1+\underline{z}L.$$

Let $N = p^{i+i'}(i+\mathbb{N})$ be the structure defined in the proof of lemma 2.1. Then N is a model of $I\Delta_0 + \Omega_1$. We shall identify $[\underline{u}]$ with u for $u \le i$. Let i'' be the largest such that $\omega_1^{i''-1}(4) \le [c^i]$ in N. So, there is a code of the sequence $\langle 4, \omega_1(4), \ldots, \omega_1^{i''-1}(4)\rangle$ in N. Thus, there are constants $K, L < L^n(i+m_0)$ for an $m_0 \in \mathbb{N}$ such that the pair $\langle [K], [L]\rangle$ codes the above sequence via the Gödel β function in N. Fix K, L. We shall show that K, L are as required. Let $z \le i$. Since $K, L, \underline{z} < L^n(i+m_0)$, by lemma 2.1 there are constants $q^z, r^z < L^n(i+m_0+m)$ such that

$$p^{i+i'} \Vdash K - r^z = q^z(1+\underline{z}L) \& r^z < 1+\underline{z}L.$$

We shall show by induction on z (in M) that for $z \le i$

$$p^{i+i'} \Vdash K - c^z = q^z(1+\underline{z}L) \& c^z < 1+\underline{z}L.$$

For the sake of induction assume the above for $z-1$.

By the inductive assumption and the choice of K, L, $N \models [c^{z-1}] = [r^{z-1}] = \omega_1^{z-2}(4)$. Since

$$p^{i+i'} \Vdash c^z = \omega_1(c^{z-1}) \& c^z \le c^i,$$

$$N \models [c^z] = \omega_1([c^{z-1}]) \& [c^z] \le [c^i].$$

Thus $\omega_1^{z-1}(4) \le [c^i]$ in N. Hence $i'' \ge z$. Moreover, $N \models \omega_1^{z-1}(4) = [c^z]$. We have $N \models [r^z] = \omega_1^{z-1}(4)$, by the choice of K, L, and hence $N \models [r^z] = [c^z]$. Thus $p^{i+i'} \Vdash r^z = c^z$. Hence

$$p^{i+i'} \Vdash K - c^z = q^z(1+\underline{z}L) \& c^z < 1+\underline{z}L.$$

For $z = 1$ the above holds by the choice of K, L and by a similar reasoning. This completes the induction.

Now consider the formula $\psi'(\underline{i}, c^i, K, L)$:

$$\begin{aligned}
&\forall u < \underline{i} \forall t_1, \ldots, \forall t_m \exists q, r, q', r', q'', q''' \\
&\big(K - r = q(1+uL) \& r < 1+uL \& \\
&\& K - r' = q'(1+(u+1)L) \& r' < 1+(u+1)L \& r' = \omega_1(r) \& \\
&\& K - 4 = q''(1+L) \& 4 < 1+L \& K - c^i = q'''(1+\underline{i}L) \& c^i < 1+\underline{i}L\big).
\end{aligned}$$

To show that $p^{i+i'}$ forces this formula we put it in the form required by def.2.1 and we apply def.2.1. We choose any constants for the redundant existential quantifiers. When dealing with the quantifier $\forall u < \underline{i}$, we may restrict ourselves to considering the constants $\underline{u}$ for $u < i$, by lemma 2.5. For $\underline{u}$ we interpret the quantifiers $\exists q, r, q', r', q'', q'''$ by the constants $q^u, c^u, q^{u+1}, c^{u+1}, q^1, q^i$, respectively. Since all c^u are $< L^n(i)$, $q^u < L^n(i + m_0 + m)$, it follows that

$$p^{i+i'} \Vdash \psi'(\underline{i}, c^i, K, L).$$

$\square$.

Let $c = c^i$. Interpreting in def.2.1 applied to the formula $(*)$ the quantifier $\exists u$ by the constant $\underline{i}$ and the quantifiers $\exists k, l$ in ψ by the constants K, L, we infer

$$p^{i+i'} \Vdash \psi(\underline{i}, c).$$

This completes the proof of the lemma.

Corollary 2.2

Let $b \in J$ and assume that there is a an $b + 2x(b)$-condition w.r.t. $IE_n + \Omega_1$ forcing φ Then there is an $x(b)$-condition forcing "$\varphi \& \underline{b} \in J$".

That is,

$$\sigma_2^{b+2x(b)}(\varphi) \& b \in J \Rightarrow \sigma_1^b(\varphi \& \underline{b} \in J).$$

Proof.

We have $b + 2x(b) - \varepsilon_{x(b)}(\log^3 b) \leq b - 1$ since $\varepsilon_{x(b)}(\log^3 b) > 2x(b)$. The formula $b + 2x(b) - \varepsilon_{x(b)}(\log^3 b) \leq b - 1$, as a formula of b, can be bounded by b and hence, by lemma 2.6, $p^{b+x(b)}$ forces $\underline{b} + 2x(\underline{b}) - \varepsilon_{x(\underline{b})}(\log^3 \underline{b}) \leq \underline{b} - 1$.

Let A', K', L' be the constants c, K, L from claim 2.1. Fix p – a $b + 2x(b)$-condition w.r.t. $IE_n + \Omega_1$ forcing φ.

Thus, by lemma 2.7, we get

$$p \Vdash \varphi \& \underline{b} + 2x(\underline{b}) - \varepsilon_{x(\underline{b})}(\log^3 \underline{b}) \leq \underline{b} - 1 \& \psi'(\underline{b}, A', K', L').$$

We have $A', \underline{b}, K', L' < L^{x(b)}(b + x(b))$. Note that here the constants $A', \underline{b}, K', L'$ are defined as particular forcing constants different from the constants $1, 2 \ldots$.

Hence, interpreting in def.2.1 applied to the formula '$(*)$ the quantifier $\exists u$ by the constant $\underline{b}$ and the quantifiers $\exists k, l$ in ψ by the constants K, L, we infer

$$\exists p^{x(b)}(p^{x(b)} \Vdash \varphi + 2^{2^{2^{\underline{b}+2x(\underline{b})-\varepsilon_{x(\underline{b})}(\log^3 \underline{b})}}} \leq A),$$

whence, interpreting in def.2.1 applied to the formula '$\underline{b} \in J$" the quantifier $\exists y'$ by the constant A, we infer

$$\exists p^{x(b)}\big(p^{x(b)} \Vdash (\varphi \& \underline{b} \in J)\big),$$

whence

$$\sigma_1^b(\varphi \& \underline{b} \in J).\square$$

Corollary 2.3
We have

$$\sigma_2^{b+2x(b),x(b)}(\neg\tau'(\underline{b}))\&b \in J \Rightarrow \sigma_1^b(\neg\tau'(\underline{b}))\&\underline{b} \in J).$$

Proof
We apply corollary 2.2 to the formula $\varphi = \neg\tau'(\underline{b})$).

Hence, immediately we get

Corollary 2.4

$$iii) \qquad \sigma_2^{b+2x(b),x(b)}(\neg\tau'(\underline{b}))\&b \in J \Rightarrow \sigma_1^b(\exists y \in J \neg\tau'(y)).$$

This is (iii) from the introduction.

Finally we prove (i) from the introduction.
Assume that $\mathbb{N} \models \sigma_1(\varphi)$. Note that, for every $y \in \mathbb{N}$, $\exp^3(y+2x(y)-\varepsilon_x(\log^3 y)$ exists in $\mathbb{N}$. Hence $J^{\mathbb{N}} = \mathbb{N}$. Moreover, for every standard x there is an y (in $\mathbb{N}$) such that the condition $\varepsilon_x(\log^3 y) > 2x$ is satisfied. This is true since ε_x diverges to infinity for any fixed x (so, after fixing x we find y). Hence we have arbitrarily large conditions p^x in $\mathbb{N}$ with the property required in σ_1. Thus $\sigma_1(\varphi)$ is a consistency statement and so, since it is true in $\mathbb{N}$, the theory whose consistency it claims is truly consistent – this gives to (i) of the introduction.

Proof of theorem 1.1

Here we repeat the proof from the introduction with the application of (iii) to a special choice of b'.
We consider the Gödel diagonal sentence τ equivalent in $I\Delta_0$ to $\sigma_1(\neg\tau)$. That is, τ is equivalent to $\neg\pi_1(\tau)$. This is the same sentence as in [HP], where we take our π_1.
Note that $\neg\tau$ is

$$\exists y \in J \neg\tau'(y)$$

– see the introduction.
The sentence τ is true in $\mathbb{N}$. So, by (i), the theory

$$I\Delta_0 + \Omega_1 + \neg\tau$$

is consistent.
From now on we shall work in this theory.

So, we have

$$I\Delta_0 + \Omega_1 + b \in J + \neg\tau'(b),$$

for a b.

On the other hand (see introduction) we have

1′) $$\neg\sigma_1^b(\neg\tau).$$

Suppose "ad absurdum" that $I\Delta_0+\Omega_1\vdash Tabcons(I\Delta_0+\Omega_1)$. Then

2′) $$\sigma_2(0=0).$$

This, together with $\neg\tau'(b)$ gives by (ii) of the introduction and by lemma 2.6

3′) $$\sigma_2(\neg\tau'(\underline{b}))$$

(note that the assumptions of lemma 2.6 are satisfied).

Note that $\exp^3 b+2x(b)-\varepsilon_{x(b)}(\log^3 b+2x(b))$ exists since $b+2x(b)-\varepsilon_{x(b)}(\log^3 b+2x(b))\leq b+2x(b)-\varepsilon_{x(b)}(\log^3 b)$ and $\exp^3 b+2x(b)-\varepsilon_{x(b)}(\log^3 b)$ exists, since $b\in J$. Thus, from 2') we have

4′) $$\sigma_2^{b+2x(b),x(b)}(\neg\tau'(\underline{b})).$$

Hence, by corollary 2.4,

5′) $$\sigma_1^b(\neg\tau).$$

Contradiction (with 1')).

Thus the proof of the theorem has been completed.

Proof of theorem 1.2

Consider the following definable cut J:

$$J=\{y:\ \exists y'\forall x\leq y\big(\varepsilon_x(\log^3 y)>2x\&\exp^{m+2}(y+2x-\varepsilon_x(\log^3 y))\leq y'\big)\}$$

and the appropriate predicates σ_1, π_1.

We consider the same Gödel diagonal sentence τ as before for the new predicate σ_1.

Let $y=\exp^{m+2}(x-1)$ be defined by the (counterpart in the language of arithmetic) of the following formula $\psi(x,y)$:

$$\begin{aligned}&\exists k,l\forall u<x\exists r,r'\\&\big(k\equiv r\operatorname{mod}1+ul\&r<1+ul\&\\&\&k\equiv r'\operatorname{mod}1+(u+1)l\&r'<1+(u+1)l\&r'=\omega_m(r)\&\\&\&k\equiv\exp^m 2\operatorname{mod}1+l\&\exp^m 2<1+l\&k\equiv y\operatorname{mod}1+xl\&y<1+xl\big).\end{aligned}$$

The formula $\exp^{m+2}(y+2x-\varepsilon_x(\log^3 y)) \le y'$ should be read as

$$\exists u\big(y+2x-\varepsilon_x(\log^3 y) \le u-1 \& \psi(u,y')\big).$$

We use here the fact that $\exp^{m+2} x$ is the $\omega_m^x(\exp^m 2)$. Now we repeat the proof of theorem 1.1.

§3 Final remarks

We can define a slightly different function $L(i)$, not depending on n:
Let $L(0)=2$ and

$$L(i+1) = L(i) + iL(i) + (i-1)L(i)L(i-1) + (i-2)L(i)L(i-1)L(i-2) + \dots \\ + L(i)L(i-1)\cdot \ldots \cdot L(1).$$

The first term here $L(i)$ reflects the fact that all constants of the i-th stage are included into the $i+1$-th. The second term $iL(i)$ reflects the fact that to every axiom of the i first axioms of T and to its first existential quantifier possibly preceded by a universal quantifier and to every constant of the i-th stage substituted for the variable in the universal quantifier there is a Skolem constant corresponding to the existential quantifier.

The third term $(i-1)L(i)L(i-1)$ corresponds to the fact that to every axiom of the $i-1$ first axioms of T and to its second existential quantifier possibly preceded by a prefix $\forall\exists\forall$ and to every constant of the $i-1$-th stage substituted for the variable in the first universal quantifier and to every constant of the i-th stage substituted for the variable in the second universal quantifier there is a Skolem constant corresponding to the second existential quantifier while the suitable constant from the i-th stage is substituted for the variable in the first universal quantifier.

And so on.

The function $L(i)$ leads to another canonicity notion for a Tableau proof. In the former notion used in def.1.4, we first choose n, i.e. a fragment IE_n if we deal with $I\Delta_0+\Omega_1$, and then we build a Tableau proof determined canonically by this fragment. Now, we do not choose the fragment IE_n but we canonically deal with the whole theory, whose axioms are possibly of arbitrary complexity (arbitrary number of blocks of quantifiers). Now any canonical Tableau proof of "rank" i is longer, since in the ith stage one has to run over more axioms of T. If we define the function $\varepsilon(i)$ for this function L then again we can prove that it is increasing however it does not diverge to infinity (it is bounded by 1). This is due to the fact, that if we consider the sequence $a_i = \log L^n(i)$, then there is a roughly linear dependence between a_{i+1} and $a_i, \dots, a_{i-n+1}$ and for the sequence $a_i = \log L(i)$ it is no more so.

Hence the above reasoning cannot be run.

So, if instead of def.1.4 we can consider definition 3.1 below, we cannot prove our theorems in the above way.

Def.3.1

Let $TabconsT$ mean

$$\forall i\big(2^{L(i)} \text{ exists} \Rightarrow \exists p(p \text{ is an } i\text{-condition w.r.t. } T)\big).$$

We cannot infer theorem 1.1 and 1.2 for the notion 3.1 of Tableau consistency. However, the version of theorem 1.1 for $I\Delta_0 + \Omega_m$ for $m \geq 2$ does hold.

On the other hand, the argument works easily to prove

$$I\Delta_0 + exp \not\vdash Tabcons(I\Delta_0 + exp).$$

Also, we can prove similarly the following theorem

$$IE_n + \Omega_1 \not\vdash Tabcons(IE_n + \Omega_1).$$

for a fixed $n \in \mathbb{N}$ and also an analogue of theorem 1.2 for $IE_n + \Omega_m$.

We can get similar results for the suitable theories in other languages, e.g. in Buss's language, e.g. for Buss's T_2 instead of $I\Delta_0 + \Omega_1$ and T_2^n instead of IE_n.

The following questions seem to be left open:

I. If we do not agree to def.1.5 of a Tableau proof but require it to be a whole proof in the sense of def.0.1 (we may drop the canonicity requirement instead) – note that in this case the existence of a Tableau inconsistency proof for T is not equivalent in $I\Delta_0 + \Omega_1$ to Herbrand incosistency of T given by definition 1.6 – the question weather $I\Delta_0 + \Omega_m$ proves its own Tableau consistency is left open for the case $m = 1$.

II. If we consider def.3.1 instead of 1.4 the question weather $I\Delta_0 + \Omega_m$ proves its own Tableau consistency again is left open for the case $m = 1$ and the appropriate version of theorem 1.2 is open for all m.

This question is somehow related to the question of the collapsing of the hierarchy. It may be interesting since it refers to some "analytical" property of the functions L^n and L, as indicated above. The crucial thing seems to be the distance from $L(i)$ to 2^{2^i}.

We may also ask the following question

III. Can one have an analogue of lemma 2.1 for more complicated formulas φ than the $\forall\exists$ ones?

Or, one may ask some other questions concerning the structure of conditions, for instance

IV. Assume that an i-condition p does not force φ. Can anything be said about the existence of a (possibly smaller) condition forcing $\neg\varphi$?

The last question is somehow related to the "cut" rule in proof theory.

Also, let us recall here the problem following remark 1.1

V. Assume that T is "$I\Delta_0 + f$ is total", where f is a Δ_0 definable function. Assume $S \supseteq T$ and $S \vdash TabconsT$. Does it follow that there exists a satisfaction formula for the axioms of T in S? Is it true that

$$S \vdash \forall i(2^{L^n(i)} \text{ exists} \Rightarrow f^i \text{ exists})?$$

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