On an inequality for a free boundary problem for equations of a viscous compressible heat conducting capillary fluid

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Abstract. We derive an inequality for a local solution of a free boundary problem for a viscous compressible heat-conducting capillary fluid. This inequality is crucial in proving the global existence of solutions belonging to certain anisotropic Sobolev-Slobodetskii space and close to an equilibrium state.

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1. Introduction

In this paper we obtain some crucial inequality for a local solution of equations of the motion of a viscous compressible heat-conducting capillary fluid bounded by a free surface. The motion of such a fluid in a bounded domain $\Omega_t \subset \mathbb{R}^3$ (which depends on time $t \in \mathbb{R}_+$) is described by the following system with the boundary and initial conditions (see [6], [7]):

$$
\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p) = 0 && \text{in } \tilde{\Omega}^T\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \tilde{\Omega}^T,\\
&\varrho c_v(\theta_t + v\cdot\nabla\theta) + \theta p_\theta \operatorname{div} v - \varkappa\Delta\theta \\
&\quad -\frac{\mu}{2}\sum_{i,j=1}^{3}(v_{ix_j}+v_{jx_i})^2 - (\nu-\mu)(\operatorname{div} v)^2 = \varrho r && \text{in } \tilde{\Omega}^T,\\
&\mathbb{T}\bar{n} - \sigma H\bar{n} = -p_0\bar{n} && \text{on } \tilde{S}^T,\\
&v\cdot\bar{n} = -\frac{\phi_t}{|\nabla\phi|} && \text{on } \tilde{S}^T,\\
&\frac{\partial\theta}{\partial n} = \bar{\theta} && \text{on } \tilde{S}^T,\\
&\varrho\big|_{t=0} = \varrho_0, \quad \theta\big|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\tilde{\Omega}^T \equiv \bigcup_{t\in(0,T)}\Omega_t\times\{t\}$, $\Omega_0 = \Omega$ is an initial domain, $\tilde{S}^T \equiv \bigcup_{t\in(0,T)} S_t\times\{t\}$, $\varphi(x,t) = 0$ describes S_t, $\bar{n}$ is the unit outward vector normal to the boundary, i.e. $\bar{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$. Moreover, $v = v(x,t)$ is the velocity of the fluid, $\varrho = \varrho(x,t)$ the density, $\theta = \theta(x,t)$ the temperature, $r = r(x,t)$ the heat sources per unit mass, $\bar{\theta} = \bar{\theta}(x,t)$ the heat flow per unit surface, $p = p(\varrho,\theta)$ the pressure, $c_v = c_v(\varrho,\theta)$ the specific heat at constant volume, μ and ν viscosity coefficients, $\varkappa$ the coefficient of heat conductivity, σ the constant coefficient of the surface tension, and p_0 the external (constant) pressure.

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From the thermodynamic considerations we have

$$\nu > \frac{1}{3}\mu > 0, \qquad \varkappa > 0, \qquad c_v > 0, \qquad \sigma > 0.$$

Further, $\mathbb{T} = \mathbb{T}(v,p)$ denotes the stress tensor of the form

$$\mathbb{T}(v,p) = \{T_{ij}\}_{i,j=1,2,3} = \{\mu S_{ij}(v) + (\nu - \mu)\delta_{ij}\operatorname{div} v - p\delta_{ij}\}_{i,j=1,2,3}$$

where $\mathbb{S}(v) = \{v_{ix_j} + v_{jx_i}\}_{i,j=1,2,3}$ is the velocity deformation tensor.

Finally, we denote by H the double mean curvature of S_t which is negative for convex domains and can be expressed in the form

$$H\bar{n} = \Delta_{S_t}(t)x, \qquad x = (x_1, x_2, x_3),$$

where $\Delta_{S_t}(t)$ is the Laplace-Beltrami operator on S_t. Let S_t be determined by $x = x(s_1, s_2, t)$, $(s_1, s_2) \in U \subset \mathbb{R}^2$. Then we have

$$\Delta_{S_t}(t) = g^{-\frac{1}{2}}\left(\frac{\partial}{\partial s_\gamma} g^{\frac{1}{2}} g^{\gamma\delta}\frac{\partial}{\partial s_\delta}\right) \quad (\gamma, \delta = 1, 2),$$

where the convention summation over repeated indices is assumed, $g = \det\{g_{\gamma\delta}\}_{\gamma,\delta=1,2}$, $g_{\gamma\delta} = \frac{\partial x}{\partial s_\gamma} \cdot \frac{\partial x}{\partial s_\delta}$, $\{g^{\gamma\delta}\}$ is the inverse matrix to $\{g_{\gamma\delta}\}$.

Assume that the domain Ω is given. Then by $(1.1)_5$, $\Omega_t = \{x \in \mathbb{R}^3 \ : \ x = x(\xi, t), \ \xi \in \Omega\}$, where $x = x(\xi, t)$ is the solution of the Cauchy problem

$$\frac{\partial x}{\partial t} = v(x,t), \qquad x\big|_{t=0} = \xi \in \Omega, \qquad \xi = (\xi_1, \xi_2, \xi_3). \tag{1.2}$$

Integrating (1.2) we obtain

$$x = \xi + \int_0^t u(\xi, t')\mathrm{d}t' \equiv X_u(\xi, t),$$

where $u(\xi,t) = v(X_u(\xi,t))$ and $x = X_u(\xi,t)$ describes the relation between the Eulerain x and the Lagrangian ξ coordinates. Moreover, by $(1.1)_5$, $S_t = \{x \ : \ x = x(\xi,t), \ \xi \in S = \partial\Omega\}$.

By the continuity equation $(1.1)_2$ and the kinematic condition $(1.1)_5$ the total mass ins conserved, i.e.

$$\int_{\Omega_t} \varrho(x,t)\mathrm{d}x = \int_{\Omega} \varrho_o(\xi)\mathrm{d}\xi = M.$$

Now, assume that $p_\varrho > 0$, $p_\theta > 0$ for $\varrho, \theta \in \mathbb{R}_+$ and consider the equation

$$p\left(\frac{M}{\frac{4}{3}\pi R_e^3}, \theta_e\right) = p_0 + \frac{2\sigma}{R_e}. \tag{1.3}$$

We assume that there exist $R_e > 0$ and $\theta_e > 0$ satisfying (1.3). Then we have the following definition.

Definition 1.1. Let $r = \bar{\theta} = 0$. By an equilibrium state we mean a solution $(v, \theta, \varrho, \Omega_t)$ of problem (1.1) such that $v = 0$, $\theta = \theta_e$, $\varrho = \varrho_e$, $\Omega_t = \Omega_e$ for $t \geq 0$, where $\varrho_e = \frac{M}{\frac{4}{3}\pi R_e^3}$, Ω_e is a ball of radius R_e, $R_e > 0$ and $\theta_e > 0$ satisfy equation (1.3).

Next, we introduce

$$\varrho_\sigma = \varrho - \varrho_e, \qquad \theta_\sigma = \theta - \theta_e, \qquad p_\sigma = p - \frac{2\sigma}{R_e} - p_0. \tag{1.4}$$

We can write p_σ in the form

$$p_\sigma(\varrho, \theta) = p_1\varrho_\sigma + p_2\theta_\sigma, \tag{1.5}$$

where

$$p_1(\varrho,\theta)=\int\limits_0^1 p_\varrho(\varrho_e+s(\varrho-\varrho_e),\theta)\mathrm{d}s,$$

$$p_2(\theta)=\int\limits_0^1 p_\theta(\varrho_e,\theta_e+s(\theta-\theta_e))\mathrm{d}s.$$

Then by using (1.4) problem (1.1) can be rewritten in the Lagrangian coordinates as follows

$$\begin{aligned}
&\eta u_t-\operatorname{div}_u\mathbb{T}_u(u,p_\sigma)=0 && \text{in }\ \Omega^T=\Omega\times(0,T),\\
&\eta_{\sigma t}+\eta\operatorname{div}_u u=0 && \text{in }\ \Omega^T,\\
&\eta c_v\vartheta_{\sigma t}+\vartheta p_\vartheta\operatorname{div}_u u-\varkappa\nabla_u^2\vartheta_\sigma && \\
&\quad=\frac{\mu}{2}\sum_{i,j=1}^3(\xi_{x_i}\cdot\nabla_\xi u_j+\xi_{x_j}\cdot\nabla_\xi u_i)^2 && \\
&\quad-(\nu-\mu)(\operatorname{div}_u u)^2+\eta k && \text{in }\ \Omega^T,\\
&\mathbb{T}_u(u,p_\sigma)\bar n_u-\sigma(H+H_e)\bar n_u=0 && \text{on }\ S^T,\\
&\bar n_u\cdot\nabla_u\vartheta_\sigma=\bar\vartheta && \text{on }\ S^T,\\
&\eta_\sigma\big|_{t=0}=\varrho_{\sigma0},\quad \vartheta_\sigma\big|_{t=0}=\theta_{\sigma0},\quad u\big|_{t=0}=v_0, && \text{in }\ \Omega,
\end{aligned}\tag{1.6}$$

where $\eta(\xi,t)=\varrho(X_u(\xi,t),t)$, $\vartheta(\xi,t)=\theta(X_u(\xi,t),t)$, $\eta_\sigma=\eta-\varrho_e$, $\vartheta_\sigma=\vartheta-\theta_e$, $k(\xi,t)=r(X_u(\xi,t),t)$, $\bar\vartheta(\xi,t)=\bar\theta(X_u(\xi,t),t)$, $\bar n_u(\xi,t)=\bar n(X_u(\xi,t),t)$, $\nabla_u=\xi_{ix}\partial_{\xi i}=\{\xi_{ix_j}\partial_{\xi_i}\}_{j=1,2,3}$, $\mathbb{T}_u(u,p)=-pI+\mathbb{D}_u(u)$, $I=\{\delta_{ij}\}_{i,j=1,2,3}$, $\mathbb{D}_u(u)=\{\mathbb{D}_{uij}(u)\}_{i,j=1,2,3}=\{\mu(\partial_{x_i}\xi_k\partial_{\xi_k}u_j+\partial_{x_j}\xi_k\partial_{\xi_k}u_i)+(\nu-\mu)\delta_{ij}\operatorname{div}_u u\}_{i,j=1,2,3}$, $\operatorname{div}_u u=\nabla_u\cdot u=\partial_{x_i}\xi_k\partial_{\xi_k}u_i$, $\operatorname{div}_u\mathbb{T}_u(u,p)=\{\partial_{x_j}\xi_k\partial_{\xi_k}T_{uij}(u,p)\}_{i=1,2,3}$ and $\partial_{x_i}\xi_k$ are elements of matrix ξ_x which is inverse to the matrix $x_\xi=I+\int_0^t u_\xi(\xi,t')\mathrm{d}t'$.

The aim of our considerations is to prove the global existence of solutions to problem (1.1) which are sufficiently close to the equilibrium solution (see Definition 1.1) for all $t>0$. Since the proof is very long we have divided it into the following parts:

- 1^0 the proof of the local existence of solutions to problem (1.1) (see [13]);
- 2^0 the global energy estimate and the estimate which shows an increase of regularity of solution after some time (see [14]);
- 3^0 the estimate proved in this paper;
- 4^0 the global existence of solutions to problem (1.1) which is proved by an appropriate combining estimates from points 2^0–3^0 and by extending step by step the local solution to all $t>0$ (see [12]).

In this paper we derive an estimate for the local solution $(u,\vartheta_\sigma,\eta_\sigma)\in W_2^{2+\alpha,1+\frac{\alpha}{2}}(\Omega^T)\times W_2^{2+\alpha,1+\frac{\alpha}{2}}(\Omega^T)\times W_2^{1+\alpha,\frac12+\frac{\alpha}{2}}(\Omega^T)\cap C(0,T;W_2^{1+\alpha}(\Omega))$, $\alpha\in\left(\frac34,1\right)$ (see definitions in Section 2) in which the norms of $u,\vartheta_\sigma,\eta_\sigma$ are estimated by the sum of L_2-norms of these functions, the nonlinear terms consisting of the products of the highest order norms of $u,\vartheta_\sigma,\eta_\sigma$ and the norms of data. The main result is formulated in Theorem 3.1.

Moreover, the free boundary problem for a viscous barotropic compressible capillary fluid has been considered in [9], [10] and [15].

2. Notation and auxiliary results

In Section 3 we use a partition of unity $(\{\tilde\Omega_i\},\{\zeta_i\})$, $\Omega=\bigcup_i\tilde\Omega_i$. Let $\tilde\Omega$ be one of the $\tilde\Omega_i's$ and $\zeta(\xi)=\zeta_i(\xi)$ be the corresponding function. If $\tilde\Omega$ is an interior subdomain then let $\tilde\omega\subset\tilde\Omega$ and $\zeta(\xi)=1$ for $\xi\in\tilde\omega$. Otherwise we assume that $\bar{\tilde\Omega}\cap S\neq\varnothing$, $\bar{\tilde\omega}\cap S\neq\varnothing$, $\bar{\tilde\omega}\subset\bar{\tilde\Omega}$. Let $\beta\in\bar{\tilde\omega}\cap S\subset\bar{\tilde\Omega}\cap S$, $\tilde S\equiv S\cap\tilde\Omega$.

Introduce local coordinates connected with $\{\xi\}$ by

$$y_k=\alpha_{kl}(\xi_l-\beta_l)\qquad \alpha_{3k}=n_k(\beta),\qquad k=1,2,3,\tag{2.1}$$

where α_{kl} is a constant orthogonal matrix such that $\tilde{S}$ is determined by

$$y_3 = F(y_1, y_2), \qquad F \in W_2^{\frac{5}{2}+\alpha}$$

and

$$\tilde{\Omega} = \{y \ : \ |y_i| < d, \quad i = 1, 2, \quad F(y') < y_3 < F(y') + d, \quad y' = (y_1, y_2)\}. \tag{2.2}$$

Next, we introduce functions u', ϑ' and η' by

$$u'_i(y) = \alpha_{ij} u_j(\xi)\big|_{\xi=\xi(y)}, \quad \vartheta'(y) = \vartheta(\xi)\big|_{\xi=\xi(y)}, \quad \eta'(y) = \eta(\xi)\big|_{\xi=\xi(y)},$$

where $\xi = \xi(y)$ is the inverse transformation to (2.1). Further we introduce new variables by

$$z_i = y_i, \qquad i = 1, 2, \qquad z_3 = y_3 - \tilde{F}(y), \qquad y \in \tilde{\Omega} \tag{2.3}$$

which will be denoted by $z = \Phi(y)$, where $\tilde{F}$ is an extension of F to $\tilde{\Omega}$.

Let

$$\hat{\Omega} = \Phi(\tilde{\Omega}) = \{z \ : \ |z_i| < d, \quad i = 1, 2, \quad 0 < z_3 < d\} \tag{2.4}$$

and $\hat{S} = \Phi(\tilde{S})$.

Define

$$\hat{f}(z) = f'(y)\big|_{y=\Phi^{-1}(z)}, \qquad f \in \{u, \vartheta, \eta\}.$$

Introduce $\hat{\nabla}_k = \xi_{lx_k} z_{i\xi_l} \nabla_{z_i}\big|_{\xi=\chi^{-1}(z)}$, where $\chi(\xi) = \Phi(\psi(\xi))$ and $y = \psi(\xi)$ is defined by (2.1). Introduce also the notation

$$\begin{aligned}
&\tilde{u}(\xi) = u(\xi)\zeta(\xi), && \tilde{\vartheta}_\sigma(\xi) = \vartheta_\sigma(\xi)\zeta(\xi), \\
&\tilde{\eta}_\sigma(\xi) = \eta_\sigma(\xi)\zeta(\xi), && \xi \in \tilde{\Omega}, \quad \tilde{\Omega} \cap S = \varnothing, \\
&\tilde{u}(z) = \hat{u}(z)\hat{\zeta}(z), && \tilde{\vartheta}_\sigma(z) = \hat{\vartheta}_\sigma(z)\hat{\zeta}(z), \\
&\tilde{\eta}_\sigma(z) = \hat{\eta}_\sigma(z)\hat{\zeta}(z), && z \in \hat{\Omega} = \Phi(\tilde{\Omega}), \quad \bar{\tilde{\Omega}} \cap S \neq \varnothing,
\end{aligned}$$

where $\hat{\zeta}(z) = \zeta(\xi)\big|_{\xi=\chi^{-1}(z)}$.

Under the above notation problem (1.6) has the following form in an interior subdomain:

$$\begin{aligned}
&\eta\tilde{u}_{it} - \nabla_j T_{ij}(\tilde{u}, \tilde{p}_\sigma) = -\nabla_{u_i} B_{uij}(u, \zeta) - T_{uij}(u, p_\sigma)\nabla_{u_j}\zeta \\
&\quad - (\nabla_i T_{ij}(\tilde{u}, \tilde{p}_\sigma) - \nabla_{u_i} T_{uij}(\tilde{u}, \tilde{p}_\sigma)) \equiv k_{1i}, \quad i = 1, 2, 3, \\
&\tilde{\eta}_{\sigma t} + \varrho_e \nabla \cdot \tilde{u} = \varrho_e u \cdot \nabla_u \zeta - \eta_\sigma \nabla_u \cdot u\zeta + \varrho_e(\nabla \cdot \tilde{u} - \nabla_u \cdot \tilde{u}) \equiv k_2, \\
&\eta c_v(\eta, \vartheta)\tilde{\vartheta}_{\sigma t} - \varkappa \nabla^2 \tilde{\vartheta}_\sigma + \theta_e p_\vartheta(\varrho_e, \theta_e)\nabla \cdot \tilde{u} \\
&\quad = \eta\tilde{k} + \left[\frac{\mu}{2}\sum_{i,j=1}^{3}(\xi_{kx_i}\partial_{\xi_k} u_j + \xi_{kx_j}\partial_{\xi_k} u_i)^2 + (\nu - \mu)(\nabla_u \cdot u)^2\right]\zeta \\
&\quad + \theta_e p_\vartheta(\varrho_e, \theta_e) u \cdot \nabla_u \zeta + (\theta_e p_\vartheta(\varrho_e, \theta_e) - \vartheta p_\vartheta(\varrho, \vartheta))\nabla_u \cdot u\zeta \\
&\quad + \theta_e p_\vartheta(\varrho_e, \theta_e)(\nabla \cdot \tilde{u} - \nabla_u \cdot \tilde{u}) - \varkappa(\nabla_u^2 \zeta \vartheta_\sigma + 2\nabla_u \zeta \cdot \nabla_u \vartheta_\sigma) - \varkappa(\nabla^2 \tilde{\vartheta}_\sigma - \nabla_u^2 \tilde{\vartheta}_\sigma) \equiv \eta\tilde{k} + k_3,
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
&\mathbb{B}_u(u, \zeta) = \{B_{uij}(u, \zeta)\}_{i,j=1,2,3} = \{\mu(u_i \nabla_{u_i}\zeta + u_j \nabla_{u_i}\zeta) + (\nu - \mu)\delta_{ij} u \cdot \nabla_u \zeta\}_{i,j=1,2,3}, \\
&\nabla_{u_j} = \xi_{ix_j}\partial_{\xi_i}, \qquad \tilde{p}_\sigma = p_\sigma \zeta
\end{aligned}$$

and in a boundary subdomain:

$$
\begin{aligned}
&\hat\eta\tilde u_{it} - \nabla_j T_{ij}(\tilde u,\tilde p_\sigma) = -\hat\nabla_j \hat B_{ij}(\hat u,\hat\zeta) - \hat T_{ij}(\hat u,\hat p_\sigma)\hat\nabla_j\hat\zeta \\
&-(\nabla_i T_{ij}(\tilde u,\tilde p_\sigma) - \hat\nabla_j \hat T_{ij}(\tilde u,\tilde p_\sigma)) \equiv k_{4i}, \quad i=1,2,3, \\
&\tilde\eta_{\sigma t} + \varrho_e\nabla\cdot\tilde u = \varrho_e\hat u\cdot\hat\nabla\hat\zeta - \hat\eta_\sigma\hat\nabla\cdot\hat u\hat\zeta + \varrho_e(\nabla\cdot\tilde u - \hat\nabla\cdot\tilde u) \equiv k_5, \\
&\hat\eta c_v(\hat\eta,\hat\vartheta)\tilde\vartheta_{\sigma t} - \varkappa\Delta\tilde\vartheta_\sigma + \theta_e p_{\hat\vartheta}(\varrho_e,\theta_e)\nabla\cdot\tilde u \\
&= \hat\eta\tilde k + \left[\frac{\mu}{2}\sum_{i,j=1}^3(\hat\nabla_i\hat u_j + \hat\nabla_j\hat u_i)^2 + (\nu-\mu)(\hat\nabla\cdot\hat u)^2\right]\hat\zeta \\
&+ \theta_e p_{\hat\vartheta}(\varrho_e,\theta_e)\hat u\cdot\hat\nabla\hat\zeta + (\theta_e p_{\hat\vartheta}(\varrho_e,\theta_e) - \hat\vartheta p_{\hat\vartheta}(\hat\eta,\hat\vartheta))\hat\nabla\cdot\hat u\hat\zeta \\
&+ \theta_e p_{\hat\vartheta}(\varrho_e,\theta_e)(\nabla\cdot\tilde u - \hat\nabla\cdot\tilde u) - \varkappa(\hat\nabla^2\hat\zeta\hat\vartheta_\sigma + 2\hat\nabla\hat\zeta\cdot\hat\nabla\hat\vartheta_\sigma) \\
&-\varkappa(\nabla^2\tilde\vartheta_\sigma - \hat\nabla^2\tilde\vartheta_\sigma) \equiv \hat\eta\tilde k + k_6, \\
&\mathbb{T}(\tilde u,\tilde p_\sigma)\hat n' = \sigma\Delta_{\hat S}\hat\xi\hat\zeta + \sigma\Delta_{\hat S}\int_0^t \tilde u\,\mathrm{d}t' \\
&+ \mathbb{T}(\tilde u,\tilde p_\sigma)\hat n' - \hat{\mathbb{T}}(\tilde u,\tilde p_\sigma)\hat n + \hat{\mathbb{B}}(\hat u,\hat\zeta)\hat n \\
&-\sigma\Big(2\hat\nabla\int_0^t \hat u\,\mathrm{d}t'\hat\nabla\hat\zeta + \int_0^t \hat u\,\mathrm{d}t'\hat\nabla^2\hat\zeta\Big) + \frac{2\sigma}{R_e}\hat n\hat\zeta \equiv \frac{2\sigma}{R_e}\hat n\hat\zeta + k_7, \\
&\hat n'\cdot\nabla\tilde\vartheta_\sigma = \tilde{\bar\vartheta} + \hat n\cdot\hat\nabla\hat\zeta\hat\vartheta_\sigma + \hat n'\cdot\nabla\tilde\vartheta_\sigma - \hat n\cdot\hat\nabla\tilde\vartheta_\sigma \equiv \tilde{\bar\vartheta} + k_8,
\end{aligned}
\tag{2.6}
$$

where $\hat{\mathbb{T}}$ and $\hat{\mathbb{B}}$ indicate that the operator ∇_u is replaced by $\hat\nabla$; $\hat n' = (0,0,1)$, $\hat n$ is the vector $\bar n_u = \bar n(x(\xi,t),t)$ written in z coordinates and

$$
\Delta_{\hat S}(t) = \frac{1}{\sqrt{g_{\hat u}}}\frac{\partial}{\partial z_\gamma}\left(\sqrt{g_{\hat u}}g_{\hat u}^{\gamma\delta}\frac{\partial}{\partial z_\delta}\right), \tag{2.7}
$$

$\{g_{\hat u}^{\gamma\delta}\}$ is the inverse matrix to $\{g_{\hat u}^{\gamma\delta}\}$,

$$
g_{\hat u}^{\gamma\delta} = \frac{\partial x}{\partial z_\gamma}\cdot\frac{\partial x}{\partial z_\delta}, \qquad x = \hat\xi + \int_0^t \hat u(z,t')\mathrm{d}t', \qquad g_{\hat u} = \det\{g_{\hat u_{\gamma\delta}}\}.
$$

From (2.7) we have

$$
\Delta_{\hat S}(0) = \frac{1}{\sqrt g}\frac{\partial}{\partial z_\gamma}\left(\sqrt g g^{\gamma\delta}\frac{\partial}{\partial z_\delta}\right),
$$

where by (2.1) and (2.3) $g = \det\{g_{\gamma\delta}\} = \det\left\{\frac{\partial\xi}{\partial z_\gamma}\cdot\frac{\partial\xi}{\partial z_\delta}\right\} = 1 + F_{z_1}^2 + F_{z_2}^2$ and

$$
g^{11} = g^{-1}(1+F_{z_2}^2), \qquad g^{22} = g^{-1}(1+F_{z_1}^2), \qquad g^{12} = g^{21} = -g^{-1}F_{z_1}F_{z_2}.
$$

We assume that d from (2.2) and (2.4) is so small that

$$
|F_{z_i}| \le \frac14, \quad i=1,2. \tag{2.8}
$$

Now, we introduce the notation and the spaces used in the paper. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with the boundary S.

By $W_2^{k+\alpha,\frac{k}{2}+\frac{\alpha}{2}}(\Omega^T)$, $k\in\mathbb{N}\cup\{0\}$, $\alpha\in(0,1)$ we denote Sobolev-Slobodetskii space with the norm

$$
\begin{aligned}
\|u\|_{W_2^{k+\alpha,\frac{k}{2}+\frac{\alpha}{2}}(\Omega^T)} &= \sum_{|\beta|+2i\le k}\|\partial_x^\beta\partial_t^i u\|^2_{L_2(\Omega^T)} \\
&\quad+\sum_{|\beta|=k}\int_0^T\int_\Omega\int_\Omega \frac{|\partial_x^\beta u(x,t)-\partial_{x'}^\beta u(x',t)|^2}{|x-x'|^{3+2\alpha}}\mathrm{d}x\mathrm{d}x'\mathrm{d}t \\
&\quad+\int_\Omega\int_0^T\int_0^T \frac{|\partial_t^{[\frac{k}{2}]}u(x,t)-\partial_{t'}^{[\frac{k}{2}]}u(x,t')|^2}{|t-t'|^{1+\alpha+k-2[\frac{k}{2}]}}\mathrm{d}x\mathrm{d}t\mathrm{d}t',
\end{aligned}
$$

where $\partial_x^\beta=\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}\partial_{x_3}^{\beta_3}$, $\beta=(\beta_1,\beta_2,\beta_3)$ is a multi-index, $|\beta|=\beta_1+\beta_2+\beta_3$. Similarly we can define the norms in spaces $W_2^{k+\alpha}(\Omega)$ and $W_2^{k+\alpha,\frac{k}{2}+\frac{\alpha}{2}}(S^T)$.

Moreover, we shall use the notation:

$$
\begin{aligned}
&\|u\|_{W_2^{k+\alpha,\frac{k}{2}+\frac{\alpha}{2}}(\Omega^T)}=\|u\|_{k+\alpha,\Omega^T}, \\
&\|u\|_{W_2^{k+\alpha}(\Omega)}=\|u\|_{k+\alpha,Q}, && Q\in\{\Omega,S,S^1\}\ \ (S^1 \text{ is the unit sphere}); \\
&\|u\|_{L_p(Q)}=|u|_{p,Q}, && p\in[1,\infty],\ \ Q\in\{\Omega,S\}; \\
&\|u\|_{L_2(Q)}=\|u\|_{0,Q}, && q\in\{\Omega,S,\Omega^T,S^T\}; \\
&\|u\|_{\Omega^T}^{(\alpha+2,\frac{\alpha}{2}+1)}=\Bigg[\|u\|^2_{2+\alpha,\Omega^T}+T^{-\alpha}\Bigg(\|u_t\|^2_{0,\Omega^T}+\sum_{|\beta|=2}\|\partial_x^\beta u\|^2_{0,\Omega^T}\Bigg) \\
&\qquad\qquad+\sup_{t\le T}\|u(\cdot,t)\|^2_{\alpha+1,\Omega}\Bigg]^{\frac{1}{2}}; \\
&\|u\|_{Q^T}^{(\alpha,\frac{\alpha}{2})}=(\|u\|^2_{\alpha,Q^T}+T^{-\alpha}\|u\|^2_{0,Q^T})^{1/2}, && Q\in\{\Omega,S\}.
\end{aligned}
$$

Next, we define the isotropic Besov spaces by introducing the norm (see [3], Sect. 18)

$$
\|u\|_{B_p^l(\mathbb{R}^n)}=\|u\|_{L_p(\mathbb{R}^n)}+\sum_{i=1}^n\left(\int_0^{h_0}\mathrm{d}h\int_{\mathbb{R}^n}\mathrm{d}x\frac{|\Delta_i^m(h)\partial_{x_i}^k u|^p}{h^{1+(l-k)p}}\right)^{1/p}, \tag{2.9}
$$

where $p\in[1,+\infty]$, $\Delta_i^m(h)f(x)=\sum_{j=0}^m(-1)^{m-j}c_{jm}f(x+jhe_i)$, $c_{jm}=\binom{m}{j}=\frac{m!}{j!(m-j)!}$, $x\in\mathbb{R}^n$, e_i is the unit vector of the coordinate axis i, $i=1,\dots,n$ and $m>l-k$, $m,k\in\mathbb{N}\cup\{0\}$, $l\in\mathbb{R}_+$, $l\notin\mathbb{Z}$.

It is proved in [4] that the Besov spaces defined by (2.9) are equivalent for all m,k satisfying $m>l-k$.

Now, we define the Sobolev-Slobodetskii spaces by introducing the norm

$$
\|u\|_{W_p^l(\mathbb{R}^n)}=\|u\|_{L_p(\mathbb{R}^n)}+\sum_{i=1}^n\left(\int_0^{h_0}\mathrm{d}h\int_{\mathbb{R}^n}\mathrm{d}x\frac{|\Delta_i(h)\partial_{x_i}^{[l]}u|^p}{h^{1+p(l-[l])}}\right)^{1/p},
$$

where $\Delta_i(h)=\Delta_i^1(h)$, $l\notin\mathbb{Z}$, $[l]$ is the integer part of l.

By the Golovkin theorem from [4] the norms of spaces $B_p^l(\mathbb{R}^n)$ and $W_p^l(\mathbb{R}^n)$ are equivalent.

We also define the spaces $\tilde{B}_p^l(\mathbb{R}^n)$ and $\tilde{W}_p^l(\mathbb{R}^n)$ by introducing the norms

$$
\|u\|_{\tilde{B}_p^l(\mathbb{R}^n)}=\|u\|_{L_p(\mathbb{R}^n)}+\left(\int_{\mathbb{R}^n}\mathrm{d}x\int_{\mathbb{R}^n}\mathrm{d}y\frac{|\Delta^m(x-y)\partial_y^k u(y)|^p}{|x-y|^{n+p(l-k)}}\right)^{1/p}
$$

and

$$\|u\|_{\tilde{W}_p^l(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \left(\int\limits_{\mathbb{R}^n} \mathrm{d}x \int\limits_{\mathbb{R}^n} \mathrm{d}y \frac{|\Delta(x-y)\partial_y^{[l]}u(y)|^p}{|x-y|^{n+p(l-[l])}}\right)^{1/p},$$

where $\Delta^m(x-y)f(y) = \sum_{i=1}^m \Delta_i^m(h)f(y)$, $(x-y)\cdot e_i = h$, $\Delta(x-y)f(y) = \Delta^1(x-y)f(y)$ and $m > l-k$, $m,k \in \mathbb{N}\cup\{0\}$, $l \in \mathbb{R}_+$, $l \notin \mathbb{Z}$.

By Lemma 7.44 of [2] the spaces $\tilde{B}_p^l(\mathbb{R}^n)$, $B_p^l(\mathbb{R}^n)$ (with $h_0 = \infty$), $\tilde{W}_p^l(\mathbb{R}^n)$ and $W_p^l(\mathbb{R}^n)$ (with $h_0 = \infty$) are equivalent.

Now, we recall the following imbedding for the Besov spaces (2.9) (see [3, Sect. 18]):

$$\partial_x^\sigma B_p^l(\mathbb{R}^n) \subset B_q^\varrho(\mathbb{R}^n) \qquad \text{for} \qquad \frac{n}{p} - \frac{n}{q} + |\sigma| + \varrho \le l. \tag{2.10}$$

Moreover, for $\varkappa = \frac{1}{l}\left(\frac{n}{p} - \frac{n}{q} + |\sigma| + \varrho\right) > 0$ and $\varkappa_1 = \frac{3}{p} - \frac{3}{r}$ we have the interpolation inequality

$$\|\partial_x^\sigma u\|_{B_q^\varrho(\mathbb{R}^n)} \le (\varepsilon^{1-\varkappa} + c\varepsilon^{-\varkappa}\varepsilon_1^{1-\varkappa_1})\|u\|_{B_p^l(\mathbb{R}^n)} + c\varepsilon^{-\varkappa}\varepsilon_1^{-\varkappa_1}\|u\|_{L_r(\mathbb{R}^n)}$$

with $r \ge 1$.

In the above notation $B_p^l(\mathbb{R}^n)$ with $l \in \mathbb{Z}_+$ is the Sobolev space.

All above remarks can be applied for spaces of functions defined on a bounded domain $\Omega \subset \mathbb{R}^n$, and by using a partition of unity we can also define spaces of traces on the boundary of Ω and formulate corresponding trace theorems.

3. Inequality for global existence

In order to derive global estimates we assume the following condition: Ω_t is diffeomorphic to a ball, so S_t can be described by

$$|x| = r = R(\omega, t), \qquad \omega \in S^1.$$

Then the following theorem holds.

Theorem 3.1. *Assume $k \in W_2^{\alpha,\frac{\alpha}{2}}(\Omega^T)$, $\bar{\vartheta} \in W_2^{\alpha+\frac{1}{2},\frac{\alpha}{2}+\frac{1}{4}}(S^T)$, $\alpha \in \left(\frac{3}{4}, 1\right)$ and there exists the local solution determined by Theorem 3.4 of [13]. Then the following inequality holds*

$$\begin{aligned} &\phi(t,\Omega) + \bar{c}\|u\|^2_{2+\alpha,\Omega^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t} \\ &\le c(t,Z_1,Z_2)\Big[\|u\|^2_{0,\Omega^t} + \|\eta_\sigma\|^2_{0,\Omega^t} + \|\vartheta_\sigma\|^2_{0,\Omega^t} + \|k\|^2_{\alpha,\Omega^t} + \|\bar{\vartheta}\|^2_{\alpha+\frac{1}{2},S^t} \\ &+ \varepsilon_1 Z_4 + Z_1 Z_3 + (Z_1 + Z_4)Z_5\Big] + \phi(0,\Omega), \end{aligned} \tag{3.1}$$

where $t \le T$, T is the time of the local existence, $\phi(t,\Omega)$ is defined by (3.86), $c = c(t,Z_1,Z_2)$ is a positive continuous function nondecreasing with respect to its arguments, $\bar{c} \le 1$ is a positive constant, $\varepsilon_1 \in (0,1)$ is a sufficiently small constant and

$$\begin{aligned} Z_1 &= \|u\|^2_{2+\alpha,\Omega^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t}, \\ Z_2 &= \sup_{0\le t'\le t}\|u\|^2_{1+\alpha,\Omega} + \sup_{0\le t'\le t}\|\vartheta_\sigma\|^2_{1+\alpha,\Omega} + \sup_{0\le t'\le t}\|\eta_\sigma\|^2_{1+\alpha,\Omega}, \\ Z_3 &= \left[\|u\|^{(\alpha+2,\frac{\alpha}{2}+1)}_{\Omega^t} + \|\vartheta_\sigma\|^{(\alpha+2,\frac{\alpha}{2}+1)}_{\Omega^t}\right]^2 + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t} + \sup_{0\le t'\le t}\|\eta_\sigma\|^2_{1+\alpha,\Omega}, \\ Z_4 &= t\left\|H(\cdot,0) + \frac{2}{R_e}\right\|^2_{\alpha+1/2,S^1} + \|R(\cdot,t') - R(\cdot,0)\|^2_{2+\alpha,S^1\times(0,t)}, \\ Z_5 &= \Big\|\int\limits_0^t u\mathrm{d}t'\Big\|^2_{2+\alpha,S} = \sum_{i\in\mathfrak{N}}\Big\|\int\limits_0^t \tilde{u}\mathrm{d}t'\Big\|^2_{2+\alpha,\hat{S}_i}. \end{aligned}$$

Proof. Similarly as in [5] we first obtain the inequality for interior subdomains. Let us consider an interior subdomain $\tilde{\Omega}$ and let $\zeta \in C_0^\infty(\mathbb{R}^n)$ be the corresponding function from the partition of unity. We can assume that $\tilde{\Omega}$ is a ball with the centre at a point ξ_0 and assume that $\zeta(\xi) = 1$ for $\xi \in B_\lambda(\xi_0) = \{\xi \in \mathbb{R}^3 : |\xi - \xi_0| < \lambda\}$, $\zeta(\xi) = 0$ for $\xi \in \mathbb{R}^3 \setminus B_{2\lambda}(\xi_0)$, where $B_\lambda(\xi_0) \subset \tilde{\Omega}$ and $\bar{\tilde{\Omega}} \subset B_{2\lambda}(\xi_0)$.

Denote by $\Delta^s(z)f(\xi)$ the s-th finite difference of f such that

$$\Delta^s(z)f(\xi) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} f(\xi + kz).$$

Similarly as in [5] we apply $\Delta^2(z)$ to $(2.5)_1$–$(2.5)_3$ and multiply the resulting equations by $\Delta^2(z)\tilde{u}_i$, $\frac{p_1}{\varrho_e}\Delta^2(z)\tilde{\eta}_\sigma$ and $\frac{p_2}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\Delta^2(z)\tilde{\vartheta}_\sigma$, respectively. Then integrating each of the equations over $B_{2\lambda}(\xi_0)$ and with respect to z over $\mathbb{R}^3$ with the measure $\frac{1}{|z|^{3+2(\alpha+1)}}$ and using (1.5), $(1.6)_2$ and the Korn inequality we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\mathrm{d}z\int_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\Big(\frac{\eta|\Delta^2(z)\tilde{u}|^2}{|z|^{3+2(1+\alpha)}} + \frac{p_1|\Delta^2(z)\tilde{\eta}_\sigma|^2}{\varrho_e|z|^{3+2(1+\alpha)}} + \frac{p_2\eta c_v}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(z)\tilde{\vartheta}_\sigma|^2}{|z|^{3+2(1+\alpha)}}\Big) + \\
&+ c_1\int_{\mathbb{R}^3}\mathrm{d}z\frac{\|\Delta^2(z)\tilde{u}\|^2_{1,B_{2\lambda}(\xi_0)}}{|z|^{3+2(1+\alpha)}} + \varkappa\int_{\mathbb{R}^3}\mathrm{d}z\int_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\frac{p_2|\Delta^2(z)\nabla\tilde{\vartheta}_\sigma|^2}{|z|^{3+2(1+\alpha)}} \\
&\le I_1 + \int_{\mathbb{R}^3}\mathrm{d}z\int_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\frac{p_2\Delta^2(z)k_3\Delta^2(z)\tilde{\vartheta}_\sigma}{|z|^{3+2(1+\alpha)}},
\end{aligned} \tag{3.2}$$

where $c_1 > 0$ is a constant and

$$\begin{aligned}
&|I_1| \le \varepsilon\left(\|\tilde{u}\|^2_{2+\alpha,\tilde{\Omega}} + \|\vartheta_\sigma\|^2_{2+\alpha,\tilde{\Omega}} + \int_{\mathbb{R}^3}\mathrm{d}z\int_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\frac{|\Delta^2(z)\tilde{\eta}_\sigma|^2}{|z|^{3+2(1+\alpha)}}\right) \\
&+ \psi_1\left(\frac{1}{\varepsilon}, a\right)(\|\tilde{u}\|^2_{1+\alpha,\tilde{\Omega}} + \|\eta_\sigma\|^2_{\alpha,\tilde{\Omega}} + \|\vartheta_\sigma\|^2_{\alpha,\tilde{\Omega}}) \\
&+ \psi_2\left(\frac{1}{\varepsilon}, a\right)\Big[\int_0^t \|u\|^2_{2+\alpha,\Omega}\mathrm{d}t'(\|\tilde{u}\|^2_{2+\alpha,\tilde{\Omega}} + \|\eta_\sigma\|^2_{1+\alpha,\tilde{\Omega}} \\
&+ \|\vartheta_\sigma\|^2_{1+\alpha,\tilde{\Omega}} + \|\tilde{\eta}_\sigma\|^2_{1+\alpha,\tilde{\Omega}}\|\tilde{u}\|^2_{2+\alpha,\tilde{\Omega}} + \|\tilde{\vartheta}_\sigma\|^2_{1+\alpha,\tilde{\Omega}}\|\tilde{u}\|^2_{2+\alpha,\tilde{\Omega}}) \\
&+ \|u\|^2_{2+\alpha,\Omega}\|\tilde{u}\|^2_{2+\alpha,\tilde{\Omega}} + \|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\tilde{u}_t\|^2_{\alpha,\tilde{\Omega}} + \\
&+ (\|\tilde{\eta}_\sigma\|^2_{1+\alpha,\tilde{\Omega}} + \|\tilde{u}\|^2_{2+\alpha,\tilde{\Omega}} + \|\tilde{\vartheta}_\sigma\|^2_{1+\alpha,\tilde{\Omega}})(\|\eta_\sigma\|^2_{1+\alpha,\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\Omega}) \\
&+ (\|\vartheta_{\sigma t}\|^2_{1+\alpha,\Omega} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega})\|\tilde{\vartheta}_\sigma\|^2_{1+\alpha,\tilde{\Omega}} + \|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\tilde{k}\|^2_{\alpha,\tilde{\Omega}}\Big],
\end{aligned} \tag{3.3}$$

where $a = T^{1/2}(\int_0^T \|u\|^2_{2+\alpha,\Omega}\mathrm{d}t')^{1/2}$, ψ_i $(i = 1, 2)$ and all functions ψ_i occuring below are positive continuous functions.

Next, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{p_2\Delta^2(z)k_3\Delta^2(z)\tilde\vartheta_\sigma}{|z|^{3+2(1+\alpha)}} = -\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{\Delta(z)k_3\Delta(z)p_2\Delta^2(z)\tilde\vartheta_\sigma}{|z|^{3+2(1+\alpha)}}\\
&-\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{p_2\Delta(z)k_3\Delta^3(z)\tilde\vartheta_\sigma}{|z|^{3+2(1+\alpha)}} \le\\
&\varepsilon \int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta^3(z)\tilde\vartheta_\sigma|^2}{|z|^{3+2(2+\alpha)}} + c(\varepsilon)\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)k_3|^2}{|z|^{3+2\alpha}}\\
&+c\left[\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)\eta_\sigma|^{p_1}}{|z|^{3+\frac{p_1}{2}(1+\alpha)}} + \int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)\vartheta_\sigma|^{p_1}}{|z|^{3+\frac{p_1}{2}(1+\alpha)}}\right]^{1/p_1}\cdot\\
&\cdot\left(\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)k_3|^{p_2}}{|z|^{3+\frac{p_2}{2}(1+\alpha)}}\right)^{1/p_2}\left(\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta^2(z)\tilde\vartheta_\sigma|^{p_3}}{|z|^{3+p_3(1+\alpha)}}\right)^{1/p_3}\\
&\le \varepsilon\|\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega} + I_2 + I_3
\end{aligned}
\tag{3.4}
$$

with

$$
\begin{aligned}
|I_2| \le c &\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{1}{|z|^{3+2\alpha}}(|\Delta(z)\xi_x^2 u_\xi^2|^2 + |\Delta(z)u\cdot\nabla_u\zeta|^2\\
&+|\Delta(z)(\theta_e p_\vartheta(\varrho_e,\theta_e) - \vartheta p_\vartheta(\eta,\vartheta))\nabla_u\cdot u\zeta|^2 + |\Delta(z)(\nabla\cdot\tilde u - \nabla_u\cdot\tilde u)|^2\\
&+|\Delta(z)(\nabla_u^2\zeta\vartheta_\sigma + 2\nabla_u\zeta\cdot\nabla_u\vartheta_\sigma)|^2 + |\Delta(z)(\nabla^2\tilde\vartheta_\sigma - \nabla_u^2\tilde\vartheta_\sigma)|^2) = \sum_{i=1}^6 I_2^i
\end{aligned}
\tag{3.5}
$$

and

$$
|I_3| \le \varepsilon\|\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega} + c(\varepsilon)[\|k_3\|^2_{\alpha,\tilde\Omega}(\|\eta_\sigma\|^2_{1+\alpha,\tilde\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega})],
\tag{3.6}
$$

where $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$ and we have used imbeddings $B_2^{1+\alpha}(\Omega)\subset B_{p_1}^{(1+\alpha)/2}(\Omega)$, $B_2^\alpha(\Omega)\subset B_{p_2}^{(1+\alpha)/2}(\Omega)$, $B_2^{2+\alpha}(\Omega)\subset B_{p_3}^{1+\alpha}(\Omega)$ valid for $\frac{3}{2}-\frac{3}{p_1}+\frac{1+\alpha}{2}\le 1+\alpha$, $\frac{3}{2}-\frac{3}{p_2}+\frac{1+\alpha}{2}\le\alpha$, $\frac{3}{2}-\frac{3}{p_3}+1+\alpha\le 2+\alpha$ which all hold for suitable p_i if $\alpha>\frac{1}{2}$.

Now, we estimate the terms on the right-hand side of (3.5). We have

$$
\begin{aligned}
|I_2^1| \le\ & c\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)u_\xi u_\xi|^2}{|z|^{3+2\alpha}} + c\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)\int_0^t u_\xi dt'\psi_3(b)u_\xi^2|^2}{|z|^{3+2\alpha}}\\
&+c\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\int_0^t u_\xi dt'\Delta(z)\psi_3(b)u_\xi^2|^2}{|z|^{3+2\alpha}} + c\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\int_0^t u_\xi dt'\psi_3(b)\Delta(z)u_\xi u_\xi|^2}{|z|^{3+2\alpha}}\\
\le\ & c|u_\xi|^2_{\infty,\Omega}\int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)u_\xi|^2}{|z|^{3+2\alpha}} + c\psi_4(a)|u_\xi|^2_{\infty,\Omega}\left[\left(\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|u_\xi|^{2p}}{|z|^{3-\delta p}}\right)^{\frac{1}{2p}}\right]^2\cdot\\
&\cdot\left[\left(\int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)\int_0^t u_\xi dt'|^{2p'}}{|z|^{3+2p'\alpha+\delta p'}}\right)^{\frac{1}{2p'}}\right]^2 + c\psi_5(a)|\int_0^t u_\xi(\xi,t')dt'|^2_{\infty,\Omega}|u_\xi|^2_{\infty,\Omega}\int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)u_\xi|^2}{|z|^{3+2\alpha}},
\end{aligned}
\tag{3.7}
$$

where $u_\xi^2=\sum_{i,j=1}^3 u_{i\xi_j}^2$, $\delta>0$ is sufficiently small and $\frac{1}{p}+\frac{1}{p'}=1$, $b=\int_0^t u_\xi \mathrm{d}t'$ and we used that $\xi_x=I+b\psi_3(b)$, I is the unit matrix and $\psi_3(0)\neq 0$, ψ_3 is a positive continuous increasing function.

Using the imbeddings $B_2^\alpha(\Omega)\subset L_{2p}(\Omega)$ and $B_2^{1+\alpha}(\Omega)\subset B_{2p'}^{\alpha+\frac{\delta}{2}}(\Omega)$ which hold simultaneously for $\alpha>\frac{1}{2}$ and some δ, p, p' such that $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{3}{2}-\frac{3}{2p}\le\alpha$ and $\frac{3}{2}-\frac{3}{2p'}+\alpha+\frac{\delta}{2}\le 1+\alpha$ (see imbedding (2.10)) we get

$$|I_2^1|\le c\|u\|_{2+\alpha,\Omega}^2\|u\|_{1+\alpha,\Omega}^2+\psi_6(a)\|u\|_{2+\alpha,\Omega}^2\|u\|_{1+\alpha,\Omega}^2\int_0^t\|u\|_{2+\alpha,\Omega}^2\mathrm{d}t'. \tag{3.8}$$

In the same way we obtain

$$|I_2^2|+|I_2^5|\le c(\|u\|_{\alpha,\Omega}^2+\|\vartheta_\sigma\|_{1+\alpha,\Omega}^2)+\psi_7(a)\int_0^t\|u\|_{2+\alpha,\Omega}^2\mathrm{d}t'(\|u\|_{1+\alpha,\Omega}^2+\|\vartheta_\sigma\|_{2+\alpha,\Omega}^2). \tag{3.9}$$

Next, we estimate

$$\begin{aligned}|I_2^6|\le& c\int\limits_{B_{2\lambda}(0)}\mathrm{d}z\int\limits_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\frac{|\Delta(z)\int_0^t u_\xi\mathrm{d}t'\psi_3(b)\tilde\vartheta_{\sigma\xi\xi}|^2}{|z|^{3+2\alpha}}\\ &+c\int\limits_{B_{2\lambda}(0)}\mathrm{d}z\int\limits_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\frac{|\int_0^t u_\xi\mathrm{d}t'\Delta(z)\psi_3(b)\tilde\vartheta_{\sigma\xi\xi}|^2}{|z|^{3+2\alpha}}\\ &+c\int\limits_{B_{2\lambda}(0)}\mathrm{d}z\int\limits_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\frac{|\int_0^t u_\xi\mathrm{d}t'\psi_3(b)\Delta\tilde\vartheta_{\sigma\xi\xi}|^2}{|z|^{3+2\alpha}}=I_4^1+I_4^2+I_4^3,\end{aligned} \tag{3.10}$$

where

$$|I_4^1|\le\psi_8(a)\left[\left(\int\limits_{B_{2\lambda}(0)}\mathrm{d}z\int\limits_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\frac{|\tilde\vartheta_{\sigma\xi\xi}|^{2p}}{|z|^{3-\delta p}}\right)^{\frac{1}{2p}}\right]^2\cdot\left[\left(\int\limits_{\mathbb{R}^3}\mathrm{d}z\int\limits_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\frac{|\Delta(z)\int_0^t u_\xi\mathrm{d}t'|^{2p'}}{|z|^{3+2p'\alpha+\delta p'}}\right)^{\frac{1}{2p'}}\right]^2, \tag{3.11}$$

$\delta>0$ is sufficiently small and $\frac{1}{p}+\frac{1}{p'}=1$. Hence, we get

$$|I_4^1|\le\psi_9(a)\|\tilde\vartheta_\sigma\|_{2+\alpha,\tilde\Omega}^2\int_0^t\|u\|_{2+\alpha,\Omega}^2\mathrm{d}t'. \tag{3.12}$$

I_4^2 we estimate in the same way, and

$$|I_4^3|\le\psi_{10}(a)|\int_0^t u_\xi\mathrm{d}t'|_{\infty,\Omega}\int\limits_{\mathbb{R}^3}\mathrm{d}z\int\limits_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\frac{|\Delta(z)\tilde\vartheta_{\sigma\xi\xi}|^2}{|z|^{3+2\alpha}}\le\psi_{11}(a)\int_0^t\|u\|_{2+\alpha,\Omega}^2\mathrm{d}t'\|\tilde\vartheta_\sigma\|_{2+\alpha,\tilde\Omega}^2. \tag{3.13}$$

Hence

$$|I_2^6|\le\psi_{12}(a)\int_0^t\|u\|_{2+\alpha,\Omega}^2\mathrm{d}t'\|\tilde\vartheta_\sigma\|_{2+\alpha,\tilde\Omega}^2. \tag{3.14}$$

Finally,

$$|I_2^3| \le \psi_{13}(a)(\|\vartheta_\sigma\|^2_{\alpha,\tilde\Omega} + \|\eta_\sigma\|^2_{\alpha,\tilde\Omega} + 1)(\|\eta_\sigma\|^2_{1+\alpha,\tilde\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega})\|u\|^2_{2+\alpha,\tilde\Omega} \int_0^t \|u\|^2_{2+\alpha,\Omega} \mathrm{d}t'. \tag{3.15}$$

Taking into estimates (3.2)–(3.15) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \mathrm{d}z \int_{B_{2\lambda}(\xi_0)} \mathrm{d}\xi \left(\frac{\eta|\Delta^2(z)\tilde u|^2}{|z|^{3+2(1+\alpha)}} + \frac{p_1|\Delta^2(z)\tilde\eta_\sigma|^2}{\varrho_e|z|^{3+2(1+\alpha)}} + \frac{p_2\eta c_v}{\theta_e p_\vartheta(\varrho_e,\theta_e)} \frac{|\Delta^2(z)\tilde\vartheta_\sigma|^2}{|z|^{3+2(1+\alpha)}} \right) \\
&+ c_1 \int_{\mathbb{R}^3} \mathrm{d}z \frac{\|\Delta^2(z)\tilde u\|^2_{1,B_{2\lambda}(\xi_0)}}{|z|^{3+2(1+\alpha)}} + \varkappa \int_{\mathbb{R}^3} \mathrm{d}z \int_{B_{2\lambda}(\xi_0)} \mathrm{d}\xi \frac{p_2|\nabla(\Delta^2(z)\tilde\vartheta_\sigma)|^2}{|z|^{3+2(1+\alpha)}} \\
&\le \varepsilon(\|u\|^2_{2+\alpha,\tilde\Omega} + \|\eta_\sigma\|^2_{1+\alpha,\tilde\Omega} + \|\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega}) \\
&+ \psi_{14}\left(\frac{1}{\varepsilon}, a\right) \left[X_1X_2 + X_1(1+X_2^2) \int_0^t \|u\|^2_{2+\alpha,\Omega}\mathrm{d}t' + \|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\tilde u_t\|^2_{\alpha,\tilde\Omega} \right] \\
&+ \psi_{15}\left(\frac{1}{\varepsilon}, a\right) \left(\|u\|^2_{1+\alpha,\tilde\Omega} + \|\tilde\eta_\sigma\|^2_{\alpha,\tilde\Omega} + \|\tilde\vartheta_\sigma\|^2_{\alpha,\tilde\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega} \right) + \psi_{16}\left(\frac{1}{\varepsilon}, a\right) \|\tilde k\|^2_{\alpha,\tilde\Omega},
\end{aligned} \tag{3.16}$$

where

$$X_1 = \|u\|^2_{2+\alpha,\Omega} + \|\eta_\sigma\|^2_{1+\alpha,\Omega} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega}, \tag{3.17}$$

$$X_2 = \|u\|^2_{1+\alpha,\Omega} + \|\eta_\sigma\|^2_{1+\alpha,\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\Omega}. \tag{3.18}$$

Now, we shall derive the energy type estimate for the local solution of problem (2.5).

In order to do this multiply equations $(2.5)_1$ by $\tilde u_i$ $(i = 1,2,3)$ and equations $(2.5)_2$–$(2.5)_3$ by $\frac{p_1}{\varrho_e}\tilde\eta_\sigma$ and $\frac{p_2}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\tilde\vartheta_\sigma$, respectively. Then integrating the resulting equations over $\tilde\Omega$, using the boundary condition $\tilde u\big|_{\partial\tilde\Omega} = 0$, the equation of continuity, the Korn inequality and relation (1.5) and next adding these equations we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\tilde\Omega} \left(\eta\tilde u^2 + \frac{p_1}{\varrho_e}\tilde\eta_\sigma^2 + \frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\tilde\vartheta_\sigma^2 \right) \mathrm{d}\xi \\
&+ c_2(\|\tilde u\|^2_{1,\tilde\Omega} + \|\nabla\tilde\vartheta_\sigma\|^2_{0,\tilde\Omega}) \le \varepsilon(\|\tilde u\|^2_{1,\tilde\Omega} + \|\eta_\sigma\|^2_{0,\tilde\Omega} \\
&+ \|\vartheta_\sigma\|^2_{0,\tilde\Omega} + \|\tilde\vartheta_\sigma\|^2_{1,\tilde\Omega}) + \psi_{17}\left(\frac{1}{\varepsilon}, a\right) \{(\|\tilde u\|^2_{2,\Omega} + \|\eta_\sigma\|^2_{1,\tilde\Omega} \\
&+ \|\vartheta_\sigma\|^2_{2,\tilde\Omega}) \int_0^t \|u\|^2_{2,\Omega}\mathrm{d}t' + (\|\tilde\vartheta_\sigma\|^2_{2,\tilde\Omega} + 1)\|\tilde u\|^2_{1,\Omega} \int_0^t \|u\|^2_{2+\alpha,\Omega}\mathrm{d}t' \\
&+ \|\eta_\sigma\|^2_{1,\tilde\Omega}[\|\vartheta_{\sigma t}\|^2_{1,\tilde\Omega} + \|\tilde\vartheta_\sigma\|^2_{2,\tilde\Omega} + \|u\|^2_{2,\tilde\Omega}(1 + \|\tilde\eta_\sigma\|^2_{1,\tilde\Omega} \\
&+ \|\tilde\vartheta_\sigma\|^2_{1,\tilde\Omega})] + \|\vartheta_\sigma\|^2_{1,\tilde\Omega}(\|u\|^2_{2,\tilde\Omega} + \|\tilde\vartheta_\sigma\|^2_{2,\tilde\Omega} + \|\vartheta_{\sigma t}\|^2_{1,\tilde\Omega}) \\
&+ \|u\|^2_{1,\tilde\Omega}\|\tilde u\|^2_{1,\tilde\Omega} + \|u\|^2_{1,\tilde\Omega} + \|\vartheta_\sigma\|^2_{0,\tilde\Omega} + \|\tilde k\|^2_{0,\tilde\Omega}\},
\end{aligned} \tag{3.19}$$

where $c_3 > 0$ is a constant.

Now, using the equivalence of norms for fractional spaces (see [4]) and estimates (3.16) and (3.19) we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\tilde\Omega}\left(\eta\tilde u^2+\frac{p_1}{\varrho_e}\tilde\eta_\sigma^2+\frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\tilde\vartheta_\sigma^2\right)\mathrm{d}\xi\\
&+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}\mathrm{d}z\int\limits_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\biggl(\frac{\eta|\Delta^2(z)\tilde u|^2}{|z|^{3+2(1+\alpha)}}+\frac{p_1|\Delta^2(z)\tilde\eta_\sigma|^2}{\varrho_e|z|^{3+2(1+\alpha)}}\\
&+\frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(z)\tilde\vartheta_\sigma|^2}{|z|^{3+2(1+\alpha)}}\biggr)+c_3(\|\tilde u\|^2_{2+\alpha,\tilde\Omega}+\|\nabla\tilde\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega})\\
&\le\varepsilon(\|u\|^2_{2+\alpha,\tilde\Omega}+\|\eta_\sigma\|^2_{1+\alpha,\tilde\Omega}+\|\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega})+\\
&+c\|\tilde k\|^2_{\alpha,\tilde\Omega}+c(\|u\|^2_{1+\alpha,\tilde\Omega}+\|\tilde\eta_\sigma\|^2_{\alpha,\tilde\Omega}\\
&+\|\tilde\vartheta_\sigma\|^2_{\alpha,\tilde\Omega}+\|\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega}+\|\eta_\sigma\|^2_{0,\tilde\Omega})\\
&+cX_1[X_2(1+X_2)+(1+X_2^2)\int\limits_0^t\|u\|^2_{2+\alpha,\Omega}\mathrm{d}t']\\
&+c\|\eta_\sigma\|^2_{1+\alpha,\Omega}\|u_t\|^2_{\alpha,\tilde\Omega},
\end{aligned}
\tag{3.20}
$$

where X_1 and X_2 are given by (3.17) and (3.18) respectively, $c_4>0$ is a constant.

To examine the third term on the r.h.s. of (3.20) we apply the interpolation inequality

$$
\begin{aligned}
&\|u\|^2_{1+\alpha,\tilde\Omega}+\|\tilde\eta_\sigma\|^2_{\alpha,\tilde\Omega}+\|\tilde\vartheta_\sigma\|^2_{\alpha,\tilde\Omega}+\|\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega}\\
&\le\varepsilon(\|u\|^2_{2+\alpha,\tilde\Omega}+\|\tilde\eta_\sigma\|^2_{1+\alpha,\tilde\Omega}+\|\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega})\\
&+c(\varepsilon)(\|u\|^2_{0,\tilde\Omega}+\|\tilde\eta_\sigma\|^2_{0,\tilde\Omega}+\|\tilde\vartheta_\sigma\|^2_{0,\tilde\Omega}).
\end{aligned}
\tag{3.21}
$$

From $(2.5)_1$ we have

$$
\begin{aligned}
&\|\tilde\eta_\sigma\|^2_{1+\alpha,\tilde\Omega}\le\\
&\varepsilon(\|u\|^2_{2+\alpha,\tilde\Omega}+\|\eta_\sigma\|^2_{1+\alpha,\tilde\Omega}+\|\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega})+c(\|\tilde u\|^2_{2+\alpha,\tilde\Omega}+\|\tilde u_t\|^2_{\alpha,\tilde\Omega}\\
&+\|u\|^2_{0,\tilde\Omega}+\|\eta_\sigma\|^2_{0,\tilde\Omega}+\|\vartheta_\sigma\|^2_{0,\tilde\Omega})+c\int\limits_0^t\|u\|^2_{2+\alpha,\Omega}\mathrm{d}t'(X_1+X_2^2)+cX_2^2.
\end{aligned}
\tag{3.22}
$$

Inequalities (3.20)–(3.22) imply

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\tilde\Omega}\left(\eta\tilde u^2+\frac{p_1}{\varrho_e}\tilde\eta_\sigma^2+\frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\tilde\vartheta_\sigma^2\right)\mathrm{d}\xi\\
&+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}\mathrm{d}z\int\limits_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\biggl(\frac{\eta|\Delta^2(z)\tilde u|^2}{|z|^{3+2(1+\alpha)}}+\frac{p_1|\Delta^2(z)\tilde\eta_\sigma|^2}{\varrho_e|z|^{3+2(1+\alpha)}}\\
&+\frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(z)\tilde\vartheta_\sigma|^2}{|z|^{3+2(1+\alpha)}}\biggr)+c_4(\|\tilde u\|^2_{2+\alpha,\tilde\Omega}\\
&+\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega}+\|\tilde\eta_\sigma\|^2_{2+\alpha,\tilde\Omega})\le\varepsilon\|\tilde u_t\|^2_{\alpha,\tilde\Omega}\\
&+\varepsilon(\|u\|^2_{2+\alpha,\tilde\Omega}+\|\eta_\sigma\|^2_{1+\alpha,\tilde\Omega}+\|\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega})\\
&+c\|\tilde k\|^2_{\alpha,\tilde\Omega}+c(\varepsilon)(\|u\|^2_{0,\tilde\Omega}\\
&+\|\tilde\eta_\sigma\|^2_{0,\tilde\Omega}+\|\tilde\vartheta_\sigma\|^2_{0,\tilde\Omega})+cX_1[X_2(1+X_2)\\
&+(1+X_2^2)\int\limits_0^t\|u\|^2_{2+\alpha,\Omega}\mathrm{d}t']+c\|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\tilde u_t\|^2_{\alpha,\tilde\Omega},
\end{aligned}
\tag{3.23}
$$

where $c_5 > 0$ is a constant.

To examine the norm $\|\tilde u_t\|^2_{\alpha,\tilde\Omega}$ from the r.h.s. of (3.23) we use the interpolation inequality (see [8]):

$$\int_0^t \|\tilde u_{t'}\|^2_{\alpha,\tilde\Omega}\mathrm{d}t \le \varepsilon\int_0^t \|\tilde u\|^2_{2+\alpha,\tilde\Omega}\mathrm{d}t + c(\varepsilon)\int_{\tilde\Omega} \|\tilde u_{t'}\|^2_{\frac{\alpha}{2},(0,t)}\mathrm{d}\xi, \tag{3.24}$$

where $c(\varepsilon)$ does not depend on t.

To estimate the last term on the r.h.s. of (3.24) we consider the time differences

$$\Delta_t^k(h)f(\xi,t) = \sum_{j=0}^k c_k^j(-1)^{k-j}f(\xi,t+jh), \quad k > \alpha.$$

Applying $\Delta_t(h)$ to $(2.5)_1$ we obtain

$$\eta\Delta_t\tilde u_{it} - \nabla_j T_{ij}(\Delta_t\tilde u,\Delta_t\tilde p_\sigma) = \eta\Delta_t\tilde u_{it} - \Delta_t(\eta\tilde u_{it}) + \Delta_t k_{1i}. \tag{3.25}$$

In the same way as in [5] from (3.25) we get

$$\begin{aligned}
&\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\eta|\Delta_{t'}\tilde u_{t'}|^2\mathrm{d}\xi\\
&+\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_{\tilde\Omega}\left(\frac{\mu}{4}|\mathbb{S}(\Delta_{t'}\tilde u)|^2 + \frac{\nu-\mu}{2}|\mathrm{div}\,\Delta_{t'}\tilde u|^2\right)\mathrm{d}\xi\Big|_{t'=t-h_0}\\
&-\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_{\tilde\Omega}\left(\frac{\mu}{4}|\mathbb{S}(\Delta_{t'}\tilde u)|^2 + \frac{\nu-\mu}{2}|\mathrm{div}\,\Delta_{t'}\tilde u|^2\right)\mathrm{d}\xi\Big|_{t'=0}\\
&=\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}[\eta\Delta_{t'}\tilde u_{t'} - \Delta_{t'}(\eta\tilde u_{t'})]\cdot\Delta_{t'}\tilde u_{t'}\mathrm{d}\xi\\
&+\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\Delta_{t'}k_1\cdot\Delta_{t'}\tilde u_{t'}\mathrm{d}\xi\\
&-\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\nabla(\Delta_{t'}\tilde p_\sigma)\cdot\Delta_t\tilde u_t\mathrm{d}\xi \equiv I_5+I_6+I_7,
\end{aligned}\tag{3.26}$$

where $\mathbb{S}(v) = \{v_{i\xi_j} + v_{j\xi_i}\}_{i,j=1,2,3}$, $0 < h_0 < t$.

First, we estimate I_5. Using the continuity equation $(1.6)_2$ we have

$$\begin{aligned}
|I_5| &\le \int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}|\Delta_{t'}(\eta_\sigma)\tilde u_{t'}||\Delta_{t'}\tilde u_{t'}|\mathrm{d}\xi\\
&\le \varepsilon\|\tilde u\|^2_{2+\alpha,\tilde\Omega^t} + c(\varepsilon)\int_0^{h_0}\mathrm{d}h\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\mathrm{d}\xi\frac{|\int_{t'}^{t'+h}u_\xi\mathrm{d}\tau|^2|\tilde u_t|^2}{h^{1+\alpha}}\\
&\le \varepsilon\|\tilde u\|^2_{2+\alpha,\tilde\Omega^t} + c(\varepsilon)t^{1-\alpha}\int_0^t|u_\xi|^2_{\infty,\tilde\Omega}\mathrm{d}t'\int_0^t|\tilde u_{t'}|^2_{2,\tilde\Omega}\mathrm{d}t'\\
&\le \varepsilon\|\tilde u\|^2_{2+\alpha,\tilde\Omega^t} + c(\varepsilon)t^{1-\alpha}\|u\|^4_{2+\alpha,\tilde\Omega^t}.
\end{aligned}\tag{3.27}$$

Next, we get

$$
\begin{aligned}
|I_6| &\le \varepsilon\|\tilde u\|^2_{2+\alpha,\tilde\Omega^t}+c(\varepsilon)\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\mathrm{d}\xi(|\Delta_{t'}u|^2+|\Delta_{t'}\eta_\sigma|^2\\
&\quad+|\Delta_{t'}\vartheta_\sigma|^2+|\Delta_t u_\xi|^2)+c(\varepsilon)\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\mathrm{d}\xi|\int_{t'}^{t'+h}u_\xi\mathrm{d}\tau|^2(|u|^2+|\eta_\sigma|^2+|\vartheta_\sigma|^2\\
&\quad+|u_\xi|^2+|\tilde u_{\xi\xi}|^2+|\tilde\eta_{\sigma\xi}|^2+|\tilde\vartheta_{\sigma\xi}|^2)+c(\varepsilon)\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\mathrm{d}\xi|\int_0^{t'}u_\xi\mathrm{d}\tau|^2(|\Delta_t\tilde u_{\xi\xi}|^2\\
&\quad+|\Delta_t\tilde\eta_{\sigma\xi}|^2+|\Delta_t\tilde\vartheta_{\sigma\xi}|^2)+c(\varepsilon)\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\mathrm{d}\xi|\int_{t'}^{t'+h}u_\xi\mathrm{d}\tau|^2|\int_0^t u_{\xi\xi}\mathrm{d}\tau|^2(|\tilde u_\xi|^2\\
&\quad+|\tilde\eta_\sigma|^2+|\tilde\vartheta_\sigma|^2)+c(\varepsilon)\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\mathrm{d}\xi|\int_{t'}^{t'+h}u_{\xi\xi}\mathrm{d}\tau|^2(|\tilde u_\xi|^2\\
&\quad+|\tilde\eta_\sigma|^2+|\tilde\vartheta_\sigma|^2)+c(\varepsilon)\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\mathrm{d}\xi|\int_0^{t'}u_{\xi\xi}\mathrm{d}\tau|^2(|\Delta_t\tilde u_\xi|^2\\
&\quad+|\Delta_t\tilde\eta_\sigma|^2+|\Delta_t\tilde\vartheta_\sigma|^2)\equiv\varepsilon\|\tilde u\|^2_{2+\alpha,\tilde\Omega^t}+\sum_{i=1}^6 I_6^i.
\end{aligned}
\tag{3.28}
$$

Let us estimate the terms on the r.h.s. of (3.28). We obtain

$$
\begin{aligned}
|I_6^1| &\le c\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\mathrm{d}\xi(|\int_{t'}^{t'+h}u_\tau\mathrm{d}\tau|^2+|\int_{t'}^{t'+h}\vartheta_{\sigma\tau}\mathrm{d}\tau|^2)\\
&\le ct^{1-\alpha}(\|u_{t'}\|^2_{0,\tilde\Omega^t}+\|\vartheta_{\sigma t'}\|^2_{0,\tilde\Omega^t}).
\end{aligned}
\tag{3.29}
$$

From equations $(1.6)_1$ and $(1.6)_3$ we get respectively

$$
\begin{aligned}
&\|u_{t'}\|^2_{0,\Omega}\le\varepsilon_1^{1-\varkappa}(\|u\|^2_{2+\alpha,\Omega}+\|\eta_\sigma\|^2_{1+\alpha,\Omega}+\|\vartheta_\sigma\|^2_{1+\alpha,\Omega})\\
&+c\varepsilon_1^{-\varkappa}(\|u\|^2_{0,\Omega}+\|\eta_\sigma\|)^2_{0,\Omega}+\|\vartheta_\sigma\|^2_{0,\Omega})+\\
&+c\|u\|^2_{2+\alpha,\Omega^t}\|u\|^2_{1+\alpha,\Omega}
\end{aligned}
\tag{3.30}
$$

and

$$
\begin{aligned}
&\|\vartheta_{\sigma t'}\|^2_{0,\Omega}\le\varepsilon_1^{1-\varkappa}(\|u\|^2_{1+\alpha,\Omega}+\|\vartheta_\sigma\|^2_{2+\alpha,\Omega})\\
&+c\varepsilon_1^{-\varkappa}(\|u\|^2_{0,\Omega}+\|\vartheta_\sigma\|^2_{0,\Omega})\\
&+c(\|k\|^2_{0,\Omega}+\|u\|^2_{2+\alpha,\Omega}\|u\|^2_{1,\Omega}).
\end{aligned}
\tag{3.31}
$$

Using (3.30)–(3.31) in (3.29) and assuming $\varepsilon_1=\big(\frac{\varepsilon}{ct^{1-\alpha}}\big)^{\frac{1}{1-\varkappa}}$ yields

$$
\begin{aligned}
|I_6^1| &\le\varepsilon(\|u\|^2_{2+\alpha,\Omega^t}+\|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t}+\|\eta_\sigma\|^2_{1+\alpha,\Omega^t})\\
&\quad+c(\varepsilon)t^{(1-\alpha)/(1-\varkappa)}(\|u\|^2_{0,\Omega^t}+\|\eta_\sigma\|^2_{0,\Omega^t}+\|\vartheta_\sigma\|^2_{0,\Omega^t}+\|k\|^2_{0,\Omega^t}\\
&\quad+\|u\|^2_{2+\alpha,\Omega^t}\sup_{0\le t'\le t}\|u\|^2_{1+\alpha,\Omega}),
\end{aligned}
$$

where $\varepsilon\in(0,1)$ is sufficiently small.

Next, we have

$$
\begin{aligned}
|I_6^2|&\le ct^{1-\alpha}|\int_0^t|u|^2_{\infty,\tilde\Omega}\mathrm{d}t'(\int_0^t|u|^2_{2,\tilde\Omega}\mathrm{d}t'+\int_0^t|\eta_\sigma|^2_{2,\tilde\Omega}\mathrm{d}t'+\int_0^t|\vartheta_\sigma|^2_{2,\tilde\Omega}\mathrm{d}t'\\
&+\int_0^t|u_\xi|^2_{2,\tilde\Omega}\mathrm{d}t'+\int_0^t|\tilde u_{\xi\xi}|^2_{2,\tilde\Omega}\mathrm{d}t'+\int_0^t|\tilde\eta_{\sigma\xi}|^2_{2,\tilde\Omega}\mathrm{d}t'\\
&+\int_0^t|\tilde\vartheta_{\sigma\xi}|^2_{2,\tilde\Omega}\mathrm{d}t')\le ct^{1-\alpha}|\|u\|^2_{2+\alpha,\tilde\Omega^t}(\|u\|^2_{2,\tilde\Omega^t}\\
&+\|\eta_\sigma\|^2_{1,\tilde\Omega^t}+\|\vartheta_\sigma\|^2_{1,\tilde\Omega^t}).
\end{aligned}
$$

I_6^3 we estimate in the following way:

$$
\begin{aligned}
|I_6^3|&\le ct\int_0^t|u_\xi|^2_{\infty,\tilde\Omega}\mathrm{d}t'\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'(|\Delta_t\tilde u_{\xi\xi}|^2_{2,\tilde\Omega}\\
&+|\Delta_t\tilde\eta_{\sigma\xi}|^2_{2,\tilde\Omega}+|\Delta_t\tilde\vartheta_{\sigma\xi}|^2_{2,\tilde\Omega})\le ct\|u\|^2_{2+\alpha,\tilde\Omega^t}(\|\tilde u\|^2_{2+\alpha,\Omega^t}\\
&+\|\tilde\eta_\sigma\|^2_{1+\alpha,\tilde\Omega^t}+\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\Omega^t}).
\end{aligned}
$$

Next, we get

$$
\begin{aligned}
|I_6^4|&\le ct^{2-\alpha}\int_0^t|u_\xi|^2_{\infty,\tilde\Omega}\mathrm{d}t'\int_0^t|u_{\xi\xi}|^2_{2,\tilde\Omega}\mathrm{d}t'(\int_0^t|\tilde u_\xi|^2_{\infty,\tilde\Omega}\mathrm{d}t'\\
&+\int_0^t|\tilde\eta_\sigma|^2_{\infty,\tilde\Omega}\mathrm{d}t'+\int_0^t|\tilde\vartheta_\sigma|^2_{\infty,\tilde\Omega}\mathrm{d}t')\\
&\le ct^{2-\alpha}\|u\|^4_{2+\alpha,\tilde\Omega^t}(\|\tilde u\|^2_{2+\alpha,\tilde\Omega^t}+\|\tilde\eta_\sigma\|^2_{1+\alpha,\tilde\Omega^t}\\
&+\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t}).
\end{aligned}
$$

Estimating I_6^5 in the same way we obtain

$$
|I_6^5|\le ct^{1-\alpha}\|u\|^2_{2+\alpha,\tilde\Omega^t}(\|\tilde u\|^2_{2+\alpha,\tilde\Omega^t}+\|\tilde\eta_\sigma\|^2_{1+\alpha,\tilde\Omega^t}+\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t}).
$$

Finally, we have

$$
\begin{aligned}
|I_6^6|&\le ct\int_0^t|u_{\xi\xi}|^2_{4,\tilde\Omega}\mathrm{d}t'\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'(|\Delta_t\tilde u_\xi|^2_{4,\tilde\Omega}\\
&+|\Delta_t\tilde\eta_\sigma|^2_{4,\tilde\Omega}+|\Delta_t\tilde\vartheta_\sigma|^2_{4,\tilde\Omega})\\
&\le ct\|u\|^2_{2+\alpha,\tilde\Omega^t}\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t[\varepsilon_1(|\Delta_t\tilde u_{\xi\xi}|^2_{2,\tilde\Omega}\\
&+|\Delta_t\tilde\eta_{\sigma\xi}|^2_{2,\tilde\Omega}+|\Delta_t\tilde\vartheta_{\sigma\xi}|^2_{2,\tilde\Omega})+c(\varepsilon_1)(|\Delta_t\tilde u|^2_{2,\tilde\Omega}\\
&+|\Delta_t\tilde\eta_\sigma|^2_{2,\tilde\Omega}+|\Delta_t\tilde\vartheta_\sigma|^2_{2,\tilde\Omega})]\\
&\le ct\|u\|^2_{2+\alpha,\tilde\Omega^t}(\|\tilde u\|^2_{2+\alpha,\tilde\Omega^t}+\|\tilde\eta_\sigma\|^2_{1+\alpha,\tilde\Omega^t}+\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t}).
\end{aligned}
$$

Summarizing the above considerations we get

$$
\begin{aligned}
|I_6| &\le \varepsilon(\|u\|^2_{2+\alpha,\Omega^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t}) \\
&\quad + c(\varepsilon,t)[\|u\|^2_{0,\Omega^t} + \|\vartheta_\sigma\|^2_{0,\Omega^t} + \|\eta_\sigma\|^2_{0,\Omega^t} + \|k\|^2_{0,\Omega^t} \\
&\quad + \|u\|^2_{2+\alpha,\Omega^t} \sup_{0\le t'\le t} \|u\|^2_{1+\alpha,\Omega} + \|u\|^2_{2+\alpha,\tilde\Omega^t}(\|u\|^2_{2+\alpha,\tilde\Omega^t} \\
&\quad + \|\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t})(1 + \|u\|^2_{1+\alpha,\tilde\Omega^t})],
\end{aligned}
\tag{3.32}
$$

where $c(\varepsilon,t)$ is a positive continuous function increasing with respect to t.

In the same way we estimate I_7 and we get

$$
\begin{aligned}
|I_7| &\le \varepsilon(\|u\|^2_{2+\alpha,\Omega^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t}) + \mathrm{d}\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t} + c(\varepsilon,t)[\|u\|^2_{0,\Omega^t} \\
&\quad + \|\vartheta_\sigma\|^2_{0,\Omega^t} + \|k\|^2_{0,\Omega^t} + \sup_{0\le t'\le t}\|u\|^2_{1+\alpha,\tilde\Omega}(\|\vartheta_\sigma\|^2_{\alpha,\tilde\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\tilde\Omega^t}) \\
&\quad + \sup_{0\le t'\le t}\|u\|^2_{1+\alpha,\Omega^t}\|u\|^2_{2+\alpha,\tilde\Omega^t} + \|u\|^2_{2+\alpha,\tilde\Omega^t}\cdot(\|u\|^2_{2+\alpha,\tilde\Omega^t} + \|\tilde\vartheta_\sigma\|^2_{2,\tilde\Omega^t} + \|\tilde\eta_\sigma\|^2_{1,\tilde\Omega^t}) \\
&\quad + \|\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t}(\|\tilde\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega^t} + \|\tilde\eta_\sigma\|^2_{1+\alpha,\tilde\Omega^t})],
\end{aligned}
\tag{3.33}
$$

where d is given in (2.4).

Thus, taking into account (3.26), (3.27), (3.32) and (3.33) we get

$$
\begin{aligned}
&\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\tilde\Omega}\eta|\Delta_{t'}\tilde u_{t'}|^2\mathrm{d}\xi \\
&+\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_{\tilde\Omega}\left(\frac{\mu}{4}|\mathbb{S}(\Delta_{t'}\tilde u)|^2 + \frac{\nu-\mu}{2}|\mathrm{div}\,\Delta_{t'}\tilde u|^2\right)\mathrm{d}\xi\Big|_{t'=t-h_0} \\
&\le \int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_{\tilde\Omega}\left(\frac{\mu}{4}|\mathbb{S}(\Delta_{t'}\tilde u)|^2 + \frac{\nu-\mu}{2}|\mathrm{div}\,\Delta_{t'}\tilde u|^2\right)\mathrm{d}\xi\Big|_{t'=0} \\
&+ \varepsilon(\|u\|^2_{2+\alpha,\Omega^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t}) + \mathrm{d}\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t} \\
&+ c(\varepsilon,t)[\|u\|^2_{0,\Omega^t} + \|\vartheta_\sigma\|^2_{0,\Omega^t} + \|\eta_\sigma\|^2_{0,\Omega^t} + \|k\|^2_{0,\Omega^t} \\
&+ \|u\|^2_{2+\alpha,\Omega^t}\sup_{0\le t'\le t}\|u\|^2_{1+\alpha,\Omega} + \|u\|^2_{2+\alpha,\tilde\Omega^t}(\|u\|^2_{2+\alpha,\tilde\Omega^t} \\
&+ \|\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t})(1 + \|u\|^2_{1+\alpha,\tilde\Omega^t}) + \\
&+ \sup_{0\le t'\le t}\|u\|^2_{1+\alpha,\tilde\Omega}(\|\vartheta_\sigma\|^2_{\alpha,\tilde\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\tilde\Omega^t}) + \|\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t}\cdot \\
&\cdot(\|\tilde\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega^t} + \|\tilde\eta_\sigma\|^2_{1+\alpha,\tilde\Omega^t})],
\end{aligned}
\tag{3.34}
$$

where $c(\varepsilon,t)$ is positive continuous function increasing with respect to t

Now, we obtain an estimate for $\int_0^t\|\tilde\vartheta_{\sigma t'}\|^2_{\alpha,\tilde\Omega}\mathrm{d}t'$. In order to do this we use (3.24) and to estimate $\int_{\tilde\Omega}\|\tilde\vartheta_{\sigma t'}\|^2_{\frac{\alpha}{2},(0,t)}\mathrm{d}\xi$ we apply $\Delta_{t'}(h)$ to $(2.5)_3$ and then multiply the equation by $\Delta_{t'}\tilde\vartheta_{\sigma t'}$. We get

$$
\begin{aligned}
&\int\limits_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int\limits_0^{t-h_0}\mathrm{d}t'\int\limits_{\tilde\Omega}\eta c_v(\eta,\vartheta)|\Delta_{t'}\tilde\vartheta_\sigma|^2\mathrm{d}\xi\\
&+\frac{\varkappa}{2}\int\limits_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int\limits_{\tilde\Omega}(|\nabla(\Delta_{t'}\tilde\vartheta_\sigma)|^2)\big|_{t'=t-h_0}\mathrm{d}\xi\\
&\le\frac{\varkappa}{2}\int\limits_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int\limits_{\tilde\Omega}(|\nabla(\Delta_{t'}\tilde\vartheta_\sigma)|^2)\big|_{t'=0}\mathrm{d}\xi\\
&+\varepsilon(\|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t}+\|u\|^2_{2+\alpha,\Omega^t}+\|\eta_\sigma\|^2_{1+\alpha,\Omega^t})\\
&+\mathrm{d}\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t}+c(\varepsilon,t)[\|u\|^2_{0,\Omega^t}+\|\vartheta_\sigma\|^2_{0,\Omega^t}\\
&+\|\eta_\sigma\|^2_{0,\Omega^t}+\|k\|^2_{\alpha,\Omega^t}+\|\bar\vartheta\|^2_{\alpha+1/2,S^t}+\|u\|^2_{2+\alpha,\tilde\Omega^t}\cdot\\
&\cdot(\|u\|^2_{2+\alpha,\tilde\Omega^t}+\sup_{0\le t'\le t}\|\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega}+\sup_{0\le t'\le t}\|\eta_\sigma\|^2_{1+\alpha,\tilde\Omega}\\
&+\sup_{0\le t'\le t}\|u\|^2_{1+\alpha,\tilde\Omega}+\|u\|^2_{1+\alpha,\tilde\Omega^t}\cdot\sup_{0\le t'\le t}\|u\|^2_{1+\alpha,\tilde\Omega}+\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t}\\
&+\|u\|^2_{2+\alpha,\tilde\Omega^t}\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\Omega^t})+\|u\|^2_{2+\alpha,\Omega^t}\cdot\|u\|^2_{1+\alpha,\Omega^t}\\
&+\sup_{0\le t'\le t}\|\eta\|^2_{1+\alpha,\tilde\Omega}\|\tilde k\|^2_{\alpha,\tilde\Omega^t}+\|\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t}\cdot\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t}],
\end{aligned}
\tag{3.35}
$$

where $c(\varepsilon,t)$ is a positive continuous function increasing with respect to t.

Let us introduce the quantity

$$
\begin{aligned}
\phi_1(t,\tilde\Omega)&=\int\limits_{\tilde\Omega}\left(\eta\tilde u^2+\frac{p_1}{\varrho_e}\tilde\eta_\sigma^2+\frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\tilde\vartheta_\sigma^2\right)\mathrm{d}\xi\\
&+\int\limits_{\mathbb{R}^3}\mathrm{d}z\int\limits_{B_{2\lambda}(\xi_0)}\mathrm{d}\xi\Big(\frac{\eta|\Delta^2(z)\tilde u|^2}{|z|^{3+2(1+\alpha)}}+\frac{p_1|\Delta^2(z)\tilde\eta_\sigma|^2}{\varrho_e|z|^{3+2(1+\alpha)}}+\frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(z)\tilde\vartheta_\sigma|^2}{|z|^{3+2(1+\alpha)}}\Big)\\
&+\int\limits_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int\limits_{\tilde\Omega}\left(\frac{\mu}{4}|\mathbb{S}(\Delta_{t'}\tilde u)|^2+\frac{\nu-\mu}{2}|\mathrm{div}\,\Delta_{t'}\tilde u|^2\right)\big|_{t'=t-h_0}\mathrm{d}\xi\\
&+\frac{\varkappa}{2}\int\limits_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int\limits_{\tilde\Omega}(|\nabla(\Delta_{t'}\tilde\vartheta_\sigma)|^2)\big|_{t'=t-h_0}\mathrm{d}\xi,
\end{aligned}
\tag{3.36}
$$

which is such that

$$
\begin{aligned}
&\bar c_1(\|\tilde u\|^2_{1+\alpha,\tilde\Omega}+\|\tilde\eta_\sigma\|^2_{1+\alpha,\tilde\Omega}+\|\tilde\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega})\le\\
&\le\phi_1(t,\tilde\Omega)\le\bar c_2(\|\tilde u\|^2_{1+\alpha,\tilde\Omega}+\|\tilde\eta_\sigma\|^2_{1+\alpha,\tilde\Omega}+\|\tilde\vartheta_\sigma\|^2_{1+\alpha,\tilde\Omega}),
\end{aligned}
$$

where $\bar c_1,\bar c_2$ depend on the bound from the estimate for the local solution.

Integrating (3.23) with respect to time, using (3.24) (for $\tilde u$ and $\tilde\vartheta_\sigma$) (3.34) and (3.35) we obtain for $t\le T$.

$$
\begin{aligned}
&\phi_1(t,\tilde\Omega)+\bar c(\|\tilde u\|^2_{2+\alpha,\tilde\Omega^t}+\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t}+\|\tilde\eta_\sigma\|^2_{1+\alpha,\tilde\Omega^t})\\
&\le\varepsilon(\|u\|^2_{2+\alpha,\tilde\Omega^t}+\|\vartheta_\sigma\|^2_{2+\alpha,\tilde\Omega^t}+\|\eta_\sigma\|^2_{1+\alpha,\tilde\Omega^t})\\
&+c(\varepsilon,t)\{\|u\|^2_{0,\tilde\Omega^t}+\|\vartheta_\sigma\|^2_{0,\tilde\Omega^t}+\|\eta_\sigma\|^2_{0,\tilde\Omega^t}\\
&+(1+\sup_{t'}\|\eta\|^2_{1+\alpha,\Omega})\|k\|^2_{\alpha,\tilde\Omega^t}+\|\bar\vartheta\|^2_{1/2+\alpha,S^t}\\
&+X_3[X_4(1+X_4)+(1+X_4^2)X_3]\}+\phi_1(0,\tilde\Omega),
\end{aligned}
\tag{3.37}
$$

where

$$X_3 = \|u\|^2_{2+\alpha,\Omega^t} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t}$$

and

$$X_4 = \sup_{t'\le t}\|u\|^2_{1+\alpha,\Omega} + \sup_{t'\le t}\|\eta_\sigma\|^2_{1+\alpha,\Omega} + \sup_{t'\le t}\|\vartheta_\sigma\|^2_{1+\alpha,\Omega}, \tag{3.39}$$

T is the time of the local existence, $c(\varepsilon,t)$ is a positive continous function increasing with respect to t.

Now, we shall derive an estimate in a boundary subdomain. We apply $\Delta^2(\tau)$ (where $\Delta^s(\tau)f(z) = \sum_{k=0}^s\binom{s}{k}(-1)^{s-k}f(z'+k\tau,z_3)$, $z'=(z_1,z_2)$, $\tau=(\tau_1,\tau_2)$) to $(2.6)_1$–$(2.6)_3$ and multiply the resulting equations by $\Delta^2(\tau)\tilde u_i$, $\frac{p_1}{\varrho_e}\Delta^2(\tau)\tilde\eta_\sigma$ and $\frac{p_2}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\Delta^2(\tau)\tilde\vartheta_\sigma$, respectively. Then integrating each of the equations over $\hat\Omega(\xi_0)$ and with respect to τ over $\mathbb{R}^2$ with the measure $\frac{1}{|\tau|^{2+2(1+\alpha)}}$ and using (1.5), the Korn inequality and the equation of continuity in $\hat\Omega$, i.e. $\hat\eta_t + \hat\eta\hat\nabla\cdot\hat u = 0$ we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\mathrm{d}\tau\int_{\hat\Omega(\xi_0)}\mathrm{d}z\Big(\frac{\hat\eta|\Delta^2(\tau)\tilde u|^2}{|\tau|^{2+2(1+\alpha)}} + \frac{p_1}{\varrho_e}\frac{|\Delta^2(\tau)\tilde\eta_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}\\
&+\frac{p_2\hat\eta c_v}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(\tau)\tilde\vartheta_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}\Big) + c_5\int_{\mathbb{R}^2}\mathrm{d}\tau\frac{\|\Delta^2(\tau)\tilde u\|^2_{1,\hat\Omega(\xi_0)}}{|\tau|^{2+2(1+\alpha)}}\\
&+\varkappa\int_{\mathbb{R}^2}\mathrm{d}\tau\int_{\hat\Omega(\xi_0)}\mathrm{d}z\frac{p_2|\Delta^2(\tau)\nabla\tilde\vartheta_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}+\\
&-\int_{\mathbb{R}^2}\mathrm{d}\tau\int_{\hat S(\xi_0)}\mathrm{d}z_{\hat s}\frac{\mathbb{T}(\Delta^2(\tau)\tilde u,\Delta^2(\tau)\tilde p_\sigma)\hat n'\cdot\Delta^2(\tau)\tilde u}{|\tau|^{2+2(1+\alpha)}}\\
&-\int_{\mathbb{R}^2}\mathrm{d}\tau\int_{\hat S(\xi_0)}\mathrm{d}z_{\hat s}\frac{\hat n'\cdot\nabla(\Delta^2(\tau)\tilde\vartheta_\sigma)\cdot\Delta^2(\tau)\tilde\vartheta_\sigma}{|\tau|^{2+2(1+\alpha)}}\le\\
&\le\varepsilon(\|\hat u\|^2_{2+\alpha,\hat\Omega} + \|\hat\eta_\sigma\|^2_{1+\alpha,\hat\Omega} + \|\hat\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega})\\
&+\psi_{18}\left(\frac{1}{\varepsilon},a\right)[X_1X_2 + X_1(1+X_2^2)\int_0^t\|u\|^2_{2+\alpha,\Omega}\mathrm{d}t'\\
&+\|\hat\eta_\sigma\|^2_{1+\alpha,\Omega}\|\tilde u_t\|^2_{\alpha,\hat\Omega}] + \psi_{19}\left(\frac{1}{\varepsilon},a\right)(\|\tilde u\|^2_{1+\alpha,\hat\Omega} + \|\tilde\eta_\sigma\|^2_{\alpha,\hat\Omega}\\
&+\|\tilde\vartheta_\sigma\|^2_{\alpha,\hat\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\hat\Omega}) + \psi_{20}\left(\frac{1}{\varepsilon},a\right)\|\tilde k\|^2_{\alpha,\hat\Omega},
\end{aligned}\tag{3.40}$$

where X_1 and X_2 are given by (3.17) and (3.18) respectively.

Using boundary condition $(2.6)_4$, definition (2.4) and imbedding (2.10) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}\mathrm{d}\tau\int_{\hat S(\xi_0)}\mathrm{d}z_{\hat S}\frac{\mathbb{T}(\Delta^2(\tau)\tilde u,\Delta^2(\tau)\tilde p_\sigma)\hat n'\cdot\Delta^2(\tau)\tilde u}{|\tau|^{2+2(1+\alpha)}}\\
&=-\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\mathrm{d}\tau\int_{\hat S(\xi_0)}\mathrm{d}z_{\hat S}\frac{g_{\hat u}^{\gamma\delta}\int_0^t\Delta^2(\tau)\frac{\partial\tilde u}{\partial z_\gamma}\mathrm{d}t'\cdot\int_0^t\Delta^2(\tau)\frac{\partial\tilde u}{\partial z_\delta}\mathrm{d}t'}{|\tau|^{2+2(1+\alpha)}}\\
&+\sigma\int_{\mathbb{R}^2}\mathrm{d}\tau\int_{\hat S(\xi_0)}\mathrm{d}z_{\hat S}\frac{\Delta^2(\tau)\left(\left(\Delta_{\hat S}\hat\xi + \frac{2}{R_e}\hat n\right)\hat\zeta\right)\cdot\Delta^2(\tau)\tilde u}{|\tau|^{2+2(1+\alpha)}}\\
&+I_8\equiv J_1+J_2+I_8.
\end{aligned}\tag{3.41}$$

where

$$|I_8| \le \psi_{21}(a, \|F\|_{\frac{5}{2}+\alpha,\hat{S}})[(\varepsilon + d)\|\tilde{u}\|^2_{2+\alpha,\hat{\Omega}} + \varepsilon(\|\tilde{\eta}_\sigma\|^2_{1+\alpha,\hat{\Omega}}$$

$$+ \|\tilde{\vartheta}_\sigma\|^2_{1+\alpha,\hat{\Omega}}) + \varepsilon_1 \Big\| \int_0^t \hat{u}\mathrm{d}t' \Big\|^2_{2+\alpha,\hat{S}} + \varepsilon_1 \Big\| \int_0^t \hat{u}\mathrm{d}t' \Big\|^4_{2+\alpha,\hat{S}}$$

$$+ \|\tilde{u}\|^2_{2+\alpha,\hat{\Omega}}\Big(\int_0^t \|\hat{u}\|^2_{2+\alpha,\hat{\Omega}}\mathrm{d}t' + \Big\| \int_0^t \hat{u}\mathrm{d}t' \Big\|^2_{2+\alpha,\hat{S}}\Big) \tag{3.42}$$

$$+ \int_0^t \|\tilde{u}\|^2_{2+\alpha,\hat{\Omega}}\mathrm{d}t' \Big\| \int_0^t \hat{u}\mathrm{d}t' \Big\|^2_{2+\alpha,\hat{\Omega}} \Big(1 + \Big\| \int_0^t \hat{u}\mathrm{d}t' \Big\|^2_{2+\alpha,\hat{S}}\Big) + \|\tilde{u}\|^2_{0,\hat{\Omega}}],$$

where ε_1 and ε are sufficiently small positive constants, d is given in (2.4).

Now, we estimate the second term on the right-hand side of (3.41). We have

$$J_2 = \sigma \int_{\mathbb{R}^2} \mathrm{d}\tau \int_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \frac{\Delta^2(\tau)\left(\left(H(\cdot,0) + \frac{2}{R_e}\right)\hat{n}_0\hat{\zeta}\right) \cdot \Delta^2(\tau)\tilde{u}}{|\tau|^{2+2(1+\alpha)}}$$

$$+ \frac{2\sigma}{R_e} \int_{\mathbb{R}^2} \mathrm{d}\tau \int_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \frac{\Delta^2(\tau)(\hat{n} - \hat{n}_0) \cdot \Delta^2(\tau)\tilde{u}}{|\tau|^{2+2(1+\alpha)}} \equiv J_2^1 + J_2^2, \tag{3.43}$$

where $\hat{n}_0(z') = \hat{n}(z'_1 0)$ and

$$|J_2^2| \le c \int_{\mathbb{R}^2} \mathrm{d}\tau \int_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \frac{|\Delta^2(\tau) \int_0^t \hat{u}_{z'}\mathrm{d}t'||\Delta^2(\tau)\tilde{u}|}{|\tau|^{2+2(1+\alpha)}}$$

$$\le \varepsilon_1 \Big\| \int_0^t \hat{u}\mathrm{d}t' \Big\|^2_{2+\alpha,\hat{S}} + c(\varepsilon_1)(\varepsilon\|\tilde{u}\|^2_{2+\alpha,\hat{\Omega}} + c(\varepsilon)\|\tilde{u}\|^2_{0,\hat{\Omega}}). \tag{3.44}$$

Next, we obtain

$$J_2^1 = -\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \mathrm{d}\tau \int_{\hat{S}(\xi_0)} \frac{2\Delta(\tau)\left(\left(H(\cdot,0) + \frac{2}{R_e}\right)\hat{n}_0\hat{\zeta}\right) \cdot \int_0^t \Delta^3(\tau)\tilde{u}\mathrm{d}t'}{|\tau|^{2+2+(1+\alpha)}}$$

$$= -\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \mathrm{d}\tau \int_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \Big(4\sqrt{\hat{c}} \frac{\Delta(\tau)\left(\left(H(\cdot,0) + \frac{2}{R_e}\right)\hat{n}_0\hat{\zeta}\right)}{|\tau|^{1+\alpha}} \tag{3.45}$$

$$+ \frac{1}{4\sqrt{\hat{c}}} \frac{\int_0^t \Delta^3(\tau)\tilde{u}\mathrm{d}t'}{|\tau|^{1+2+\alpha}}\Big)^2 + \frac{\sigma}{32\bar{c}}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \mathrm{d}\tau \int_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \frac{(\int_0^t \Delta^3(\tau)\tilde{u}\mathrm{d}t')^2}{|\tau|^{2+2(2+\alpha)}},$$

where $\hat{c} > 0$ is a constant such that

$$\int_{\mathbb{R}^2} \mathrm{d}\tau \int_{\hat{S}(\xi)} \mathrm{d}z_{\hat{S}} \frac{(\int_0^t \Delta^3(\tau)\tilde{u}\mathrm{d}t')^2}{|\tau|^{2+2(2+\alpha)}} \le \hat{c} \int_{\mathbb{R}^2} \mathrm{d}\tau \int_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \frac{\sum_{\gamma=1}^2 (\int_0^t \Delta^2(\tau)\tilde{u}_{z_\gamma}\mathrm{d}t')^2}{|\tau|^{2+2(1+\alpha)}} \tag{3.46}$$

(see [3], [4]).

By (3.41)–(3.45) we get

$$
\begin{aligned}
&-\int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \frac{\mathbb{T}(\Delta^2(\tau)\tilde{u}, \Delta^2(\tau)\tilde{p}_\sigma)\hat{n}' \cdot \Delta^2(\tau)\tilde{u}}{|\tau|^{2+2(1+\alpha)}} \\
&= \frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \frac{g_{\hat{u}}^{\gamma\delta} \int_0^t \Delta^2(\tau)\tilde{u}_{z_\gamma}\mathrm{d}t' \cdot \int_0^t \Delta^2(\tau)\tilde{u}_{z_\delta}\mathrm{d}t'}{|\tau|^{2+2(1+\alpha)}} \\
&+ \frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \Big(4\sqrt{\hat{c}}\frac{\Delta(\tau)\left(\left(H(\cdot,0)+\frac{2}{R_e}\right)\hat{n}_0\hat{\zeta}\right)}{|\tau|^{1+\alpha}} \\
&+ \frac{1}{4\sqrt{\hat{c}}}\frac{\int_0^t \Delta^3(\tau)\tilde{u}\mathrm{d}t'}{|\tau|^{1+2+\alpha}}\Big)^2 - \frac{\sigma}{32\hat{c}}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \frac{(\int_0^t \Delta^3(\tau)\tilde{u}\mathrm{d}t')^2}{|\tau|^{2+2(2+\alpha)}} \\
&+ I_8 + J_2^2,
\end{aligned}
\tag{3.47}
$$

where estimates for I_8 and J_2^2 are given by (3.42) and (3.44), respectively.

Next, using boundary condition $(2.6)_5$ we have

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \frac{\hat{n}' \cdot \nabla(\Delta^2(\tau)\tilde{\vartheta}_\sigma) \cdot \Delta^2(\tau)\tilde{\vartheta}_\sigma}{|\tau|^{2+2(1+\alpha)}} \le \psi_{22}(a, \|F\|_{2+\alpha,\hat{S}})[(\varepsilon+d)\|\hat{\vartheta}_\sigma\|^2_{2+\alpha,\hat{\Omega}} + \|\tilde{\bar{\vartheta}}\|_{\alpha+\frac{1}{2},\hat{S}} \\
&+ \|\hat{\vartheta}_\sigma\|^2_{0,\hat{\Omega}} + \|\hat{\vartheta}_\sigma\|^2_{0,\hat{\Omega}} \Big\| \int\limits_0^t \hat{u}\mathrm{d}t' \Big\|^2_{2+\alpha,\hat{S}}].
\end{aligned}
\tag{3.48}
$$

Taking into account (3.40), (3.42), (3.46), (3.44) and the considerations for interior subdomains we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\phi_2(t,\hat{\Omega}) + \Phi_1(t,\hat{\Omega}) \le cX,
\tag{3.49}
$$

where

$$
\begin{aligned}
\phi_2(t,\hat{\Omega}) &= \frac{1}{2}\int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat{\Omega}(\xi_0)} \mathrm{d}z \Big(\frac{\hat{\eta}|\Delta^2(\tau)\tilde{u}|^2}{|\tau|^{2+2(1+\alpha)}} + \frac{p_1}{\varrho_e}\frac{|\Delta^2(\tau)\tilde{\eta}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}} \\
&+ \frac{p_2\hat{\eta}c_v(\hat{\eta},\hat{\vartheta})}{\theta_e p_{\hat{\vartheta}}(\varrho_e,\theta_e)}\frac{|\Delta^2(\tau)\tilde{\vartheta}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}\Big) + \frac{1}{2}\int\limits_{\hat{\Omega}(\xi_0)} \left(\hat{\eta}\tilde{u}^2 + \frac{p_1}{\varrho_e}\tilde{\eta}_\sigma^2 + \frac{p_2\hat{\eta}c_v(\hat{\eta},\hat{\vartheta})}{\theta_e p_{\hat{\vartheta}}(\varrho_e,\theta_e)}\tilde{\vartheta}_\sigma^2\right)\mathrm{d}z \\
&+ \frac{\sigma}{2}\int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \frac{g_{\hat{u}}^{\gamma\delta} \int_0^t \Delta^2(\tau)\frac{\partial\tilde{u}}{\partial z_\gamma}\mathrm{d}t' \cdot \int_0^t \Delta^2(\tau)\frac{\partial\tilde{u}}{\partial z_\delta}\mathrm{d}t'}{|\tau|^{2+2(1+\alpha)}}, \\
&\frac{\sigma}{2}\int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \left(2\sqrt{\hat{c}}\frac{\Delta(\tau)\left(\left(H(\cdot,0)+\frac{2}{R_e}\right)\hat{n}_0\hat{\zeta}\right)}{|\tau|^{1+\alpha}} + \frac{1}{2\sqrt{\hat{c}}}\frac{\int_0^t \Delta^3(\tau)\tilde{u}\mathrm{d}t'}{|\tau|^{1+2+\alpha}}\right)^2 \\
&- \frac{\sigma}{8\bar{c}}\int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat{S}(\xi_0)} \mathrm{d}z_{\hat{S}} \frac{(\int_0^t \Delta^3(\tau')\tilde{u}\mathrm{d}t')^2}{|\tau|^{2+2(2+\alpha)}},
\end{aligned}
\tag{3.50}
$$

$$
\begin{aligned}
\Phi_1(t,\hat{\Omega}) &= c\Big(\|\tilde{u}\|^2_{1,\hat{\Omega}} + \|\tilde{\vartheta}_{\sigma x}\|^2_{0,\hat{\Omega}} + \\
&+ \int\limits_{\mathbb{R}^2} \mathrm{d}\tau \frac{\|\Delta^2(\tau)\tilde{u}\|^2_{1,\hat{\Omega}(\xi_0)}}{|\tau|^{2+2(1+\alpha)}} + \int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat{\Omega}(\xi_0)} \mathrm{d}z \frac{p_2|\nabla(\Delta^2(\tau)\tilde{\vartheta}_\sigma)|^2}{|\tau|^{2+2(1+\alpha)}}\Big),
\end{aligned}
\tag{3.51}
$$

$$
(3.52)\qquad \begin{aligned}
X=\psi_{23}(a,\|F\|_{\frac{5}{2}+\alpha,\hat S})\{&\mathrm{d}(\|\tilde u\|^2_{2+\alpha,\hat\Omega}+\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega})+\varepsilon(\|\hat u\|^2_{2+\alpha,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega})\\
&+\varepsilon\|\hat\eta_\sigma\|^2_{1+\alpha,\hat\Omega}+\varepsilon_1\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^2_{2+\alpha,\hat S}+\varepsilon_1\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^4_{2+\alpha,\hat S}+\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^2_{2+\alpha,\hat S}[\|\tilde u\|^2_{2+\alpha,\hat\Omega}\\
&+\int_0^t\|\tilde u\|^2_{2+\alpha,\hat\Omega}\mathrm{d}t'(1+\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^2_{2+\alpha,\hat S})]+A+\|\hat\vartheta_\sigma\|^2_{0,\hat\Omega}\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^2_{2+\alpha,\hat S}\\
&+X_1X_2+X_1(1+X_2^2)\int_0^t\|\hat u\|^2_{2+\alpha,\Omega}\mathrm{d}t'+\|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\tilde u_t\|^2_{\alpha,\tilde\Omega}\},
\end{aligned}
$$

$$
(3.53)\qquad A=\|\hat u\|^2_{0,\hat\Omega}+\|\hat\eta_\sigma\|^2_{0,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{0,\hat\Omega}+\|\hat k\|^2_{\alpha,\hat\Omega}+\|\tilde{\tilde\vartheta}\|^2_{\alpha+\frac12,\hat S},
$$

X_1 and X_2 are given by (3.17) and (3.18), respectively.

Now, repeating the argument from Lemma 4.4 of [15] we obtain the estimate

$$
(3.54)\qquad \begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\mathrm{d}\tau\int_{\hat\Omega(\xi_0)}\mathrm{d}z\left(\hat\eta\frac{|\Delta(\tau)\tilde u_z|^2}{|\tau|^{2+2\alpha}}+\frac{p_1}{\varrho_e}\frac{|\Delta(\tau)\tilde\eta_\sigma|^2}{|\tau|^{2+2\alpha}}\right)\\
&+\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\mathrm{d}\tau\int_{\hat S(\xi_0)}\mathrm{d}z_{\hat S}\frac{g_{\hat u}^{\gamma\delta}\int_0^t\Delta(\tau)\tilde u_{z_\beta z_\gamma}\mathrm{d}t'\cdot\int_0^t\Delta(\tau)\tilde u_{z_\beta z_\delta}\mathrm{d}t'}{|\tau|^{2+2\alpha}}\\
&\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\mathrm{d}\tau\int_{\hat S(\xi_0)}\mathrm{d}z_{\hat S}\left(\frac{4\Delta(\tau)\left(H(\cdot,0)+\frac{2}{R_e}\right)\hat n_0\hat\zeta}{|\tau|^{1+\alpha}}+\frac14\frac{\sum_{\gamma=1}^2\int_0^2\Delta(\tau)\tilde u_{z_\gamma z_\gamma}\mathrm{d}t'}{|\tau|^{1+\alpha}}\right)^2\\
&-\frac{\sigma}{32}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\mathrm{d}\tau\int_{\hat S(\xi_0)}\mathrm{d}z_{\hat S}\frac{(\sum_{\gamma=1}^2\int_0^t\Delta(\tau)\tilde u_{z_\gamma z_\gamma}\mathrm{d}t')^2}{|\tau|^{2+2\alpha}}+\frac{\mu}{2}\int_{\mathbb{R}^2}\frac{\|\Delta(\tau)\tilde u\|^2_{2,\hat\Omega}}{|\tau|^{2+2\alpha}}\mathrm{d}\tau+\int_{\mathbb{R}^2}\frac{\|\Delta(\tau)\tilde\eta_{\sigma z}\|^2_{0,\hat\Omega}}{|\tau|^{2+2\alpha}}\mathrm{d}\tau\\
&\le\varepsilon\Big(\|\int_0^t\tilde u\mathrm{d}t'\|^2_{2+\alpha,\hat S}+\|\hat u\|^2_{2+\alpha,\hat\Omega}+\Big\|H(\cdot,0)+\frac{2}{R_e}\Big\|^2_{\alpha,S^1}+\|R(\cdot,t)-R(\cdot,0)\|^2_{2+\alpha,S^1}\Big)\\
&+c(\|\hat u\|^2_{0,\hat\Omega}+\|\hat\eta_\sigma\|^2_{0,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{0,\hat\Omega}+cX_2(X_1+\int_0^t\|u\|^2_{2+\alpha,\Omega}\mathrm{d}t').
\end{aligned}
$$

Next, boundary condition $(2.6)_4$ yields

$$
(3.55)\qquad \begin{aligned}
&\Big\|\int_0^t\tilde u\mathrm{d}t'\Big\|^2_{2+\alpha,\hat S}\le\int_{\mathbb{R}^2}\frac{\|\Delta(\tau)\int_0^t\tilde u\mathrm{d}t'\|^2_{2,\hat S}}{|\tau|^{2+2\alpha}}\mathrm{d}\tau+\Big\|\int_0^t\tilde u\mathrm{d}t'\Big\|^2_{0,\hat S}\\
&\le\varepsilon(\|\tilde u\|^2_{2+\alpha,\hat\Omega}+\|\hat\eta_\sigma\|^2_{1+\alpha,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{1+\alpha,\hat\Omega})\\
&+c\Big(\|\tilde u\|^2_{0,\hat\Omega}+\|q_\sigma\|^2_{0,\hat\Omega}+\Big\|\int_0^t\hat u\mathrm{d}t'\Big\|^2_{0,\hat\Omega}+\Big\|H(\cdot,0)+\frac{2}{R_e}\Big\|^2_{\alpha,S^1}+\|R(\cdot,t)-R(\cdot,0)\|^2_{2+\alpha,S^1}\Big)\\
&+c(\|\hat u\|^2_{2+\alpha,\hat\Omega}+\|\hat\eta_\sigma\|^2_{1+\alpha,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{1+\alpha,\hat\Omega})\int_0^t\|\hat u\|^2_{2+\alpha,\hat\Omega}\mathrm{d}t'.
\end{aligned}
$$

Now, we consider the problem

$$
\begin{aligned}
\varkappa\Delta\tilde{\vartheta}_\sigma =&\hat{\eta}c_v(\hat{\eta},\hat{\vartheta})\tilde{\vartheta}_{\sigma t}+\theta_e p_{\hat{\vartheta}}(\varrho_e,\theta_e)\nabla\cdot\tilde{u}-\hat{\eta}\tilde{k}-k_6\equiv K_3 \qquad \text{in } \hat{\Omega},\\
\hat{n}\cdot\nabla\tilde{\vartheta}_\sigma =& \tilde{\tilde{\vartheta}}+k_8\equiv K_4 \text{ on } \hat{S}.
\end{aligned}
\tag{3.56}
$$

From Agmon-Douglis-Nirenberg theory (see [1]) we get

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^2}\frac{\|\Delta(\tau)\tilde{\vartheta}_\sigma\|^2_{2,\hat{\Omega}}}{|\tau|^{2+2\alpha}}\mathrm{d}\tau\le c\int\limits_{\mathbb{R}^2}\frac{\|\Delta(\tau)\nabla\cdot\tilde{u}\|_{1,\hat{\Omega}}}{|\tau|^{2+2\alpha}}\mathrm{d}\tau+\\
&+c\int\limits_{\mathbb{R}^2}\frac{\|\Delta(\tau)K_3\|^2_{0,\hat{\Omega}}}{|\tau|^{2+2\alpha}}\mathrm{d}\tau+c\int\limits_{\mathbb{R}^2}\frac{\|\Delta(\tau)K_4\|^2_{\frac{1}{2},\hat{S}}}{|\tau|^{2+2\alpha}}\mathrm{d}\tau\\
&\le c(X+\|\tilde{\vartheta}_{\sigma t}\|^2_{\alpha,\hat{\Omega}})\equiv c\bar{X}.
\end{aligned}
\tag{3.57}
$$

Next, applying $\Delta_3(n)$ to $(3.56)_1$ we obtain the estimate

$$
\int\limits_{\mathbb{R}^1_+}\mathrm{d}n\int\limits_{\hat{\Omega}(\xi_0)}\mathrm{d}z\frac{|\Delta_3(n)\tilde{\vartheta}_{\sigma z_3z_3}|^2}{|n|^{1+2\alpha}}\le c\int\limits_{\mathbb{R}^1_+}\mathrm{d}n\int\limits_{\hat{\Omega}(\xi_0)}\mathrm{d}z\frac{|\Delta_3(n)\tilde{\vartheta}_{\sigma zz'}|^2}{|n|^{1+2\alpha}}+C\bar{X},
\tag{3.58}
$$

where the first term on the right-hand side of (3.58) we estimate using the following interpolation inequality

$$
\int\limits_{\mathbb{R}^1_+}\mathrm{d}n\int\limits_{\hat{\Omega}(\xi_0)}\mathrm{d}z\frac{|\Delta_3(n)\tilde{\vartheta}_{\sigma zz'}|^2}{|n|^{1+2\alpha}}\le c\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat{\Omega}(\xi_0)}\frac{|\Delta(\tau)\tilde{\vartheta}_{\sigma zz'}|^2}{|\tau|^{2+2\alpha}}\mathrm{d}z+c\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat{\Omega}(\xi_0)}\frac{|\Delta(\tau)\tilde{\vartheta}_{\sigma zz}|^2}{|\tau|^{2+2\alpha}}\mathrm{d}z.
\tag{3.59}
$$

Repeating the considerations from [5] (see the proof of Lemma 3.2) and taking into account (3.49),(3.52), (3.54), (3.56)–(3.58) we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\phi_3(t,\hat{\Omega})+\Phi_2(t,\hat{\Omega})\le cY,
\tag{3.60}
$$

where

$$
\begin{aligned}
&\phi_3(t,\hat{\Omega})=\phi_2(t,\hat{\Omega})+\frac{\mu+\nu}{\varrho_e}\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat{\Omega}(\xi_0)}\mathrm{d}z\frac{|\Delta(\tau)\hat{\nabla}_3\tilde{\eta}_\sigma|^2}{|\tau|^{2+2\alpha}}\\
&+\frac{\mu+\nu}{\varrho_e}\int\limits_{\mathbb{R}^1_+}\mathrm{d}n\int\limits_{\hat{\Omega}(\xi_0)}\mathrm{d}z\frac{|\Delta_3(n)\hat{\nabla}_3\tilde{\eta}_\sigma|^2}{|n|^{1+2\alpha}}+\\
&+\frac{1}{2}\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat{\Omega}(\xi_0)}\mathrm{d}z\left(\hat{\eta}\frac{|\Delta(\tau)\tilde{u}_z|^2}{|\tau|^{2+2\alpha}}+\frac{p_1}{\varrho_e}\frac{|\Delta(\tau)\tilde{\eta}_\sigma|^2}{|\tau|^{2+2\alpha}}\right)\\
&+\frac{\sigma}{2}\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat{S}(\xi_0)}\mathrm{d}z_{\hat{S}}\frac{g^{\gamma\delta}_{\hat{u}}\int_0^t\Delta(\tau)\tilde{u}_{z_\beta z_\gamma}\mathrm{d}t'\cdot\int_0^t\Delta(\tau)\tilde{u}_{z_\beta z_\delta}\mathrm{d}t'}{|\tau|^{2+2\alpha}}\\
&+\frac{\sigma}{2}\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat{S}(\xi_0)}\mathrm{d}z_{\hat{S}}\left(\frac{4\Delta(\tau)\left(H(\cdot,0)+\frac{2}{R_2}\right)\hat{n}_0\hat{\zeta}}{|\tau|^{1+\alpha}}+\frac{1}{4}\frac{\sum_{\gamma=1}^2\int_0^t\Delta(\tau)\tilde{u}_{z_\gamma z_\gamma}\mathrm{d}t'}{|\tau|^{1+\alpha}}\right)^2\\
&-\frac{\sigma}{32}\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat{S}(\xi_0)}\mathrm{d}z_{\hat{S}}\frac{(\sum_{\gamma=1}^2\int_0^t\Delta(\tau)\tilde{u}_{z_\gamma z_\gamma}\mathrm{d}t')^2}{|\tau|^{2+2\alpha}},
\end{aligned}
$$

$$
\begin{aligned}
\Phi_2(t,\hat\Omega) &= \Phi_1(t,\hat\Omega) + \int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat\Omega(\xi_0)} \mathrm{d}z \frac{|\Delta(\tau)\hat\nabla_3\hat\eta_\sigma|^2}{|\tau|^{2+2\alpha}} \\
&+ \int\limits_{\mathbb{R}^2} \mathrm{d}\tau \int\limits_{\hat\Omega(\xi_0)} \mathrm{d}z \frac{|\Delta(\tau)\hat\nabla_3^2\tilde u_3|^2}{|\tau|^{2+2\alpha}} + \int\limits_{\mathbb{R}^1_+} \mathrm{d}n \int\limits_{\hat\Omega(\xi_0)} \mathrm{d}z \frac{|\Delta_3(n)\tilde u_{zz'}|^2}{|n|^{1+2\alpha}} \\
&+ \int\limits_{\mathbb{R}^1_+} \mathrm{d}n \int\limits_{\hat\Omega(\xi_0)} \mathrm{d}z \frac{|\Delta_3(n)\hat\nabla_3\tilde\eta_\sigma|^2}{|n|^{1+2\alpha}} + + \int\limits_{\mathbb{R}^1_+} \mathrm{d}n \int\limits_{\hat\Omega(\xi_0)} \mathrm{d}z \frac{|\Delta_3(n)\tilde u_{z_3z_3}|^2}{|n|^{1+2\alpha}} \\
&+ \int\limits_{\mathbb{R}^1_+} \mathrm{d}n \int\limits_{\hat\Omega(\xi_0)} \mathrm{d}z \frac{|\Delta_3(n)\tilde\eta_{\sigma z'}|^2}{|n|^{1+2\alpha}} + \mu \int\limits_{\mathbb{R}^2} \frac{\|\Delta(\tau)\tilde u\|^2_{2,\hat\Omega}}{|\tau|^{2+2\alpha}} \mathrm{d}\tau \\
&+ \int\limits_{\mathbb{R}^2} \frac{\|\Delta(\tau)\tilde\eta_{\sigma z}\|^2_{0,\hat\Omega}}{|\tau|^{2+2\alpha}} \mathrm{d}\tau + \int\limits_{\mathbb{R}^2} \frac{\|\Delta(\tau)\tilde\vartheta_\sigma\|^2_{2,\hat\Omega}}{|\tau|^{2+2\alpha}} \mathrm{d}\tau \\
&+ \int\limits_{\mathbb{R}^1_+} \mathrm{d}n \int\limits_{\hat\Omega(\xi_0)} \mathrm{d}z \frac{|\Delta_3(n)\tilde\vartheta_{\sigma z_3z_3}|^2}{|n|^{1+2\alpha}} + \int\limits_{\mathbb{R}^1_+} \mathrm{d}n \int\limits_{\hat\Omega(\xi_0)} \mathrm{d}z \frac{|\Delta_3(n)\tilde\vartheta_{\sigma zz'}|^2}{|n|^{1+2\alpha}},
\end{aligned} \tag{3.61}
$$

$$
\begin{aligned}
Y &= \psi_{24}(a, \|F\|_{\frac52+\alpha,\hat S}) \bigg\{ X + X_2\|u\|^2_{2+\alpha,\Omega^t} \\
&+ \|\tilde u_t\|^2_{\alpha,\hat\Omega} + \|\vartheta_{\sigma t}\|^2_{\alpha,\hat\Omega} + \varepsilon\bigg(\Big\| \int\limits_0^t \tilde u \mathrm{d}t' \Big\|^2_{2+\alpha,\hat S} + \Big\| H(\cdot,0) + \frac{2}{R_e} \Big\|^2_{\alpha,S^1} + \|R(\cdot,t) - R(\cdot,0)\|^2_{2+\alpha,S^1} \bigg) \bigg\},
\end{aligned} \tag{3.62}
$$

ϕ_2, Φ_1, X_2 A are given by (3.50), (3.51), (3.18), (3.53), respectively.

Now, using (2.8) and assuming that δ from the local existence theorem (see Theorem 3.4 of [13]) such that $T^{1/2}\|u\|_{2+\alpha,\Omega^T} \le \delta$ is sufficiently small we get

$$
g_{\hat u}^{\gamma\delta} \int\limits_0^t \Delta(\tau)\tilde u_{z_\beta z_\gamma} \mathrm{d}t' \cdot \int\limits_0^t \Delta(\tau)\tilde u_{z_\beta z_\delta} \mathrm{d}t' \cdot \ge \frac34 \sum_{\gamma=1}^2 \Big(\int\limits_0^t \Delta(\tau)\tilde u_{z_\gamma z_\gamma} \mathrm{d}t' \Big)^2. \tag{3.63}
$$

Moreover,

$$
\begin{aligned}
& g_{\hat u}^{\gamma\delta} \int\limits_0^t \Delta^2(\tau) \frac{\partial\tilde u}{\partial z_\gamma} \mathrm{d}t' \cdot \int\limits_0^t \Delta^2(\tau) \frac{\partial\tilde u}{\partial z_\delta} \mathrm{d}t' \\
& \ge \frac34 \sum_{\gamma=1}^2 \Big(\int\limits_0^t \Delta^2(\tau) \frac{\partial\tilde u}{\partial z_\gamma} \mathrm{d}t' \Big)^2.
\end{aligned} \tag{3.64}
$$

Therefore integrating (3.60) with respect to time and using (3.63), (3.64), (3,47) we obtain for $t \le T$

$$
\phi_4(t,\hat\Omega) + \int\limits_0^t \Phi_2(t',\hat\Omega) \mathrm{d}t' \le c \int\limits_0^t Y \mathrm{d}t' + \phi_4(0,\hat\Omega), \tag{3.65}
$$

where

$$
\begin{aligned}
\phi_4(t,\hat\Omega)=&\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat\Omega(\xi_0)}\mathrm{d}z\bigg(\frac{\hat\eta|\Delta^2(\tau)\tilde u|^2}{|\tau|^{2+2(1+\alpha)}}+\frac{p_1}{\varrho_e}\frac{|\Delta^2(\tau)\eta_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}\\
&+\frac{p_2\hat\eta c_v(\hat\eta,\hat\vartheta)}{\theta_e p_{\hat\vartheta}(\varrho_e,\theta_e)}\frac{|\Delta^2(\tau)\tilde\vartheta_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}\bigg)+\int\limits_{\hat\Omega(\xi_0)}\bigg(\hat\eta\tilde u^2+\frac{p_1}{\varrho_e}\tilde\eta_\sigma^2+\frac{p_2\hat\eta c_v(\hat\eta,\hat\vartheta)}{\theta_e p_{\hat\vartheta}(\varrho_e,\theta_e)}\tilde\vartheta_\sigma^2\bigg)\mathrm{d}z\\
&+\frac{2(\mu+\nu)}{\varrho_e}\Bigg[\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat\Omega(\xi_0)}\mathrm{d}z\frac{|\Delta(\tau)\hat\nabla_3\tilde\eta_\sigma|^2}{|\tau|^{2+2\alpha}}+\int\limits_{\mathbb{R}^1_+}\mathrm{d}n\int\limits_{\hat\Omega(\xi_0)}\frac{|\Delta_3(n)\hat\nabla_3\tilde\eta_\sigma|^2}{|n|^{1+2\alpha}}\Bigg]\\
&+\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat\Omega(\xi_0)}\mathrm{d}z\left(\hat\eta\frac{|\Delta(\tau)\tilde u_2|^2}{|\tau|^{2+2\alpha}}+\frac{p_1}{\varrho_e}\frac{|\Delta(\tau)\tilde\eta_\sigma|^2}{|\tau|^{2+2\alpha}}\right)\\
&+\sigma\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat S(\xi_0)}\mathrm{d}z_{\hat S}\Bigg(4\sqrt{\hat c}\frac{\Delta(\tau)\left(\left(H(\cdot,0)+\frac{2}{R_e}\right)\hat n_0\hat\zeta\right)}{|\tau|^{1+\alpha}}+\frac{1}{4\sqrt{\hat c}}\frac{\int_0^t\Delta^3(\tau)\tilde u\mathrm{d}t'}{|\tau|^{1+2+\alpha}}\Bigg)^2\\
&+\sigma\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat S(\xi_0)}\mathrm{d}z_{\hat S}\Bigg(\frac{4\Delta(\tau)\left(\left(H(\cdot,0)+\frac{2}{R_e}\right)\hat n_0\hat\zeta\right)}{|\tau|^{1+\alpha}}+\frac14\frac{\sum_{\gamma=1}^2\int_0^t\Delta(\tau)\tilde u_{z_\gamma z_\gamma}\mathrm{d}t'}{|\tau|^{1+\alpha}}\Bigg)^2\\
&+\frac{\sigma}{4}\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat S(\xi_0)}\mathrm{d}z_{\hat S}\frac{(\sum_{\gamma=1}^2\int_0^t\Delta(\tau)\tilde u_{z_\gamma z_\gamma}\mathrm{d}t')^2}{|\tau|^{2+2\alpha}}+\frac{\sigma}{4}\int\limits_{\mathbb{R}^2}\mathrm{d}\tau\int\limits_{\hat S(\xi_0)}\mathrm{d}z_{\hat S}\frac{(\sum_{\gamma=1}^2\int_0^t\Delta^2(\tau)\tilde u_{z_\gamma}\mathrm{d}t')^2}{|\tau|^{2+2(1+\alpha)}},
\end{aligned}
\tag{3.66}
$$

Φ_2 and Y are given by (3.61) and (3.62) respectively.

In order to finish the proof of the lemma we have only to estimate $\int_0^t\|\tilde u_t\|^2_{\alpha,\hat\Omega}\mathrm{d}t'$ and $\int_0^t\|\tilde\vartheta_{\sigma t}\|^2_{\alpha,\hat\Omega}\mathrm{d}t'$. We obtain these estimates in the same way as for interior subdomains but in this case we have to take into account boundary conditions $(2.6)_5$ and $(2.6)_4$ respectively. We get

$$
\begin{aligned}
&\int\limits_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int\limits_0^{t-h_0}\mathrm{d}t'\int\limits_{\hat\Omega}\hat\eta c_v(\hat\eta,\hat\vartheta)|\Delta_{t'}\tilde\vartheta_\sigma|^2\mathrm{d}z+\frac{\varkappa}{2}\int\limits_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int\limits_{\hat\Omega}(|\nabla(\Delta_{t'}\tilde\vartheta_\sigma)|^2)\big|_{t'=t-h_0}\mathrm{d}z\\
&\le\frac{\varkappa}{2}\int\limits_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int\limits_{\hat\Omega}(|\nabla(\Delta_{t'}\tilde\vartheta_\sigma)|^2)\big|_{t'=t-h_0}\mathrm{d}z+\varepsilon(\|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t}+\|u\|^2_{2+\alpha,\Omega^t}+\|\eta_\sigma\|^2_{1+\alpha,\Omega^t})\\
&+\mathrm{d}\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}+c(\varepsilon,t)[\|u\|^2_{0,\Omega^t}+\|\vartheta_\sigma\|^2_{0,\Omega^t}+\|\eta_\sigma\|^2_{0,\Omega^t}+\|k\|^2_{\alpha,\Omega^t}+\|\bar\vartheta\|^2_{\alpha+1/2,S^t}\\
&+\|\hat u\|^2_{2+\alpha,\hat\Omega^t}(\|\hat u\|^2_{2+\alpha,\hat\Omega^t}+\sup_{0\le t'\le t}\|\hat\vartheta_\sigma\|^2_{1+\alpha,\hat\Omega}+\sup_{0\le t'\le t}\|\hat\eta_\sigma\|^2_{1+\alpha,\hat\Omega}+\sup_{0\le t'\le t}\|\hat u\|^2_{1+\alpha,\hat\Omega}\\
&+\|\hat u\|^2_{1+\alpha,\hat\Omega^t}\cdot\sup_{0\le t'\le t}\|\hat u\|^2_{1+\alpha,\hat\Omega}+\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}+\|\hat u\|^2_{2+\alpha,\hat\Omega^t}\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t})+\|u\|^2_{2+\alpha,\Omega^t}\|u\|^2_{1+\alpha,\Omega^t}\\
&+\sup_{0\le t'\le t}\|\hat\eta\|^2_{1+\alpha,\hat\Omega}\|\tilde k\|^2_{\alpha,\hat\Omega^t}+\|\hat\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}\cdot\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}]
\end{aligned}
\tag{3.67}
$$

and

$$
(3.68)\quad \begin{aligned}
&\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\hat\Omega}\hat\eta|\Delta_{t'}\tilde u_{t'}|^2\mathrm{d}z+\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_{\hat\Omega}\left(\frac{\mu}{4}|\mathbb{S}(\Delta_{t'}\tilde u)|^2+\frac{\nu-\mu}{2}|\mathrm{div}\,\Delta_{t'}\tilde u|^2\right)\mathrm{d}z\Big|_{t'=t-h_0}\\
&\le\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_{\hat\Omega}\left(\frac{\mu}{2}\mathbb{S}(\Delta_{t'}\tilde u)|^2+\frac{\nu-\mu}{2}|\mathrm{div}\,\Delta_{t'},\tilde u|^2\right)\mathrm{d}z\Big|_{t'=0}+I+\sum_{i=1}^5K_i,
\end{aligned}
$$

where

$$
(3.69)\quad \begin{aligned}
|I|\le{}&\varepsilon(\|u\|^2_{2+\alpha,\Omega^t}+\|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t}+\|\eta_\sigma\|^2_{1+\alpha,\Omega^t})\\
&+\mathrm{d}\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}+c(\varepsilon,t)[\|u\|^2_{0,\Omega^t}+\|\vartheta_\sigma\|^2_{0,\Omega^t}\\
&+\|\eta_\sigma\|^2_{0,\Omega^t}+\|k\|^2_{0,\Omega^t}+\|u\|^2_{2+\alpha,\Omega^t}\sup_{0\le t'\le t}\|u\|^2_{1+\alpha,\Omega}\\
&+\|\hat u\|^2_{2+\alpha,\hat\Omega^t}(\|\hat u\|^2_{2+\alpha,\hat\Omega^t}+\|\hat\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}+\|\eta_\sigma\|^2_{1+\alpha,\Omega^t})\cdot\\
&\cdot(1+\|\hat u\|^2_{1+\alpha,\hat\Omega^t})+\sup_{0\le t'\le t}\|\hat u\|^2_{1+\alpha,\hat\Omega}(\|\hat\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}+\\
&+\|\hat\eta_\sigma\|^2_{1+\alpha,\hat\Omega^t})+\|\hat\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}\cdot(\|\tilde\vartheta_\sigma\|^2_{1+\alpha,\hat\Omega^t}+\|\tilde\eta_\sigma\|^2_{1+\alpha,\hat\Omega^t})\\
&+\|\hat u\|^2_{2+\alpha,\hat\Omega^t}(\sup_{0\le t'\le t}\|\tilde\eta_\sigma\|^2_{1+\alpha,\hat\Omega}+\sup_{0\le t'\le t}\|\tilde\vartheta_\sigma\|^2_{1+\alpha,\hat\Omega})].
\end{aligned}
$$

and

$$
(3.70)\quad K_1=\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\hat S}\Delta_{t'}\tilde p_\sigma I\hat n'\Delta_{t'}\tilde u_{t'}\mathrm{d}z_{\hat S},
$$

$$
(3.71)\quad K_2=-\sigma\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\int_{\hat S}\Delta_{t'}(\Delta_{\hat S})\int_0^{t'}\tilde u\mathrm{d}\tau\cdot\Delta_{t'}\tilde u_{t'}\mathrm{d}z_{\hat S},
$$

$$
(3.72)\quad K_3=-\sigma\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{t-h_0}\mathrm{d}t'\Delta_{\hat S}\Delta_t\int_0^t\tilde u\mathrm{d}t'\cdot\Delta_{t'}\tilde u_{t'}\mathrm{d}z_{\hat S},
$$

$$
(3.73)\quad K_4=\int_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int_0^{T-h_0}\mathrm{d}t'\int_{\hat S}\hat{\mathbb{T}}(\tilde u,\tilde p_\sigma)\Delta_{t'}\hat n\cdot\Delta_{t'}\tilde u_t\mathrm{d}z_{\hat S}.
$$

First, we have

$$
(3.74)\quad \begin{aligned}
|K_1|\le{}&\varepsilon\int_{\hat S}\mathrm{d}z_{\hat S}\int_0^{h_0}\mathrm{d}h\int_0^{t-h_0}\mathrm{d}t'\frac{|\Delta_{t'}\tilde u_{t'}|^2}{h^{1+2(\frac{\alpha}{2}-\frac14)}}+c(\varepsilon)\int_{\hat S}\mathrm{d}z_{\hat S}\int_0^{h_0}\mathrm{d}h\int_0^{t-h_0}\mathrm{d}t'\frac{|\Delta_{t'}(p_1\tilde\eta_\sigma+p_2\tilde\vartheta_\sigma)|^2}{h^{3/2+\alpha}}\\
\le{}&\varepsilon\|\tilde u\|^2_{3/2+\alpha,\hat S^t}+c(\varepsilon)\int_{\hat S}\mathrm{d}z_{\hat S}\int_0^{h_0}\mathrm{d}h\int_0^{t-h_0}\mathrm{d}t'\frac{|\Delta_{t'}p_1\tilde\eta_\sigma|^2}{h^{3/2+\alpha}}\\
&+c(\varepsilon)\int_{\hat S}\mathrm{d}z_{\hat S}\int_0^{h_0}\mathrm{d}h\int_0^{t-h_0}\mathrm{d}t'\frac{|\Delta_{t'}p_2\tilde\vartheta_\sigma|^2}{h^{3/2+\alpha}}+c(\varepsilon)\int_{\hat S}\mathrm{d}z_{\hat S}\int_0^{h_0}\mathrm{d}h\int_0^{t-h_0}\mathrm{d}t'\frac{|\Delta_t\tilde\eta_\sigma|^2}{h^{3/2+\alpha}}\\
&+c(\varepsilon)\int_{\hat S}\mathrm{d}z_{\hat S}\int_0^{h_0}\mathrm{d}h\int_0\mathrm{d}t'\frac{|\Delta_t\tilde\vartheta_\sigma|^2}{h^{3/2+\alpha}}\le\varepsilon\|\tilde u\|^2_{2+\alpha,\hat\Omega^t}+\sum_{i=1}^4K_1^i.
\end{aligned}
$$

We have to estimate the terms K_1^i $(i=1,\dots,4)$. First, we obtain

$$
\begin{aligned}
|K_1^4| &\le c(\varepsilon)\int\limits_{\hat S} \mathrm{d}z_{\hat S}\int\limits_0^{h_0}\mathrm{d}h\int\limits_0^{t-h_0}\mathrm{d}t'\frac{|\int_{t'}^{t'+h}\tilde\vartheta_{\sigma\tau}\mathrm{d}\tau|^2}{h^{3/2+\alpha}}\\
&\le c(\varepsilon)h_0^{3/2-\alpha}\int\limits_0^{t-h_0}\mathrm{d}t'\int\limits_{\hat S}\left(\sup_{h>0}\frac{1}{h}\int\limits_{t'}^{t'+h}|\tilde\vartheta_{\sigma\tau}|\mathrm{d}\tau\right)^2\\
&\le c(\varepsilon)t^{3/2-\alpha}\int\limits_0^t\|\tilde\vartheta_{\sigma t'}\|^2_{0,\hat S}\mathrm{d}t'\\
&\le c(\varepsilon)t^{3/2-\alpha}(\varepsilon_1^{1-\varkappa}\int\limits_0^t\|\tilde\vartheta_{\sigma t'}\|^2_{\alpha,\hat\Omega}\mathrm{d}t'+c\varepsilon_1^{-\varkappa}\int\limits_0^t\|\tilde\vartheta_{\sigma t'}\|^2_{0,\hat\Omega}\mathrm{d}t').
\end{aligned}
\tag{3.75}
$$

From equation $(2.6)_3$ we calculate

$$
\begin{aligned}
&\|\tilde\vartheta_{\sigma t}\|^2_{0,\hat\Omega}\le\varepsilon_2^{1-\varkappa_2}(\|u\|^2_{2+\alpha,\Omega}+\|\vartheta_\sigma\|^2_{2+\alpha,\Omega})\\
&+\mathrm{d}\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega}+c\varepsilon_2^{-\varkappa_2}[\|u\|^2_{0,\Omega}+\|\vartheta_\sigma\|^2_{0,\Omega}\\
&+\|k\|^2_{0,\Omega}+\|\hat u\|^2_{1+\alpha,\hat\Omega}(\|\hat u\|^2_{1+\alpha,\hat\Omega}+\|\hat\vartheta_\sigma\|^2_{\alpha,\hat\Omega}\\
&+\|\hat\eta_\sigma\|^2_{\alpha,\hat\Omega}+\|\hat u\|^2_{1+\alpha,\hat\Omega^t})+\|\hat u\|^2_{2+\alpha,\hat\Omega^t}\|\tilde\vartheta_\sigma\|^2_{2,\hat\Omega}].
\end{aligned}
\tag{3.76}
$$

Using (3.76) in (3.75) with appropriate choice of ε_1 and ε_2 yields

$$
\begin{aligned}
&|K_1^4|\le\varepsilon(\|u\|^2_{2+\alpha,\Omega^t}+\|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t})\\
&+c(\varepsilon,t)(d\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}+\|u\|^2_{0,\Omega^t}+\|\vartheta_\sigma\|^2_{0,\Omega^t}\\
&+\|k\|^2_{0,\Omega^t}+\sup_{0\le t'\le t}\|\hat u\|^2_{1+\alpha,\hat\Omega}(\|\hat u\|^2_{1+\alpha,\hat\Omega^t}\\
&+\|\hat\vartheta_\sigma\|^2_{\alpha,\hat\Omega^t}+\|\hat\eta_\sigma\|^2_{\alpha,\hat\Omega^t})+\|\hat u\|^2_{2+\alpha,\hat\Omega^t}\|\tilde\vartheta_\sigma\|^2_{2,\hat\Omega^t}],
\end{aligned}
\tag{3.77}
$$

where $c(\varepsilon,t)$ is a positive continuous function increasing with respect to t. Now, we estimate

$$
\begin{aligned}
|K_1^3| &\le c(\varepsilon)\int\limits_{\hat S}\mathrm{d}z_{\hat S}\int\limits_0^{h_0}\mathrm{d}h\int\limits_0^{t-h_0}\mathrm{d}t'\frac{|\int_{t'}^{t'+h}\hat u_z\mathrm{d}\tau|^2}{h^{3/2+\alpha}}\\
&\le c(\varepsilon)h_0^{3/2-\alpha}\int\limits_0^{t-h_0}\mathrm{d}t'\int\limits_{\hat S}(\sup_{h>0}\frac{1}{h}\int\limits_{t'}^{t'+h}|\hat u_z|\mathrm{d}\tau)^2\\
&\le c(\varepsilon)t^{3/2-\alpha}\int\limits_0^t\|\hat u_z\|^2_{0,\hat S}\mathrm{d}t'\\
&\le c(\varepsilon)t^{3/2-\alpha}(\varepsilon_1^{1-\varkappa}\|\hat u\|^2_{2+\alpha,\hat\Omega^t}+c\varepsilon_1^{-\varkappa}\|\hat u\|^2_{0,\hat\Omega^t})\\
&\le\varepsilon\|\hat u\|^2_{2+\alpha,\hat\Omega^t}+c(\varepsilon,t)\|\hat u\|^2_{0,\hat\Omega^t},
\end{aligned}
\tag{3.78}
$$

where we assumed $\varepsilon_1=\left(\frac{\varepsilon}{c(\varepsilon)t^{3/2-\alpha}}\right)^{\frac{1}{1-\varkappa}}$.

Next, we have

$$
\begin{aligned}
|K_1^1| &\le c(\varepsilon)\int\limits_{\hat S} \mathrm{d}z_{\hat S}\int\limits_0^{h_o}\mathrm{d}h\int\limits_0^{t-h_0}\mathrm{d}t'\frac{|\Delta_{t'}\hat\eta_\sigma|^2|\tilde\eta_\sigma|^2}{h^{3/2+\alpha}}\\
&\quad + c(\varepsilon)\int\limits_{\hat S} \mathrm{d}z_{\hat S}\int\limits_0^{h_0}\mathrm{d}h\int\limits_0^{t-h_0}\mathrm{d}t'\frac{|\Delta_{t'}\hat\vartheta_\sigma|^2|\tilde\eta_\sigma|^2}{h^{3/2+\alpha}}\\
&\le c(\varepsilon)\sup_{0\le t'\le t}\|\tilde\eta_\sigma\|^2_{1+\alpha,\hat\Omega}(|K_1^4|+|K_1^3|)\\
&\le c(\varepsilon)t^{3/2-\alpha}\sup_{0\le t'\le t}\|\tilde\eta_\sigma\|^2_{1+\alpha,\hat\Omega}(\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}+\|\hat u\|^2_{2+\alpha,\hat\Omega^t}).
\end{aligned}
\tag{3.79}
$$

In the same way we estimate $|K_1^2|$.

Thus, taking into account (3.74)–(3.79) we get

$$
\begin{aligned}
|K_1| &\le \varepsilon(\|u\|^2_{2+\alpha,\Omega^t}+\vartheta_\sigma\|^2_{2+\alpha,\Omega^t})+c(\varepsilon,t)[\mathrm{d}\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}\\
&\quad+\|u\|^2_{0,\Omega^t}+\|\vartheta_\sigma\|^2_{0,\Omega^t}+\|k\|^2_{0,\Omega^t}+\sup_{0\le t'\le t}\|\hat u\|^2_{1+\alpha,\hat\Omega}(\|\hat u\|^2_{1+\alpha,\hat\Omega^t}\\
&\quad+\|\hat\vartheta_\sigma\|^2_{\alpha,\hat\Omega^t}+\|\hat\eta_\sigma\|^2_{\alpha,\hat\Omega^t})+\|\hat u\|^2_{2+\alpha,\hat\Omega^t}\cdot\|\tilde\vartheta_\sigma\|^2_{2,\hat\Omega^t}\\
&\quad+\sup_{0\le t'\le t}\|\tilde\eta_\sigma\|^2_{1+\alpha,\hat\Omega}(\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}+\|\hat u\|^2_{2+\alpha,\hat\Omega^t})].
\end{aligned}
\tag{3.80}
$$

In the same way we estimate K_2, K_3, K_4 obtaining

$$
|K_2|\le\varepsilon\|\tilde u\|^2_{2+\alpha,\hat\Omega^t}+c(\varepsilon)t^{3/2-\alpha}\|\tilde u\|^4_{2+\alpha,\Omega^t}\cdot(1+\|\tilde u\|^2_{2+\alpha,\hat\Omega^t}),
\tag{3.81}
$$

$$
|K_3|\le\varepsilon\|\tilde u\|^2_{2+\alpha,\hat\Omega^t}+c(\varepsilon)t^{3/2-\alpha}(\|\hat u\|^2_{2+\alpha,\Omega^t}\cdot\|\tilde u\|^2_{2+\alpha,\hat\Omega^t}+\|\tilde u\|^2_{0,\hat\Omega^t})
\tag{3.82}
$$

and

$$
\begin{aligned}
|K_4| &\le\varepsilon\|\tilde u\|^2_{2+\alpha,\hat\Omega^t}+c(\varepsilon)t^{3/2-\alpha}(\sup_{0\le t'\le t}\|\tilde u\|^2_{1+\alpha,\hat\Omega}\\
&\quad+\sup_{0\le t'\le t}\|\tilde\eta_\sigma\|^2_{\alpha,\hat\Omega}+\sup_{0\le t'\le t}\|\tilde\vartheta_\sigma\|^2_{\alpha,\hat\Omega})\|\hat u\|^2_{2+\alpha,\hat\Omega^t}\cdot(1+\|\hat u\|^2_{2+\alpha,\hat\Omega^t}).
\end{aligned}
\tag{3.83}
$$

Now, inequalities (3.68), (3.69), (3.74), (3.80)–(3.83) yield

$$
\begin{aligned}
&\int\limits_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int\limits_0^{t-h_0}\mathrm{d}t'\int\limits_{\hat\Omega}\hat\eta|\Delta_{t'}\tilde u_{t'}|^2\mathrm{d}z\\
&+\int\limits_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int\limits_{\hat\Omega}\left(\frac{\mu}{4}|\mathbb{S}(\Delta_{t'}\tilde u)|^2+\frac{\nu-\mu}{2}|\mathrm{div}\,\Delta_{t'}\tilde u|^2\right)\mathrm{d}z\Big|_{t'=t-h_0}\\
&\le\int\limits_0^{h_0}\frac{\mathrm{d}h}{h^{1+\alpha}}\int\limits_{\hat\Omega}\left(\frac{\mu}{4}|\mathbb{S}(\Delta_{t'}\tilde u)|^2+\frac{\nu-\mu}{2}|\mathrm{div}\,\Delta_{t'}\tilde u|^2\right)\mathrm{d}z\Big|_{t'=0}\\
&+\varepsilon(\|u\|^2_{2+\alpha,\Omega^t}+\|\vartheta_\sigma\|^2_{2+\alpha,\Omega^t}+\|\eta_\sigma\|^2_{1+\alpha,\Omega^t})\\
&+c(\varepsilon,t)[\mathrm{d}\|\tilde\vartheta_\sigma\|^2_{2+\alpha,\hat\Omega^t}+\|u\|^2_{0,\Omega^t}+\|\vartheta_\sigma\|^2_{0,\Omega^t}\\
&+\|k\|^2_{0,\Omega^t}+(1+X_3)X_3(X_3+X_4)],
\end{aligned}
\tag{3.84}
$$

where $c(\varepsilon, t)$ is a positive continuous function nondecreasing with respect to its arguments, X_3 and X_4 are given by (3.38) and (3.39), respectively.

Finally, from $(1.6)_2$ we have

$$\tilde{\eta}_{\sigma t} + \hat{\eta} \mathrm{div}_{\hat{u}} \hat{u} \hat{\zeta} = 0,$$

so

$$\int_{\hat{\Omega}} \|\tilde{\eta}_{\sigma t}\|^2_{\frac{\alpha}{2},(0,T)} \mathrm{d}z \le \varepsilon \|u\|^2_{2+\alpha,\Omega^t} + c(\varepsilon, t)(\|u\|^2_{0,\Omega^t} + \|\vartheta_\sigma\|^2_{0,\Omega^t} + \|\eta_\sigma\|^2_{0,\Omega^t} + \|u\|^4_{2+\alpha,\Omega^t}). \tag{3.85}$$

Let us introduce a partition of unity such that $\Omega^{(k)}$, $k \in \mathfrak{M}$, is an interior subdomain and $\hat{\Omega}^{(k)}$, $k \in \mathfrak{N}$, is a boundary subdomain. By χ_k, $k \in \mathfrak{N}$, we denote the transformation χ (see Section 2) in $\Omega^{(k)}$, $k \in \mathfrak{N}$. Then we introduce the quantity

$$\phi(t, \Omega) = \sum_{k \in \mathfrak{M}} \phi_1(t, \Omega^{(k)}) + \sum_{k \in \mathfrak{N}} \phi_4(t, \hat{\Omega}^{(k)}) + \sum_{k \in \mathfrak{N}} \phi_5(t, \hat{\Omega}^{(k)}), \tag{3.86}$$

where ϕ_1 and ϕ_4 are given by (3.36) and (3.66), respectively and

$$\begin{aligned} \phi_5(t, \hat{\Omega}^{(k)}) &= \int_0^{h_0} \frac{\mathrm{d}h}{h^{1+\alpha}} \int_{\hat{\Omega}^{(k)}} \left(\frac{\mu}{4} |\mathbb{S}(\Delta_{t'} \tilde{u})|^2 + \frac{\nu - \mu}{2} |\mathrm{div}\, \Delta_{t'} \tilde{u}|^2 \right) \mathrm{d}z \Big|_{t'=t-h_0} \\ &+ \int_0^{h_0} \frac{\mathrm{d}h}{h^{1+\alpha}} \int_{\hat{\Omega}^{(k)}} (|\nabla(\Delta_{t'} \tilde{\vartheta}_\sigma)|^2) \mathrm{d}z \Big|_{t'=t-h_0}. \end{aligned}$$

Now, using estimates (3.67), (3.84) and (3.85), next passing in (3.65) to the old variables ξ and adding to (3.37) we obtain (3.1).

This completes the proof of the theorem. ■

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