Joint replenishment multiproduct inventory problem with continuous production and discrete demands

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Abstract

The basic, deterministic, multi-item inventory problem is considered. The production rates (decision variables) are represented by nonnegative functions defined on the interval $[0, T]$, $T < \infty$. The demands appear at discrete time points. The production cost consists of two parts. The first is a setup cost which depends on the joint production (or joint replenishment) rates. (The minor setup, connected with particular items, is zero.) The second part is described by a linear function of the production rates. The holding cost is linear. An auxiliary allocation problem is considered. Necessary and sufficient conditions for an optimal allocation are given. Under these conditions some qualitative properties of the multi-item joint replenishment inventory problem are obtained and a method of transformation to a one-item problem is given. A construction of the optimal allocation is presented.

1 Introduction

The basic, deterministic multiproduct inventory model is considered. In the model the production rates are variable. They are expressed by functions defined on the given interval [0,T] where T denotes a planning horizon. As in Refs 1 and 2, in the model the production rate of each product is treated as a decision variable. The demands appear at fixed discrete time points and are given by nonnegative vectors. The problem is to determine a production schedule to meet the demands and minimize the sum of production and holding costs. The production cost consists of two parts. The first is a setup cost which depends on joint production (or joint replenishment) rates. The second is expressed by a linear function of the particular production rates. The holding cost is also linear in inventory levels. The objective is

(i) to discuss the so-called allocation problem which appears as auxiliary in the multiproduct joint replenishment inventory problems,

(ii) to obtain qualitative properties of optimal schedules of a wide class of inventory problems, (it is assumed that the joint replenishment cost is expressed by a nonnegative and nondecreasing function),

(iii) to transform a multiproduct problem to a one-product problem.

In a sense the paper develops the results presented in Refs. 3, 4 and 5 for the case of variable production rate and discrete demands. Together with the solution of the allocation problem we obtain interesting qualitative properties of the inventory problem. Some of the properties (cf. Property 1 from Section 7) are similar to those presented in Refs.3 and 5 for the cases of continuous and discrete time problems, respectively. Note that in a sense the problem considered here joins both these cases because the production rates are continuous and the demands are discrete. As will turn out, some necessary conditions for optimal solutions are similar to the previous cases (cf. properties expressed by the minimal allocations in Theorem 1 from Section 5 and Property 1 from Section 7) but some are quite new (cf Proposition 4 from Section 5 and Property 2 from Section 7). An interpretation of the necessary conditions is given in Subsection 7.1.

This paper is organized as follows. In Section 2 we give a mathematical description of the basic inventory problem. In Sections 3-5 we define allocation and minimal allocation problems, establish properties of minimal allocation, and necessary and sufficient conditions, we also present a method of construction of the minimal allocation. In Section 6 the allocation results are applied to some warehouse allocation problems and in Sections 7 and 8 to some cases of multi-item joint replenishment inventory problems.

2 Inventory problem. Mathematical description

Let U be an integrable, nonnegative function defined on $[0, T]$, $T < \infty$, let $\{t_j\}$ be a sequence of time points such that $0 = t_0 < t_1 <...< t_{m-1} < t_m = T$,

$m \geq 1$, and let

$\{d_j = [d_j^1, d_j^2,...,d_j^n], \quad j = 1, 2,..., m\}$ be a sequence of nonnegative n-dimensional vectors.

Consider the minimization inventory problem $(I.P)$ of the form

$$(I.P) \qquad \min_{u_1,u_2,...u_n} \int_0^T \{c(\sum_{i=1}^n u_i(t)) + \sum_{i=1}^n a_i u_i(t) + \sum_{i=1}^n h_i I_i(t)\}dt$$

under the constraints

$$I_i(t) = \int_0^t u_i(s)ds - \sum_{\{j:t_j \leq t\}} d_j^i \geq 0 \text{ on } [0,T], \ i = 1, 2, ..., n \quad (1)$$

$$u_1 + u_2 + ... + u_n \leq U, \quad u_1 \geq 0, \quad u_2 \geq 0, ..., u_n \geq 0 \text{ a.e. on } [0,T] \quad (2)$$

where

$c : [0,T] \to R^+$ is a given nondecreasing function,

$a_i, h_i, \ i = 1, 2, ..., n$ are nonnegative constants with $h_1 > h_2 > ... > h_n$.

$u_i, \ i = 1, 2, ..., n$ are integrable decision functions.

The problem belongs to a class called in literature (cf. Ref 6.) the parallel machine multiproduct inventory problem in which

u_i denotes the continuous production rates of i-th product,

d_j^i denotes the demand of i-th product at the moment t_j,

$c(\cdot)$ describes the joint setup cost,

a_i expresses the unit production cost of the i-th product,

$U(\cdot)$ corresponds to the production capacity of the production system.

In the discussion we assume that the products have uniform units and so the sum $u_1 + u_2 + ... + u_n$ is well defined.

Together with the basic problem $(I.P)$ we will consider two additional versions of the problem. The versions are defined in Remarks 4 and 5.

2.1 Remarks on the inventory problem $(I.P)$

Remark 1 *Note that the constraint (1) means that in the model no backorders are allowed.*

Remark 2 *It is easy to see that every optimal solution of (I.P) has to satisfy the equality*

$\int\limits_0^T (\sum\limits_{i=1}^n a_i u_i(t))dt = \sum\limits_{i=1}^n \sum\limits_{j=1}^m a_i d_j^i = const$, *which means that only as much has to be produced as is really needed. So we can omit the sum* $\sum\limits_{i=1}^n a_i u_i(t)$ *in the objective function in the problem (I.P). The equivalent modified version will also be called (I.P).*

Remark 3 *The assumption* $h_1 > h_2 > \ldots > h_n$ *involves no loss of generality because the equality* $h_i = h_j$ *reduces the problem to* $n-1$ *products. (First we can solve* $n-1$ *dimensional problem without j-th product but with* $d^i := d^i + d^j$ *. Next we can allocate the i-th optimal plan to products i-th and j-th, cf. Definition 3)*

Remark 4 *In some cases it is more convenient to express the production capacity constraint (2) in the form*

$(2)'$ $u_1(t)+u_2(t)+\ldots+u_n(t) \in \widetilde{U}, 0 \in \widetilde{U},\ u_1(t) \geq 0,\ u_2(t) \geq 0, \ldots, u_n(t) \geq 0$ *a.e. on* $[0,T]$.

Note that this time $\widetilde{U} \subset R^+$ *is a set. The discussion given in the paper for the problem (I.P) is identical for the version (I.P)′ in which (2) is replaced by (2)′.*

Remark 5 *Sometimes in practical problems the setup cost is defined by the number of setup points. A setup point of a nonnegative integrable function is defined in the following way:*

Definition 1 *A time point* $t \in [0,T]$ *is called a setup point of the function* u *if there exists* $\varepsilon > 0$ *such that* $u(t) = 0$ *a.e. on* $(t-\varepsilon, t)$ *and* $u(t) > 0$ *a.e. on* $(t, t+\varepsilon)$. *The point* $t = 0$ *is called a setup point if there exists* $\varepsilon > 0$ *such that* $u(t) > 0$ *a.e. on* $(0, \varepsilon)$.

By the definition if we put $N(u) = \#\{t \in [0,T] : \quad t \quad \text{is a setup point of } u\}$ *then the setup cost on* $[0,T]$ *is given by*

$C(u) = \begin{cases} a \cdot N(u) & \text{if } N(u) < \infty \\ \infty & \text{othervise,} \end{cases}$ *where a is a nonnegative constant.*

In the paper together with (I.P) (and (I,P)′) we are able to discuss the *inventory problem (I.P)'' in the following form,*

$$(I.P)'' \qquad \min_{u_1,u_2\ldots,u_n}\{C(\sum_{i=1}^{n} u_i) + \int_0^T (\sum_{i=1}^{n} h_i I_i(t))dt\}$$

$$\begin{aligned}
I_i(t) &= \int_0^t u_i(s)ds - \sum_{\{j:t_j\le t\}} d_j^i \ge 0, \quad \textit{on} \quad [0,T], \quad i=1,2,...,n,\\
&\; u_i \textit{are integrable functions such that}\\
u_1+u_2+...+u_n &\in \widetilde{U}, \quad u_1\ge 0, \quad u_2\ge 0,...,u_n\ge 0 \textit{ a.e. on } [0,T] \textit{ where}\\
\widetilde{U} &\subset R^+ \textit{ is a given set such that } 0\in\widetilde{U}.
\end{aligned}$$

2.2 Necessary conditions for solutions of inventory problems

Let i,j be a pair of different subscripts and let $\widehat{u}_1, \widehat{u}_2,...,\widehat{u}_n$ be an optimal solution of $(I.P)$ (or $(I.P)'$ or $(I.P)''$). Put $\widehat{u} = \widehat{u}_1 + \widehat{u}_2 + \cdots + \widehat{u}_n$. Hence if $\widehat{u}$ and $\{\widehat{u}_k, k\neq i, k\neq j\}$ are fixed then the pair $(\widehat{u}_i, \widehat{u}_j)$ has to solve the following auxiliary problem *(Au.P):*

$$(Au.P) \qquad \min_{u_i,u_j}\int_0^T \{h_i I_i(t) + h_j I_j(t)\}dt \tag{3}$$

$$\begin{aligned}
I_i(t) &= \int_0^t u_i(s)ds - \sum_{\{k:\ t_k\le t\}} d_k^i \ge 0,\\
I_j(t) &= \int_0^t u_j(s)ds - \sum_{\{k:\ t_k\le t\}} d_k^j \ge 0, \qquad \textit{on} \quad [0,T],
\end{aligned}$$

$$u_i + u_j = \widehat{u} - \sum_{\{k\neq i,j\}} \widehat{u}_k = \widehat{u}_i + \widehat{u}_j, \qquad u_i\ge 0, \quad u_j\ge 0 \text{ a.e. on } [0,T].$$

This minimization problem generates the so called allocation problem $(All.P)$ which will be discussed in the next sections.

3 Allocation Problem $(ALL.P)$. Mathematical description

Let u be a given nonnegative, integrable function defined on $[0, T]$, and let $\{t_j,\ j = 0, 1, ..., m\}$ and $\{d_j^i,\ \ j = 1, 2, ..., m,\ \ i = 1, 2, ..., n\}$ be sequences as in Section 2. So $0 = t_0 < t_1 <...< t_{m-1} < t_m = T$, $d_j^i \geq 0$, $m \geq 1$ and $n \geq 2$. Put $d^i = \{d_j^i,\ j = 1, 2...,m\}$.

Definition 2 *A sequence $u, d^1, d^2, ..., d^n$ is called an allocation system or simply a system if*

$$I(t) = \int_0^t u(s)ds - \sum_{\{k:\ t_k \leq\ t\}} (d_k^1 + d_k^2 + ... + d_k^n) \geq 0.$$

Definition 3 *A sequence of integrable functions $u_1, u_2, ...u_n$ defined on $[0, T]$ is called an **allocation** of the system $u, d^1, d^2, ..., d^n$ if*

(i) $u_i \geq 0$ *a.e. on* $[0, T]$,

(ii) $u = u^1 + u^2 + \cdots + u^n$ *a.e. on* $[0, T]$,

(iii) $I_i(t) = \int_0^t u_i(s)ds - \sum_{\{k:\ t_k \leq t\}} d_k^i \geq 0, \quad i = 1, 2, ..., n, \quad t \in [0, T].$

The set of all allocations of the system $u, d^1, d^2, ..., d^n$ is denoted by $A_n(u, , d^1, d^2, ..., d^n)$ or simply A_n.

Remark 6 *Observe that any allocation from $A_n(u, , d^1, d^2, ..., d^n)$ can be obtained from elements of A_2.*

In fact, $A_n(u, , d^1, d^2, ..., d^n) = \{u_1, u_2, ..., u_n\colon \quad (u_1, u - u_1) \in A_2(u, d^1, d^2 + ... + d^n)$ and $(u_2, u_3, ..., u_n) \in A_{n-1}(u - u_1, d^2, d^3 + ... + d^n)$ provided $n > 2$. So by induction we obtain the remark.

Remark 7 *The functions $I, I_1, I_2, ..., I_n$ are piecewise continuous on $[0, T]$. On every interval (t_{j-1}, t_j) they are continuous and nondecreasing. At the points t_j, $j = 1, 2, ..., m$, they may have jumps with $I(t_j-) \geq I(t_j) = I(t_j+)$, $I_i(t_j-) \geq I_i(t_j) = I_i(t_j+)$. So the functions are right-continuous with $I(0) = I_i(0) = I(0+) = I_i(0+) = 0$ and $I(T-) \geq I(T)$, $I_i(T-) \geq I_i(T)$ (cf. Figure 1).*

Figure 1

Note that Remark 6 prompts the discussion of the two-dimensional allocations. So let u, d^1, d^2 be a system.

By Definition 3(ii) any allocation $(u_1, u_2) \in A_2(u, d^1, d^2)$ is given by $(u_1, u_2 = u - u_1)$.

Definition 4 *An allocation (u_1^*, u_2^*) is called minimal in the set $A_2(u, d^1, d^2)$ if $I_1^*(t) \leq I_1(t)$ on $[0, T]$ for all $(u_1, u_2 = u - u_1) \in A_2(u, d^1, d^2)$.*

In other words, $(u_1^*, u - u_1^*)$ is called minimal if $I_1^*(t) \leq I_1(t)$ on $[0, T]$ for all nonnegative u_1 such that

$$I_1(t) = \int_0^t u_1(s)ds - \sum_{\{k\,:t_k \leq\, t\}} d_k^1 \geq 0 \quad \text{for} \quad t \in [0, T],$$

$I_2(t) = \int_0^t u_2(s)ds - \sum_{\{k:\ t_k \leq\, t\}} d_k^2 \geq 0$ for $t \in [0, T]$ with $u_2(s) = u(s) - u_1(s) \geq 0$ a.e. on $[0, T]$.

In the next Sections 4 and 5 we will discuss some properties of minimal allocations. The Sections are of technical character. We obtain some mathematical results, which will be applied in the next Sections 6 - 8 to some warehouse allocation problems and inventory problems.

4 Properties of minimal allocations of A_2

Proposition 1 *For every allocation $(u_1, u - u_1) \in A_2(u, d^1, d^2)$ there exists an allocation $(\widehat{u}_1, u - \widehat{u}_1) \in A_2(u, d^1, d^2)$ such that $I_1(t) \geq \widehat{I}_1(t)$ and $\widehat{I}_1(T) = 0$.*

Proof. If $I_1(T) = \int_0^T u_1(s)ds - \sum_{k=1}^m d_k^1 = 0$ then the assertion follows. Suppose then that $I_1(T) > 0$. Hence there exists $\widehat{t}$ such that $\int_0^{\widehat{t}} u_1(s)ds = \sum_{k=1}^m d_k^1$. So we can put $\widehat{u}_1 = \begin{cases} u_1 \text{ on } [0, \widehat{t}], \\ 0 \text{ on } (\widehat{t}, T]. \end{cases}$

Observe that $\widehat{I}_1(t) \leq I_1(t)$, $(\widehat{u}_1, u - \widehat{u}_1) \in A_2(u, d^1, d^2)$ and $\widehat{I}_1(T) = 0$. ■

The following corollary is immediate.

Corollary 1 *If $(u_1^*, u - u_1^*)$ is minimal in $A_2(u, d^1, d^2)$ then $I_1^*(T) = 0$.*

Proposition 2 *For every system u, d^1, d^2 the minimal allocation in $A_2(u, d_1, d_2)$ is unique (provided it exists).*

Proof. Suppose $(u_1, u-u_1)$ and $(u_1^*, u-u_1^*)$ are minimal allocations. Hence for every t, $I_1(t) \leq I_1^*(t)$ and $I_1(t) \geq I_1^*(t)$ for $t \in [0,T]$. This means that $\int_0^t u_1(s)ds \leq \int_0^t u_1^*(s)ds$ and $\int_0^t u_1(s)ds \geq \int_0^t u_1^*(s)ds$ on $[0,T]$. Therefore $u_1 = u_1^*$ a.e. on $[0,T]$. ■

Now we formulate some sufficient conditions for the minimal allocation.

Proposition 3 *If $(u_1^*, u-u_1^*) \in A_2(u, d_1, d_2)$ is such that $I_1^*(t)(u-u_1^*) = 0$ a.e. on $[0,T]$ and $I_1^*(T) = 0$ then $(u_1^*, u-u_1^*)$ is the minimal allocation in $A_2(u, d^1, d^2)$.*

Proof. We need to show that for every $(u_1, u-u_1) \in A_2(u, d^1, d^2)$,

$$I_1(t) \geq I_1^*(t) \text{ } for \text{ } t \in [0,T]. \tag{4}$$

By Proposition 1 we can assume that

$$I_1(T) = 0. \tag{5}$$

If $t = t'$ is such that $I_1^*(t') = 0$ then inequality (4) follows. So consider the case

$$I_1^*(t') > 0.$$

Then by Remark 7 there exists $t_j \in \{t_1, t_2, ..., t_m\}$ and $t^* < t_j$ such that

$$t' \in (t^*, t_j),\ I_1^*(t) > 0 \text{ on } (t^*, t_j) \text{ and } I_1^*(t^*) = I_1^*(t_j) = 0. \tag{6}$$

By the assumption of this proposition $I_1^*(t)(u-u_1^*) = 0$ and so $u_1^* = u$ a.e. on (t^*, t_j).

First we show that

(i) $I_1(t) > 0$ on (t^*, t_j).

This will be done by contradiction. Suppose that $I_1(t) = 0$ for some $t \in (t^*, t_j)$. Then by the supposition, the inequality $u_1 \leq u$, and the equality $u_1^* = u$ a.e. on (t^*, t_j) we have

$$0 \leq I_1(t_j) = I_1(t) + \int_t^{t_j} u_1(s)ds - \sum_{\{k:\ t_k \in (t,t_j]\}} d_k^1 \leq 0 + \int_t^{t_j} u(s)ds - \sum_{\{k:\ t_k \in (t,t_j]\}} d_k^1$$

$$= \int_t^{t_j} u_1^*(s)ds - \sum_{\{k:\ t_k \in (t,t_j]\}} d_k^1 = \int_{t^*}^{t_j} u_1^*(s)ds - \sum_{\{k:\ t_k \in (t^*,t_j]\}} d_k^1$$

$$-\Big(\int_{t^*}^{t} u_1^*(s)ds - \sum_{\{k:\ t_k \in (t^*,t]\}} d_k^1\Big) = 0 - I_1^*(t) < 0, \text{ which is a contradiction.}$$

So (i) is true and taking into account (5) and the fact that $I_1(0)=0$ we conclude that there exists $t_i \in \{t_1, t_2, ..., t_m\}$ and $t^{**} < t_i$ with $(t^*, t_j) \subset (t^{**}, t_i)$ and $I_1(t^{**}) = 0$, $I_1(t_i) = 0$, $I_1(t) > 0$ on (t^{**}, t_i). Now in order to show (4) we prove that

(ii) $I_1(t') > I_1^*(t')$ for $t' \in (t^*, t_j)$.

For this observe that

$$0 = I_1(t_i) = 0 + \int_{t^{**}}^{t*} u_1(s)ds + \int_{t^*}^{t_j} u_1(s)ds + \int_{t_j}^{t_i} u_1(s)ds - \sum_{\{k:\ t_k \in (t^{**}, t_i]\}} d_k^1 \quad (7)$$

and

$0 = I_1^*(t_j) = \int\limits_{t^*}^{t_j} u(s)ds - \sum\limits_{\{k:\ t_k \in (t^*, t_j]\}} d_k^1.$

Therefore by these equalities

$$\int_{t^{**}}^{t*} u_1(s)ds - \sum_{\{k:\ t_k \in (t^{**}, t^*]\}} d_k^1$$
$$= -(\int_{t^*}^{t_j} u_1(s)ds - \sum_{\{k:\ t_k \in (t^*, t_j]\}} d_k^1) - (\int_{t_j}^{t_i} u_1(s)ds - \sum_{\{k:\ t_k \in (t_j, t_i]\}} d_k^1) + I_1^*(t_j)$$
$$= -(\int_{t^*}^{t_j} u_1(s)ds - \sum_{\{k:\ t_k \in (t^*, t_j]\}} d_k^1) - (\int_{t_j}^{t_i} u_1(s)ds - \sum_{\{k:\ t_k \in (t_j, t_i]\}} d_k^1)$$
$$+ \int_{t^*}^{t_j} u(s)ds - \sum_{\{k:\ t_k \in (t^*, t_j]\}} d_k^1$$
$$= \int_{t^*}^{t_j} (u(s) - u_1(s))ds - (\int_{t_j}^{t_i} u_1(s)ds - \sum_{\{k:\ t_k \in (t_j, t_i]\}} d_k^1).$$

So by this and (7) ((7) means that

$$-(\int_{t_j}^{t_i} u_1(s)ds - \sum_{\{k:\ t_k \in (t_j, t_i]\}} d_k^1) = \int_{t^{**}}^{t_j} u_1(s)ds - \sum_{\{k:\ t_k \in (t^{**}, t_j]\}} d_k^1)$$

we have

$$I_1(t') = \int_{t^{**}}^{t*} u_1(s)ds - \sum_{\{k:\ t_k \in (t^{**}, t^*]\}} d_k^1 + \int_{t^*}^{t'} u_1(s)ds - \sum_{\{k:\ t_k \in (t^*, t']\}} d_k^1$$
$$= \int_{t^*}^{t_j} (u(s) - u_1(s))ds - (\int_{t_j}^{t_i} u_1(s)ds - \sum_{\{k:\ t_k \in (t_j, t_i]\}} d_k^1) + \int_{t^*}^{t'} u_1(s)ds - \sum_{\{k:\ t_k \in (t^*, t']\}} d_k^1$$
$$= \int_{t^*}^{t'} (u(s) - u_1(s))ds + \int_{t'}^{t_j} (u(s) - u_1(s))ds + \int_{t^{**}}^{t_j} u_1(s)ds - \sum_{\{k:\ t_k \in (t^{**}, t_j]\}} d_k^1$$
$$+ \int_{t^*}^{t'} u_1(s)ds - \sum_{\{k:\ t_k \in (t^*, t']\}} d_k^1$$

$$= \int_{t^*}^{t'} (u(s) - u_1(s) + u_1(s))ds - \sum_{\{k:\ t_k \in (t^*, t']\}} d_k^1 + \int_{t'}^{t_j} (u(s) - u_1(s))ds$$
$$+ \int_{t^{**}}^{t_j} u_1(s)ds - \sum_{\{k:\ t_k \in (t^{**}, t_j]\}} d_k^1 \geq I_1^*(t') + 0 + I_1(t_j) \geq I_1^*(t').$$

We used: the fact that $u(s) = u_1^*(s)$ a.e. on (t^*, t'), $u(s) - u_1(s) \geq 0$ a.e. on (t', t_j), and $I_1(t_j) \geq 0$. Therefore (ii) is true. ■

5 A construction of the minimal allocation

We construct the allocation of $A_2(u, d^1, d^2)$ which satisfies the hypothesis of Proposition 3.

Observe that by the definition of the allocation system (Definition 2) for $k = 1, 2, ..., m$ we have

.

$$\int_0^{t_k} u(s)ds - \sum_{j \leq k} d_j^1 \geq \sum_{j \leq k} d_j^2 \tag{8}$$

and so

$$\sum_{j=1}^{k} \left(\int_{t_{j-1}}^{t_j} u(s)ds - d_j^1 \right) \geq \sum_{j \leq k} d_j^2$$

Hence if $\int_{t_{j-1}}^{t_j} u(s)ds - d_j^1 \geq 0$ for $j = 1, 2, ..., m$ then we can put

$$u_1^*(t) = \begin{cases} u(t) & \text{on} \quad (t_j^*, t_j), \\ 0 & \text{otherwise.} \end{cases}$$

where $t_j^* \in [t_{j-1}, t_j]$ is such that $\int_{t_j^*}^{t_j} u(s)ds - d_j^1 = 0$, $j = 1, 2, ..., m$, and

$u_2^*(t) = (u - u_1^*)(t)$.

It is easy to see that $(u_1^*, u_2^* = u - u_1^*) \in A_2(u, d_1, d_2)$, $I_1^*(t)(u - u_1^*) = 0$ on $[0, T]$ and $I_1^*(T) = 0$. Therefore by Proposition 3, (u_1^*, u_2^*) is the minimal allocation of $A_2(u, d^1, d^2)$.

Now let us consider the case

$$\int_{t_{j-1}}^{t_j} u(s)ds - d_j^1 < 0 \quad \text{for some } j.$$

In this case we construct a sequence of subintervals $(t_{j_1}^*,t_{j_1})$,$(t_{j_2}^*,t_{j_2})$,...,$(t_{j_r}^*,t_{j_r})$, $r\leq m$, on which we will put $u_1^*(t)=u(t)$. This will be done by backward induction.

(i) If $\int\limits_{t_{m-1}}^{t_m} u(s)ds)\geq d_m^1$ then put $t_{j1}=t_m$ and $t_{j1}^*=t'$ where $t'\in[t_{m-1},t_m]$ is such that $\int\limits_{t'}^{t_{j1}} u(s)ds=d_m^1$.

(ii) If $\int\limits_{t_{m-1}}^{t_m} u(s)ds)<d_m^1$ then put $t_{j_1}=t_m$ and $t_{j_1}^*=t'$ where t' is such that that $t'<t_{m-1},$ $\int\limits_{t'}^{t_{j_1}} u(s)ds=\sum\limits_{\{k\,:\,t_k>t'\}} d_k^1$ and $\int\limits_{t}^{t_{j_1}} u(s)ds<\sum\limits_{\{k\,:\,t_k>t\}} d_k^1$ for $t>t'$.

Observe that t'exists because by (8) $\int\limits_0^{t_m} u(s)ds-\sum\limits_{k\leq m} d_k^1\geq\sum\limits_{k\leq m} d_k^2\geq 0.$

If $\{k:\ t_k\leq t_{j_1}^*,$ and $d_k^1>0\}\neq\emptyset$ then put

$j_2=\max\{k\ ;\ t_k\leq t_{j_1}^*,$ and $d_k^1>0\}$, otherwise put $j_2=0$ and the construction is finished. If $j_2\geq 1$ then we examine two cases.

(i)$'$ If $\int\limits_{t_{j_2-1}}^{t_{j_2}} u(s)ds)\geq d_{j_2}^1$ then put $t_{j_2}^*=t'\in[t_{j_2-1},t_{j_2}]$ where $t'$is such that $\int\limits_{t'}^{t_{j_2}} u(s)ds=d_{j_2}^1$.

(ii)$'$ If $\int\limits_{t_{j_2-1}}^{t_{j_2}} u(s)ds)<d_{j_2}^1$ then put $t_{j_2}^*=t'$ where t' is such that $t'<t_{j_2-1}$ and $\int\limits_{t'}^{t_{j_2}} u(s)ds=\sum\limits_{\{k\,:\,t_k>t',k\,\leq\,j_2\}} d_k^1$ and $\int\limits_{t}^{t_{j_2}} u(s)ds<\sum\limits_{\{k\,:t_k>t,\,k\leq\,j_2\}} d_k^1$ for $t>t'$.

Observe that by (8), t' is well defined.

In this way by induction we can define the sequence $(t_{j_1}^*,t_{j_1})$,$(t_{j_2}^*,t_{j_2})$,...,$(t_{j_r}^*,t_{j_r})$, $r\leq m$.

Put

$$u_1^*(t)=\begin{cases} u(t) & \text{on} \quad (t_{j_k}^*,t_{j_k}),\ k=1,2,...,r, \\ 0 & \text{otherwise}, \end{cases}$$

$u_2^*(t)=(u-u_1^*)(t)$.

Now we show that the pair $(u_1^*(t)$,$u_2^*(t)=(u-u_1^*)(t))$ is an allocation.

From the definition of u_1^* it is not difficult to see that

$$I_1^*(t) = \int\limits_0^t u_1^*(s)ds - \sum_{\{k\,:\,t_k\,\leq\,t\}} d_k^1 \geq 0 \quad for \quad t\,\epsilon[0,T]$$

and

$$I_1^*(t) = 0 \text{ for } \ t \notin (t_{jk}^*, t_{j_k}),\ k = 1, 2, ..., r. \tag{9}$$

By the definition of the system we moreover have

$$\int\limits_0^t (u(s) - u_1^*(s))ds + \int\limits_0^t u_1^*(s)ds - \sum_{\{k\,:\,t_k\leq\,t\}} d_k^1 \geq \sum_{\{k\,:\,t_k\leq\,t\}} d_k^2.$$

and so by (9) for $t \notin (t_{jk}^*, t_{j_k})$, $k = 1, 2, ..., r$, we have

$$\int\limits_0^t (u(s) - u_1^*(s))ds + \int\limits_0^t u_1^*(s)ds - \sum_{\{k\,:\,t_k\leq\,t\}} d_k^1 \tag{10}$$

$$= \int\limits_0^t (u(s) - u_1^*(s))ds + I_1^*(t) = \int\limits_0^t (u(s) - u_1^*(s))ds = \int\limits_0^t u_2^*(s)ds \geq \sum_{\{k\,:\,t_k\leq\,t\}} d_k^2$$

Consider $t \in (t_{jk}^*, t_{j_k})$.

Since by (10) $\int\limits_0^{t_{j_k}} (u(s) - u_1^*(s))ds \geq \sum\limits_{\{i\,:\,t_i\,\leq\,t_{j_k}\}} d_i^2$ and since $u = u_1^*$ on (t_{jk}^*, t_{j_k}), we have

$$\int\limits_0^t u_2^*(s)ds = \int\limits_0^t (u(s) - u_1^*(s))ds = \int\limits_0^{t_{j_k}} (u(s) - u_1^*(s))ds \geq \sum_{\{i\,:\,t_i\,\leq\,t_{j_k}\}} d_i^2 \geq \sum_{\{i\,:\,t_i\leq\,t\}} d_i^2$$

Hence the pair $(u_1^*, u(s) - u_1^*(s))$ is an allocation.

Note that $T = t_m = t_{j_1}$. So by (9) $I_1^*(T) = 0$ and moreover for $t \notin (t_{jk}^*, t_{j_k}), k = 1, 2, ..., r$ we have $I_1^*(t) = 0$. On the interval (t_{jk}^*, t_{j_k}) we have $u - u_1^* = 0$. Therefore the hypotheses of Proposition 3 are satisfied and so the pair $(u_1^*, u(s) - u_1^*(s))$ is the minimal allocation.

Numerical example

We construct the minimal allocation for the following simple example.

Let $T=10$ and put

$$u(t)=\begin{cases} 1 & \text{for } [0,5), \\ 0 & \text{for } [5,7), \\ 1 & \text{for } [7,10]. \end{cases},$$

Put moreover $t_0=0, t_1=2, t_2=4, t_3=6, t_4=9, t_5=10$, and

$d^1=(d_1^1,d_2^1,d_3^1,d_4^1,d_5^1)=(0,1,3/2,1,1/2)$,

$d^2=(d_1^2,d_2^2,d_3^2,d_4^2,d_5^2)=(1,0,\ 1,\ \ 1,3/4)$.

It is easy to check that by Definition 2, u,d^1,d^2 is the allocation system.

Observe that $\int\limits_{t_4}^{t_5} u(s)ds = 1 \geq d_5^1 = 1/2$. So we put $t_{j_1}=t_5=10$, $t_{j_1}^*=19/2$.

Now, $j_2=\max\{k: t_k \leq t_{j_1}^* \text{ and } d_k^1>0\}=4$. So $t_{j_2}=t_4=9$. Note that $\int\limits_{t_3}^{t_{j_2}} u(s)ds = 2 > d_4^1 = 1$ and so $t_{j_2}^*=8$.

Next , $j_3=\max\{k: t_k \leq t_{j_2}^* \text{ and } d_k^1>0\}=3$. So $t_{j_3}=t_3=6$. Note that $\int\limits_{t_2}^{t_{j_3}} u(s)ds = 1 < d_3^1 = 3/2$. So $t_{j_3}^*=5/2$ because $\int\limits_{t_{j_3}^*}^{t_{j_3}} u(s)ds = d_3^1+d_2^1 = 5/2$ and for $t > t_{j_3}^* = 5/2$ we have $\int\limits_{t}^{t_{j_3}} u(s)ds = 5-t < d_3^1+d_2^1 = 5/2$ on (t,t_2) and $\int\limits_{t}^{t_{j_3}} u(s)ds = 5-t < d_3^1 = 3/2$ on $[t_2,t_{j_3}]$.

Observe that $\{k : t_k \leq t_{j_3}^* \text{ and } d_k^1>0\}=\emptyset$. So $j_4=0$ and the construction is finished.

The minimal allocation of the system u,d^1,d^2 is the following:

$$u_1^*(t)=\begin{cases} u(t) & \text{on the intervals } (5/2,6),(8,9),(19/2,10) \\ 0 & \text{otherwise} \end{cases}$$

and $u_2^*(t)=(u-u_1^*)(t)$.

As an immediate corollary to Propositions 2 and 3 and the fact that the minimal allocation exists we obtain the following theorem.

Theorem 1 *A pair $(u_1^*, u-u_1^*) \in A_2(u,d^1,d^2)$ is a minimal allocation for the system u,d^1,d^2 if and only if $I_1^*(t)(u-u_1^*)=0$ a.e. on $[0,T]$ and $I_1^*(T)=0$.*

It follows from the construction that $u_1^*=0$ a.e. on the intervals $[0,t_{j_r}^*]$, $[t_{jr},t_{j_{r-1}}^*]$, ..., $[t_{j_2},t_{j_1}^*]$. Since $u_2^*=u-u_1^*=0$ a.e. on the intervals (t_{jk}^*,t_{j_k}) $k=1,2,...r$ we have proved the next proposition.

Proposition 4 *If $(u_1^*, u_2^*=u-u_1^*)$ is a minimal allocation for a system u,d^1,d^2 then $u_1^*u_2^*=0$ a.e. on $[0,T]$.*

We note that the property formulated in the last proposition is not true in the case of discrete time production and discrete time demands (cf. Ref 3) and also in the case of continuous time production and continuous time demands (cf. Ref 5).

The next auxiliary observation follows from the construction of the minimal allocation.

Remark 8 *If u is a piecewise continuous function then the minimal allocation in $A_2(u, d^1, d^2)$ consists of a pair of piecewise continuous functions.*

6 Warehouse allocation problems with discrete demands

Observe that the minimal allocation solves the following warehouse problem.

Assume $u(t)$ denotes a given production rate of an item at the moment $t \in [0, T]$. The production has to satisfy the demands which appears at the moments $t_1, t_2, ..., t_m$, $0 = t_0 < t_1 <...< t_{m-1} < t_m = T, \quad m \geq 1$.

The demands come from two warehouses. So $d_j = d_j^1 + d_j^2$ where d_j^i denotes the demand of the i-th warehouse at the moment t_j.

Let h_1, h_2 with $h_1 > h_2$ denote the unit holding costs of warehouse 1 and 2 respectively.

The problem is to find an allocation plan u_1, u_2; $u_1 + u_2 = u$ on $[0, T]$ which satisfies the warehouse demands, so

$$I_1(t) = \int_0^t u_1(s)ds - \sum_{\{k\, : t_k \leq\, t\}} d_k^1 \geq 0 \text{ for } t \in [0, T],$$

$$I_2(t) = \int_0^t u_2(s)ds - \sum_{\{k :\ t_k \leq\, t\}} d_k^2 \geq 0 \text{ for } t \in [0, T]$$

and minimizes the following inventory holding cost:

$$C(u_1, u_2) = \int_0^T \{h_1 I_1(t) + h_2 I_2(t)\}dt = \int_0^T \{(h_1 - h_2) I_1(t) + h_2(I_1(t) + I_2(t))\}dt.$$

The second component of the integral is fixed. So 2-warehouse allocation problem $(W.ALL)_2$ takes the form.

$$
\begin{aligned}
(W.ALL)_2 \quad &: \qquad \min_{u_1,u_2} \int_0^T I_1(t)dt \\
I_1(t) &= \int_0^t u_1(s)ds - \sum_{\{k\ :t_k \leq\ t\}} d_k^1 \geq 0 \ \ for \ \ t \in [0,T], \\
I_2(t) &= \int_0^t u_2(s)ds - \sum_{\{k:\ t_k \leq\ t\}} d_k^2 \geq 0 \ \ for \ \ t \in [0,T], \\
u_1 &\geq 0, \ u_2 \geq 0; \ \ u_1 + u_2 = u \ on \ [0,T].
\end{aligned}
$$

By the definition of the minimal allocation the following remark is evident.

Remark 9 *The minimal allocation of $A_2(u, d^1, d^2)$ is the unique solution of $(W.ALL)_2$.*

Note that $(W.ALL)_2$ can be written in the form

$$
\min_{u_1} \int_0^T I_1(t)dt,
$$

$$
\begin{aligned}
\int_0^t u(s)ds - \sum_{\{k:\ \ t_k \leq\ t\}} (d_k^1 + d_k^2) \ &\geq \ I_1(t) = \int_0^t u_1(s)ds - \sum_{\{k\ :t_k \leq\ t\}} d_k^1 \geq 0 \ for \ t\epsilon[0,T], \\
0 \ &\leq \ u_1 \leq u \ \text{ a.e. on } \ [0,T].
\end{aligned}
$$

So by Remark9 if we put $\mathrm{d}_j^2 = 0$, $j = 1, 2, ..., m$, then we immediately obtain the following remark.

Remark 10 *The minimal allocation in $A_2(u, d^1, d^2 = \mathbf{0})$ is the unique solution of the following linear minimization problem (L.P):*

$$
(L.P) \qquad \min_{u_1} \int_0^T I_1(t)dt,
$$

$$
\begin{aligned}
I_1(t) &= \int_0^t u_1(s)ds - \sum_{\{k\ :t_k \leq\ t\}} d_k^1 \geq 0 \ \ for \ \ t \in [0,T], \\
0 &\leq u_1 \leq u \ \text{ a.e. on } \ [0,T].
\end{aligned}
$$

6.1 Problem $(W.All)_2^{'}$.

Let U, d^1, d^2 be a system. Consider a version of *(W.ALL)*$_2$ in which the equality $u_1 + u_2 = u$ is replaced by the inequality $u_1 + u_2 \leq U$. More precisely consider the problem $(W.ALL)_2^{'}$ of the form

$$(W.ALL)_2^{'} \qquad\qquad \min_{u_1, u_2} \int_0^T I_1(t)dt,$$

$$\begin{aligned}
I_1(t) &= \int_0^t u_1(s)ds - \sum_{\{k\ : t_k \leq\ t\}} d_k^1 \geq 0 \quad for \quad t \in [0, T], \\
I_2(t) &= \int_0^t u_2(s)ds - \sum_{\{k:\ \ t_k \leq\ t\}} d_k^2 \geq 0 \quad for \quad t \in [0, T], \\
u_1 &\geq 0,\ u_2 \geq 0, \quad u_1 + u_2 \leq U \ on\ [0, T].
\end{aligned}$$

It turns out that the solution of the problem is also given by some allocations.

Proposition 5 *If (u_1^*, u_2^*) is a solution of $(W.ALL)_2^{'}$ then (u_1^*, u_2^*) is the minimal allocation of $A_2(u^*, d^1, d^2)$ where u^* is such that $(u^*, U - u^*)$ is the minimal allocation of $A_2(U, d^1 + d^2, \mathbf{0})$.*

Proof. Observe that

(i) if in the problem $(W.ALL)_2^{'}$ we put $u_1 = u_1^*$ then by Remark 10 $(u_2^*, U - u_1^* - u_2^*)$ is the minimal allocation of $A_2(U - u_1^*, d^2, \mathbf{0})$,

(ii) if in the problem $(W.ALL)_2^{'}$ we put $u_2 = u_2^*$ then by Remark 10 $(u_1^*, U - u_2^* - u_1^*)$ is the minimal allocation of $A_2(U - u_2^*, d^1, \mathbf{0})$.

Hence by the Theorem we have

$I_2^*(t)(U - u_1^* - u_2^*)(t) = 0$ a.e. on $[0, T]$ and $I_2^*(T) = 0$;

$I_1^*(t)(U - u_1^* - u_2^*)(t) = 0$ a.e. on $[0, T]$ and $I_1^*(T) = 0$.

Therefore $(I_2^* + I_1^*)(U - u_1^* - u_2^*) = 0$ and $(I_2^* + I_1^*)(T) = 0$. So once again by the Theorem we conclude that $(u_1^* + u_2^*, U - u_1^* - u_2^*)$ is the minimal allocation of $A_2(U, d^1 + d^2, 0)$. Since the minimal allocation is unique it follows that $u^* = u_1^* + u_2^*$. This observation and the fact that (u_1^*, u_2^*) is a solution of $(W.ALL)_2^{'}$imply that the pair solves$(W.ALL)_2$for the system u^*, d^1, d^2. So Remark 9 finishes the proof. ■

6.2 n-dimensional warehouse allocation problem

It is clear that the warehouse allocation results can be extended to the case of n warehouses, with $n > 2$. We list the results without proofs because the main idea is presented for the case $n = 2$.

Let $u, d^1, d^2, ..., d^n$ be a system. Consider the warehouse allocation problem with n warehouses:

$$(W.ALL)_n \qquad \min_{u_1,u_2,...u_n} \int_0^T (h_1 I_1(t) + h_2 I_2(t) + ... + h_n I_n(t))dt,$$

$$I_i(t) = \int_0^t u_i(s)ds - \sum_{\{j:t_j \le t\}} d_j^i \ge 0 \quad \text{on } [0,T], \quad i = 1, 2, ..., n,$$

$$u_1 + u_2 + ... + u_n = u, \quad u_1 \ge 0, \quad u_2 \ge 0, ..., u_n \ge 0 \text{ a.e. on } [0,T].$$

We recall that $h_1 > h_2 > ... > h_n \ge 0$.

Proposition 6 . *Put $u_0^* = 0$. If $u_1^*, u_2^*, ..., u_n^*$ is a solution of $(W.ALL)_n$ for the system $u, d^1, d^2, ..., d^n$ then for $i = 1, 2, ..., n-1$ the pair $(u_i^*, u - u_1^* - u_2^* - \cdots - u_i^*)$ is the minimal allocation of $A_2(u - u_1^* - u_2^* - \cdots - u_{i-1}^*, d^i, d^{i+1} + d^{i+2} + \cdots + d^n)$*

Consider the version $(W.All)_n^{'}$ with $n > 2$ for the system $U, d^1, d^2, ..., d^n$

$$(W.ALL)_n^{'} \qquad \min_{u_1,u_2,...u_n} \int_0^T (h_1 I_1(t) + h_2 I_2(t) + ... + h_n I_n(t))dt,$$

$$I_i(t) = \int_0^t u_i(s)ds - \sum_{\{j:t_j \le t\}} d_j^i \ge 0 \quad \text{on } [0,T], \quad i = 1, 2, ..., n,$$

$$u_1 + u_2 + ... + u_n \le U, \quad u_1 \ge 0, \quad u_2 \ge 0, ..., u_n \ge 0 \text{ a.e. on } [0,T].$$

Proposition 7 *Put $u_0^*=0$. If $u_1^*, u_2^*, ..., u_n^*$ is a solution of $(W.ALL)_n'$ for the system $u, d^1, d^2, ..., d^n$ then for $i=1,2,...,n-1$ the pair $(u_i^*, u-u_1^*-u_2^*-\cdots-u_i^*)$ is the minimal allocation of $A_2(u^*-u_1^*-u_2^*-\cdots-u_{i-1}^*, d^i, d^{i+1}+d^{i+2}+\cdots+d^n)$ where u^* is such that $(u^*, U-u^*)$ is the minimal allocation of $A_2(U, d^1+d^2+\cdots+d^n, 0)$.*

Remark 11 *Observe that by the construction of the minimal allocation Propositions 7 and 6 give the construction of solutions of the linear n-dimensional warehouse allocation problems $(W.ALL)_n$ and $(W.ALL)_n'$.*

7 Applications of the allocation results to the inventory problems

Now let us come back to the necessary conditions for optimal solutions of the inventory problem $(I.P)$ (and $(I.P)', (I.P)''$). Consider the problem $(Au.P)$. By the description of $(W.ALL)_2$ we can observe that if $j>i$ then $(Au.P)$ is $(W.All)_2$ for the system $(\widehat{u}_i+\widehat{u}_j, d^i, d^j)$.

So Remark 9 implies that $(\widehat{u}_i, \widehat{u}_j)$ is the minimal allocation of the set $A_2(\widehat{u}_i+\widehat{u}_j, d^i, d^j)$.

Therefore Theorem 1 and Proposition 4 give the following qualitative property of optimal solutions of the inventory problem $(I.P)$ ($(I.P)'$, $(I.P)''$)).

Property 1. *If the sequence $\widehat{u}_1, \widehat{u}_2, ..., \widehat{u}_n$ is an optimal solution of (I.P) ($(I.P)'$, $(I.P)''$)) and the sequence $\widehat{I}_1, \widehat{I}_2, ...\widehat{I}_n$ denotes the corresponding inventory levels then for $i=1,2,...,n-1$ and $j>i$ we have $\widehat{I}_i(t)\cdot\, \widehat{u}_j(t)=0$ a. e. on* $[0,T]$.

Property 2. *If the sequence $\widehat{u}_1, \widehat{u}_2, ..., \widehat{u}_n$ is an optimal solution of (I.P) ($(I.P)'$, $(I.P)''$)) then for $j\neq i$ we have $\widehat{u}_i(t)\cdot\widehat{u}_j(t)=0$ a.e. on* $[0,T]$.

7.1 Remarks to Properties 1 and 2

We would like to mention that solutions of the inventory problems have properties similar to **Property 1** both for discrete time (this mans that both production and demands are discrete cf. Ref. 3), and for continuous time (both production and demands are described by density functions cf. Ref. 5). Note that the result is similar to a Wagner-Within type condition (cf. Ref. 7) for a one-product case. It follows from the result that in the case of optimal production plan if the inventory level for a product is positive then the inventory level for a product with lower holding cost is also positive provided that demand is positive. In fact, if $I_i(t')>0$ then $I_i(t)>0$ on

(t', t_k) for some k. So $u_2 = 0$ on this interval. Therefore if $d_k^2 > 0$ then it has to be $I_j = constant > 0$ on (t', t_k).

Observe moreover that in contrast to **Property 1, Property 2** is true for solutions of the specific "continuous-discrete" problem considered here. **Property 2** is true for the above mentioned problems in neither only continuous (cf. Ref. 5) nor only discrete time cases (cf. Ref. 3). The property means that only one product can be produced at any given time.

Remark that in the cases of strictly continuous or discrete models it may happen that on some intervals or at some successive time points the optimal inventories are empty, i.e. $\widehat{I}_i(t) = \widehat{I}_j(t) = 0$. This corresponds to the situations in which the corresponding optimal productions agree with the demands d_i and d_j(demands may be different from zero). In the case considered here the production cannot equal the demand because the production is continuous and the demand is discrete. This is the reason why the result obtained here is different from other ones.

Note moreover that both properties are true for any joint replenishment (or setup) cost function $c(\cdot)$ $(C(\cdot))$.

Although problem $(I.P)$ belongs to a class called in the literature the parallel-machine multiproduct problems **Property 2** reduces $(I.P)$ in a sense to a single-machine problem because the optimal solution has the property that only one product can be produced at any given time.

7.2 Transformation of $(I.P)$, $(I.P)'$, $(I.P)''$ to a one - dimensional inventory problem

By the allocation results the multi-product problem $(I.P)$ can be transformed to a one-product inventory problem $(\widehat{I.P})$ with holding cost defined by the cost of the minimal allocation. We now formulate this fact more precisely.

Proposition 8 *The functions $\widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_n$ solve (I.P) if and only if*

(1^0) $\widehat{u} = \widehat{u}_1 + \widehat{u}_2 + \cdots + \widehat{u}_n)$ *solves the following inventory problem for one product* :

$$(\widehat{I.P}) \qquad \min_u \int_0^T \{c(u(t)) + h(I(t))\}dt$$

under the constraints

$$(i) \quad I(t) = \int_0^t u(s)ds - \sum_{\{j:t_j \le t\}} (d_j^1 + d_j^2 + \cdots + d_j^n) \ge 0 \qquad on \ [0,T],$$

$$(ii) \qquad 0 \le u \le U \quad a.e. \ on \ [0,T],$$

where for fixed u satisfying (i)-(ii),

$$h(I(t)) \underset{def}{=} \sum_{i=1}^{n} h_i(\int_0^t u_i^*(s)ds - \sum_{\{k:t_k \le t\}} d_k^i),$$

$\{u_i^*, i = 1, 2, ..., n\}$ *denotes here the solution of* $(W.ALL)_n$ *for the system* $(u, d^1, d^2, ...d^n)$,

and

(2^0) the functions $\widehat{u}_1, \widehat{u}_2,...,\widehat{u}_n$ *solve* $(W.ALL)_n$ *for the system* $(\widehat{u}, d^1, d^2, ...d^n)$.

Proof. (a) Let $u_1, u_2, ..., u_n$ satisfy the constraints of $(I.P)$. Put $u(t) = u_1(t) + u_2(t) + \cdots + u_n(t)$ and $I(t) = I_1(t) + I_2(t) + \cdots + I_n(t)$. Suppose $(\widehat{u}_1, \widehat{u}_2, ..., \widehat{u}_n)$ satisfies (1^0) and (2^0). So $(\widehat{u}_1, \widehat{u}_2, ..., \widehat{u}_n)$ solves $(I.P)$ because $\int_0^T \{c(u(t)) + h_1 I_1(t) + h_2 I_2(t) + \cdots + h_n(t) I_n(t))\}dt \ge \int_0^T \{c(u(t)) + h(I(t)\}dt$

$$\ge \int_0^T \{c(\widehat{u}(t)) + h(\widehat{I}(t)\}dt = \int_0^T \{c(\widehat{u}(t)) + h_1 \widehat{I}_1(t) + h_2 \widehat{I}_2(t) + \cdots + h_n \widehat{I}_n(t)\}dt.$$

(b) Let $\widehat{u}_1, \widehat{u}_2,...,\widehat{u}_n$ solve $(I.P)$. Put $\widehat{u} = \widehat{u}_1 + \widehat{u}_2 + \cdots + \widehat{u}_n$. So it is clear that $(\widehat{u}_1, \widehat{u}_2,...,\widehat{u}_n)$ is a solution of $(W.ALL)_n$ for the system $(\widehat{u}, d^1, d^2, ..., d^n)$, which means that ($2^0$) is satisfied. Moreover, we have $\int_0^T \{c(\widehat{u}(t)) + h(\widehat{I}(t)\}dt \le \int_0^T \{c(u(t)) + h(I(t)\}$ for any u satisfying $(i), (ii)$. This means that $\widehat{u}$ solves $(\widehat{I.P})$. ■

Remark. Observe that the proposition is true for $(I.P)'$ *and* $(I.P)''$. *The proof is identical.*

8 An example. Final remarks

As an example let us consider the problem $(I.P)''$ of the form

$$(E.I.P) \qquad \min_{u_1,u_2,u_3} \{C(u_1 + u_2 + u_3) + \int_0^T (h_1 I_1(t) + h_2 I_2(t) + h_3 I_3(t))dt\},$$

$$
\begin{aligned}
I_i(t) &= \int\limits_0^t u_i(s)ds - \sum_{\{j:t_j \le t\}} d_j^i \ge 0 \qquad \text{on } \; [0,T], \quad i=1,2,3. \\
& u_1, u_2, u_3 \text{ are an integrable functions such that} \\
u_1(t) &\ge 0, \;\; u_2(t) \ge 0, \;\; u_3(t) \ge 0 \;\; \text{and} \;\; u_1(t)+u_2(t)+u_3(t) \in \{0,1\}.
\end{aligned}
$$

If a solution of the problem exists then $C(u_1+u_2+u_3) < \infty$. and so we can assume that $u_1+u_2+u_3$ is a piecewise-continuous function taking values zero or one. The equality $u_1(t)+u_2(t)+u_3(t)=1$ means that the production is going on. The equality $u_1(t)+u_2(t)+u_3(t)=0$ means that the production is stopped.

Remark 8 and **Property 2** imply that $\widehat{u}_1$,$\widehat{u}_2$,$\widehat{u}_3$ are piecewise-continuous functions such that $\widehat{u}_i \cdot \widehat{u}_j = 0$ for $i \ne j$. The observations imply that it is sufficient to consider the following $(E.I.P)^*$ instead of $(E.I.P)$.

$$
(E.I.P)^*. \qquad \min_{u_1,u_2,u_3} \{C(u_1+u_2+u_3) + \int\limits_0^T (h_1 I_1(t) + h_2 I_2(t) + h_3 I_3(t))dt\}
$$

$$
\begin{aligned}
I_i(t) &= \int\limits_0^t u_i(s)ds - \sum_{\{j:t_j \le t\}} d_j^i \ge 0 \qquad \text{on } \; [0,T], \quad i=1,2,3, \\
& u_1, u_2, u_3 \text{ are a piecewise- continuous functions such that} \\
u_i(t) &\in \{0,1\} \; i=1,2,3 \;\; \text{and} \;\; u_i(t) \cdot u_j(t) = 0 \text{ for } i \ne j.
\end{aligned}
$$

Therefore we come to the situation when only one product can be produced at a time. Note that the problem $(E.I.P)^*$ essentially reduces the set of admissible production plans. The problem $(E.I.P)^*$ is solved in the paper Ref. 4 by the transformation to the one-product case (with the allocation result) and an application of a result of Hill′s (Ref. 8) to the one-product case.

As an illustration consider the following numerical example.

Numerical example

Let $n=3$, $T=25$, $h_1=3$, $h_2=5$, $h_3=6$, $a=1$,

$k=$	0	1	2	3	4	5	6	7	8	9	10	11	12
$t_k=$	0	3	4	6	8	9	10	14	15	19	20	24	25
$d_k^1=$		1	$\frac{5}{3}$	1	0	$\frac{2}{3}$	0	$\frac{2}{3}$	0	1	$\frac{5}{3}$	$\frac{2}{3}$	0
$d_k^2=$		1	$\frac{1}{5}$	1	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{7}{5}$	$\frac{6}{5}$	1	$\frac{6}{5}$	$\frac{2}{5}$	0	$\frac{3}{5}$
$d_k^3=$		0	0	0	0	0	$\frac{1}{3}$	1	1	1	0	$\frac{1}{3}$	1

Therefore

$$d_k = d_k^1 + d_k^2 + d_k^3 = (\ 2,\ \ \tfrac{28}{15},\ \ 2,\ \ \tfrac{4}{5},\ \ \tfrac{22}{15},\ \ \tfrac{26}{15},\ \ \tfrac{43}{15},\ \ 2,\ \ \tfrac{16}{5},\ \ \tfrac{31}{15},\ \ 1,\ \ \tfrac{8}{5},\ )$$

First we treat the problem as the 1-product problem with demand $d_k = d_k^1 + d_k^2 + d_k^3$.

According to the definition of the demand batch (cf. Ref. 8) in the example considered here there are two demand batches. The first is the interval $[0, 20]$ the second the interval $[22\frac{2}{5}, 25]$. (Observe that the demand batches correspond to the minimal allocation in the set $A_2(1, d = d^1 + d^2 + d^3, 0)$.

The holding costs are high (at least 3) then it is reasonable to produce the two batches separately. So the optimal solution of the 3-product problem is given by the minimal allocations in $A_3(1, d^1, d^2, d^3)$ on $[0, 20]$ and $A_3(1, d^1, d^2, d^3)$ on $[22\frac{2}{5}, 25]$. On the interval $[20, 22\frac{2}{5}]$ the optimal plan is to produce nothing.

One can check that the minimal allocations are the following.

In the case of the second batch we have:

$$\widehat{u}_1 = \begin{cases} 1 \text{ on } (23\frac{1}{3}, 24), \\ 0 \text{ otherwise,} \end{cases} \qquad \widehat{u}_2 = \begin{cases} 1 \text{ on } (24\frac{1}{5}, 25), \\ 0 \text{ otherwise,} \end{cases}$$

$$\widehat{u}_3 = \begin{cases} 1 \text{ on } (22\frac{2}{5}, 23\frac{1}{3}),\ (24, 24\frac{2}{5}), \\ 0 \text{ otherwise.} \end{cases}$$

In the case of the first batch we have:

$$\widehat{u}_1 = \begin{cases} 1 \text{ on } (17\frac{1}{3}, 20), (13\frac{1}{3}, 14), (8\frac{1}{3}, 9), (5, 6), (1\frac{1}{3}, 4), \\ 0 \text{ otherwise,} \end{cases}$$

$$\widehat{u}_2 = \begin{cases} 1 \text{ on } (15\frac{11}{15}, 17\frac{1}{3}), (14, 15), (12\frac{2}{15}, 13\frac{1}{3}), (9, 10), \left(6\frac{1}{3}, 8\frac{1}{3}\right), (4, 5), (\frac{2}{15}, 1\frac{1}{3}), \\ 0 \text{ otherwise,} \end{cases}$$

$$\widehat{u}_3 = \begin{cases} 1 \text{ on } (15, 15\frac{11}{15}), (10, 12\frac{2}{15}), (6, 6\frac{1}{3}), (0, \frac{2}{15}) \\ 0 \text{ otherwise.} \end{cases}$$

The example shows that the allocation result gives a useful method to discuss multiproduct inventory problems with joint production (or replenishment) cost. The minimal allocation allows us to construct the production batches for every product which is the main problem in the case of ″continuous-discrete″ models considered here. We believe also that the qualitative properties given by **Property 1** and **Property 2** and the transformation to the one-product case will be useful in solutions of some special cases of the general problems considered here.

In Section 6 we showed that the allocation method solves some linear n-dimensional optimization problems. The construction of the minimal allocation and Propositions 6 and 7 give an algorithm which solve n-dimensional problem by 2-dimensional subproblems. This property essentially makes easier the construction of the solution.

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