Property Gamma for von Nemann algebras connected with general commutation relations

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Abstract: We analyze a certain class of von Neumann algebras generated by selfadjoint elements $\omega_i = a_i + a_i^+$, for a_i , a_i^+ satisfying the general commutation relations:

$$a_i a_j^+ = \sum_{r,s} t_{j\,s}^{i\,r}\ a_r^+ a_s + \delta_{ij} Id.$$

We assume that operator T for which the constans $t_{j\,s}^{i\,r}$ are matrix coefficients satisfies the braid relation. Such algebras can be continuously embedded into some closure of the set of finite linear combinations of vectors $e_{i_1} \otimes \ldots \otimes e_{i_k}$ where $\{e_i\}$ is an orthonormal basis of a Hilbert space $\mathcal{H}$. The operator which represents the vector $e_{i_1} \otimes \ldots \otimes e_{i_n}$ is denoted by $\psi(e_{i_1} \otimes \ldots \otimes e_{i_n})$ and called the Wick product of the operators $\omega_{i_1}, \ldots, \omega_{i_n}$. In [K] we described explicitly the form of this product. Also we proved that under the assumption $dim\mathcal{H} = \infty$ all the von Neumann algebras considered are II_1 factors. In this paper we prove that our algebras does not have property Gamma of Murray and von Neumann.

Key words: the general commutation relations, Fock space, von Neumann algebras

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0. Introduction

In this paper we analyze a certain class of algebras of operators. We will consider the operators which satisfy the general commutation relations of the form

$$(\star) \qquad a_i a_j^+ = \sum_{r,s\in\mathbf{I}} t_{js}^{ir}\, a_r^+ a_s + \delta_{ij} Id,$$

where $t_{js}^{ir} \in \mathbf{C}$ and $a_j^+ = a_j^*$.

The constants t_{js}^{ir} determine an operator $T \in L(\mathcal{H}\otimes\mathcal{H})$ for which these constants are matrix coefficients, i.e.

$$T(e_a \otimes e_b) = \sum_{c,d} t_{ab}^{cd}\, e_c \otimes e_d,$$

where $a, b, c, d \in \mathbf{I}$ and $\mathcal{H}$ is a Hilbert space with an orthonormal basis $\{e_i\}_{i\in\mathbf{I}}$.

For a proper T the relations $(\star)$ have a unique representation constructed from a cyclic vector Ω with the property $a_i\Omega = 0$ for all operators a_i. This representation is called the Fock representation. It was proved (see [BSp]) that if T is a selfadjoint contraction on the Hilbert space $\mathcal{H}\otimes\mathcal{H}$ and T satisfies the braid relation (or Yang-Baxter relation)

$$\textbf{(BR)} \qquad [(1\otimes T)(T\otimes 1)(1\otimes T) = (T\otimes 1)(1\otimes T)(T\otimes 1)] \quad \text{on } \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$$

then such a representation exists.

We study the von Neumann algebra $\Gamma_T(\mathcal{H})$ generated by the operators $\omega_i = a_i + a_i^+$.

For our needs we will use the following assumptions:

$$1.\, T = T^*,\ \|T\| = q < 1, \qquad 2.\, t_{js}^{ir} = t_{ij}^{rs}, \qquad 3.\, T \text{ satisfies the braid relation,}$$

$$4. \forall (i,j) \in I\times I \quad \text{the set } A_{ij} = \{(r,s); t_{js}^{ir} \neq 0\} \text{ is finite.}$$

Under these assumptions the algebra $\Gamma_T(\mathcal{H})$ can be continuously embedded into the completion of the space $\mathcal{F}(\mathcal{H}) = \mathbf{C}\Omega \bigoplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ with respect to some T-deformed scalar product $\langle\cdot\mid\cdot\rangle_T$. The map $D\colon \Gamma_T(\mathcal{H}) \to \mathcal{F}_T(\mathcal{H})$ defined as $D(X) = X\Omega$ gives the embedding. The operator $\psi(e_{i_1}\otimes\ldots\otimes e_{i_n})$ which corresponds to a vector $e_{i_1}\otimes\ldots\otimes e_{i_n}$ is called the Wick product of the operators $\omega_{i_1},\ldots,\omega_{i_n}$. Such an element of $\Gamma_T(\mathcal{H})$ exists and is unique. In [K] an explicit formula for the Wick product is given. It was also shown that under the assumption $dim\mathcal{H} = \infty$ all the algebras $\Gamma_T(\mathcal{H})$ considered are II_1 factors.

This paper is devoted to property Gamma of Murray and von Neumann. It was introduced to divide the class of non injective algebras into subclasses. Algebras which does not have property Gamma are not injective, but the reverse it is not true. A.Nou [N] showed that the algebra $\Gamma_T(\mathcal{H})$ is not injective. We are able to prove that under assumption $dim\mathcal{H} = \infty$ $\Gamma_T(\mathcal{H}$ does not have property Gamma.

1. The Fock representation and algebras $\Gamma_T(\mathcal{H})$.

This section is based on [BSp] and [Bo]. It contains all the definitions and theorems which we will use in this paper and a description of the Fock representation of the general commutation relations and the von Neumann algebra $\Gamma_T(\mathcal{H})$. Unless otherwise stated we assume that $T \in B(\mathcal{H} \otimes \mathcal{H})$ is a selfadjoint, strict contraction ($\|T\| < 1$) which satisfies the braid relation.
The following lemma is true:

Lemma 1. *For fixed $m \in \mathbf{N}$ consider the permutation group S_m and denote by π_i the transposition between i and $(i+1)$. Let $T = T^* \in B(\mathcal{H} \otimes \mathcal{H})$ and let T satisfy* **(BR)**. *Define now a function $\varphi : S_m \to B(\mathcal{F}(\mathcal{H}))$ by quasi multiplicative extension of*

$$\varphi(e) = 1 \quad , \quad \varphi(\pi_i) = T_i,$$

where

$$T_i(f_1 \otimes \ldots \otimes f_s) = f_1 \otimes \ldots \otimes f_{i-1} \otimes T(f_i \otimes f_{i+1}) \otimes \ldots \otimes f_s.$$

This means that for a reduced word $\sigma = \pi_{i_1} \cdots \pi_{i_k}$ we put $\varphi(\sigma) = T_{i_1} \cdots T_{i_k}$.
Then φ is well defined, i.e. $\varphi(\sigma)$ depends only on the permutation σ.

Now we define for each $f \in \mathcal{H}$ a creation operator $a^+(f)$ and an annihilation operator $a(f)$ on a dense subset $\mathcal{F}_{fin}(\mathcal{H})$ of the Fock space $\mathcal{F}(\mathcal{H})$, which is the set of finite linear combinations of product vectors: $e_{i_1} \otimes \ldots \otimes e_{i_k}$.

Let $\quad l^+(f)\Omega = f \quad , \quad l^+(f) f_1 \otimes \ldots \otimes f_n = f \otimes f_1 \otimes \ldots \otimes f_n$
and $\quad l(f)\Omega = 0 \quad , \quad l(f) f_1 \otimes \ldots \otimes f_n = \langle f_1 \mid f \rangle f_2 \otimes \ldots \otimes f_n.$
We set $\quad a^+(f) = l^+(f) \quad , \quad a(f) = l(f)(1 + T_1 + T_1T_2 + \ldots + T_1T_2T_3 \cdots T_{n-1})$ on $\mathcal{H}^{\otimes n}$.

Remark 2. *$a(e_i)$, $a^+(e_i)$ are bounded and adjoint to each other with respect to the scalar product $\langle \xi \mid \eta \rangle_T = \langle P^{(n)} \xi \mid \eta \rangle$ for $\xi \in \mathcal{H}^{\otimes n}, \eta \in \mathcal{H}^{\otimes m}$, where $P^{(n)} = \sum_{\sigma \in S_n} \varphi(\sigma)$.*

According to the main theorem in [BSp] this scalar product is well defined, i.e. $P^{(n)}$ is nonnegative for every n (even strictly positive in case $\|T\| < 1$). One can check that the operators $a_i = a(e_i)$ and $a_i^+ = a^+(e_i)$ satisfy the relations $(\star)$ for $t_{j\,s}^{i\,r} = \langle T(e_j \otimes e_s) \mid e_i \otimes e_r \rangle$.

Now let recall some notation and definitions.

Notation 3. *For $g = \sum \alpha_{\underline{i}} e_{i_1} \otimes \ldots \otimes e_{i_n}$ we write*

$$a^+(g) = \sum \alpha_{\underline{i}} a_{i_1}^+ \cdots a_{i_n}^+ \quad \text{and} \quad a(g) = \sum \overline{\alpha_{\underline{i}}} a_{i_1} \cdots a_{i_n}.$$

Notice that usually $a(g)$ is not adjoint to $a^+(g)$.

Definition 4. *$\mathcal{F}_T(\mathcal{H})$ is the completion of the Fock space with respect to the scalar product $\langle \cdot \mid \cdot \rangle_T$. In this notation $\Gamma_T(\mathcal{H})$ is the von Neumann algebra of operators on $\mathcal{F}_T(\mathcal{H})$ generated by selfadjoint operators $\omega_i = a_i + a_i^+$.*

Definition 5. *Normal ordered term is an expression of the form* $a^+_{i_1}\cdots a^+_{i_l}a_{i_{l+1}}\cdots a_{i_m}$. *By the Diamond Lemma each element of* $alg\{a_i, a^+_i\}_{i\in\mathbf{I}}$ *can be written as a sum of normal ordered terms.*

Proposition 6. *[BSp]*
For the algebra $\Gamma_T(\mathcal{H})$, Ω *is a cyclic and separating vector. Moreover the functional* $\tau_T(X) = \langle X\Omega\,|\Omega\rangle_T$ *is a faithfull normal state which is a trace if and only if* $t^{i\,r}_{j\,s} = t^{r\,s}_{i\,j}$.

The following result holds:

Lemma 7. *The map* $D\colon \Gamma_T(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ *defined as* $D(X) = X(\Omega)$ *is a continuous embedding of* $\Gamma_T(\mathcal{H})$ *into* $\mathcal{F}_\mathcal{T}(\mathcal{H})$ *which extends to a unitary isomorphism of* $L^2(\Gamma_T(\mathcal{H}), \tau_\mathcal{T})$ *onto* $\mathcal{F}_\mathcal{T}(\mathcal{H})$.

Definition 8. *For* $\xi \in \mathcal{F}(\mathcal{H})$ *we denote by* $\psi(\xi)$ *a unique element of* $\Gamma_T(\mathcal{H})$ *such that* $\psi(\xi)\Omega = \xi$. *The operator* $\psi(f_1 \otimes \ldots \otimes f_n)$ *is called the* T*–Wick product of the operators* $\omega(f_1), \ldots, \omega(f_n)$. *By the previous lemma such an operator exists and is unique.*

We finish this section with the formula for this product.

Definition 9. *Let* $\mathcal{A}$ *be an involutive algebra generated by* $\{a_i\}$. U_k *is a linear operator* $U_k\colon \bigoplus_{m=k}^{\infty}\mathcal{H}^{\otimes m} \to \mathcal{A}$ *defined as a linear extension of the following:*

$$U_k(e_{i_1}\otimes\ldots\otimes e_{i_n}) = a^+(e_{i_1})\ldots a^+(e_{i_k})a(e_{i_{k+1}})\ldots a(e_{i_n}) \ \text{ for } n \geq k.$$

Theorem 10. *Let* $T \in B(\mathcal{H}\otimes\mathcal{H})$ *satisfies I, II, III, IV. Let* $\psi(f_1\otimes\ldots\otimes f_n)$ *be the* T*-Wick product of* $\omega(f_1), \ldots, \omega(f_n)$. *Then*

$$\psi(f_1\otimes\ldots\otimes f_n) = \sum_{k=0}^{n}\ \sum_{\sigma\in S_n/(S_k\times S_{n-k})} U_k\big[\varphi(\sigma)(f_1\otimes\ldots\otimes f_n)\big].$$

where σ *runs over the representatives of the right cosets of* $S_k\times S_{n-k}$ *in* S_n *with minimal number of inversions and* $\sigma(i_l) = i_{\sigma^{-1}(l)}$.

The representatives are well defined since by the Coxeter theorem:

$\forall\hat\sigma\in S_n\ \exists!\sigma\in S_n$, $\tilde\sigma\in S_k\times S_{n-k}$ s.t. $\hat\sigma = \tilde\sigma\sigma$ and σ has the minimal number of inversions.

The proof of the formula can be found in [K].

2. Property Γ

Let M be a finite factor, τ the canonical trace on M. We recall that M has property Γ of Murray and von Neumann if and only if for any $x_1, \ldots, x_m \in M$ and any $\epsilon > 0$ there exists a unitary operator $u \in M$, $\tau(u) = 0$ such that $\|ux_j - x_j u\|_2 \leq \epsilon$ for $j = 1, 2, \ldots, n$.

Theorem 11. **[Con]** *Let M be a factor of type II_1 acting in a standard way in $L^2(M, \tau)$. The following conditions on M are equivalent:*
(a) M has property Γ,
(b) The $C^\star$-algebra generated by M and its commutant M', where $M' = \{Y \in B(L^2(M, \tau));\, XY = YX \text{ for all } X \in M\}$ contains no non-zero compact operator. In other words $C^\star(M, M') \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

The possibility of $C^\star(M, M') \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$ was shown for the first time by Akemann and Ostrand [AO] in case of the von Neumann algebra of the free group with two generators. In this section we show that von Neumann algebra $\Gamma_T(\mathcal{H})$, $dim(\mathcal{H}) = \infty$ does not have the property Γ.

Theorem 12. *Let T be a selfadjoint contraction of norm $q < 1$. Assume that T satisfies I, II, III, IV. Then*

$$P_\Omega \in C^\star(\Gamma_T(\mathcal{H}), (\Gamma_T(\mathcal{H}))'),$$

where $P_\Omega(f) = \langle f|\Omega\rangle\, \Omega$ is the orthogonal projection on Ω in $\Gamma_T(\mathcal{H})$.

For the proof of the theorem we will need the right T-von Neumann algebra associated to $\mathcal{H}$. First we introduce the right creators and annihilators. Let $f \in \mathcal{H}$. Define

$$a_r^+(f)\Omega = f$$
$$a_r^+(f) f_1 \otimes \ldots \otimes f_n = f_1 \otimes \ldots \otimes f_n \otimes f.$$

Then $a_r(f)$ is determined by the formulas:

$$a_r(f)\Omega = 0$$
$$a_r(f) f_1 \otimes \ldots \otimes f_n = l_r(f)(Id + T_{n-1} + T_{n-1}T_{n-2} + \ldots + T_{n-1}T_{n-2} \cdots T_1) f_1 \otimes \ldots \otimes f_n,$$

where $l_r(f)(g_1 \otimes \ldots \otimes g_n) = \langle g_n \mid f\rangle\, g_1 \otimes g_2 \otimes \ldots \otimes g_{n-1}$.

Let $w_0(f_1 \otimes \ldots \otimes f_n) = f_n \otimes \cdots \otimes f_1$. For each $f \in \mathcal{H}$ $a_r^+(f) = w_0 a^+(f) w_0$ and $a_r(f) = w_0 a(f) w_0$. But $w_0 P_T^{(n)} w_0 = P_T^{(n)}$ what implies that w_0 is selfadjoint and unitary on $\mathcal{F}_T(\mathcal{H})$. To see this take a reduced word $\pi_{\underline{i}} = \pi_{i(1)} \ldots \pi_{i(k)}$ and check that $w_0 \varphi_T(\pi_{\underline{i}}) w_0 = \hat{T}_{n-i(1)} \ldots \hat{T}_{n-i(k)}$, where $\hat{T}_k = (\hat{T})_k$ and $\hat{T} = w_0 T w_0$. Further since $t_{j\,s}^{i\,r} = t_{i\,j}^{r\,s}$ and $t_{j\,s}^{i\,r} = t_{s\,j}^{r\,i}$ (the relations $(\star)$ respect the involution) we obtain $\hat{T} = T$. Hence

$$w_0 \varphi_T(\pi_{\underline{i}}) w_0 = T_{n-i_1} \ldots T_{n-i_k} = \varphi_T(w_0 \pi_{\underline{i}} w_0). \tag{10}$$

Now using the fact that $(w_0)^2 = Id$, we find that the transformation $I(\sigma) = w_0 \sigma w_0 = (w_0)^{-1} \sigma w_0$ is an automorphism of S_n and conclude that

$$w_0 P^{(n)} w_0 = \sum_{\sigma \in S_n} w_0 \varphi(\sigma) w_0 = \sum_{\sigma \in S_n} \varphi\big(w_0 \sigma w_0\big) = \sum_{\sigma \in S_n} \varphi(\sigma) = P^{(n)}.$$

Therefore $a_r^+(f)$ and $a_r(f)$ are bounded operators on $\mathcal{F}_T(\mathcal{H})$ and $a_r^+(f)$ is the adjoint to $a_r(f)$. We define the right von Neumann algebra as: $\Gamma_{T,r}(\mathcal{H}) = vN\{\omega_r(f),\, f \in \mathcal{H}_{\mathbf{R}}\}$, where $\omega_r(f) = a_r^*(f) + a_r(f)$.

Lemma 13.

$$\Gamma_{T,r}(\mathcal{H}) \subset (\Gamma_T(\mathcal{H}))' \,.$$

proof: Clearly $a_r^+(f)a^+(g) = a^+(g)a_r^+(f)$ and $a(g)a_r(f) = a_r(f)a(g)$. But $a_r^+(f)a(g) \neq a(g)a_r^+(f)$. However $t_{j\,s}^{i\,r} = t_{i\,j}^{r\,s}$ we have

$$L_{s+1}(T_s \cdots T_2T_1) = L_1(T_2T_3 \cdots T_{s+1}) \text{ on } \mathcal{H}_R^{\otimes n} \quad n \geq s+2.$$

and therefore $L_{n+1}(T_n \cdots T_1)(g\otimes f_1\otimes\ldots\otimes f_n\otimes f) = L_1(T_2T_3\ldots T_{n+1})(g\otimes f_1\otimes\ldots\otimes f_n\otimes f)$ what is equivalent to $l_r(f)(T_n \cdots T_1)(g\otimes f_1\otimes\ldots\otimes f_n) = l(g)(T_1T_2\ldots T_n)(f_1\otimes\ldots\otimes f_n\otimes f)$. Further

$$\begin{aligned} a_r(f)a^+(g)(f_1 \otimes \ldots \otimes f_n) &= a_r(f)(g \otimes f_1 \otimes \ldots \otimes f_n) = \\ g\otimes &= a_r(f)(f_1 \otimes \ldots \otimes f_n) + l_r(f)(T_n \cdots T_1)(g \otimes f_1 \otimes \ldots \otimes f_n) = \\ &= a^+(g)a_r(f)(f_1 \otimes \ldots \otimes f_n) + l_r(f)(T_n \cdots T_1)(g \otimes f_1 \otimes \ldots \otimes f_n) \end{aligned}$$

and

$$\begin{aligned} a(g)a_r^+(f)(f_1 \otimes \ldots \otimes f_n) &= a(g)(f_1 \otimes \ldots \otimes f_n \otimes f) = \\ &= a(g)(f_1 \otimes \ldots \otimes f_n) \otimes f + l(g)(T_1T_2\ldots T_n)(f_1 \otimes \ldots \otimes f_n \otimes f) = \\ &= a_r^+(f)a(g)(f_1 \otimes \ldots \otimes f_n) + l(g)(T_1T_2\ldots T_n)(f_1 \otimes \ldots \otimes f_n \otimes f). \end{aligned}$$

This gives $\omega_r(f)\omega(g) = \omega(g)\omega_r(f)$. ∎.

Proposition 14. *Let T satisfy I, II, III, IV and $dim(\mathcal{H}) = \infty$. Then for all $f \in \mathcal{F}_T(\mathcal{H})$ we have*

$$\left|\left| \frac{1}{N}\sum_{i=1}^{N} \omega(e_i)\omega_r(e_i)f - A_N(f) \right|\right|_2^2 \leq \frac{4C_q^2}{N}\|f\|_T^2,$$

where

$$A_N(f) = \frac{1}{N}\sum_{n=0}^{\infty}\sum_{i=1}^{N}(L_{n+1}(T_n\ldots T_1)D_i)X_n$$

$$f = \sum_{n=0}^{\infty} f_n \quad \text{for} \quad f_n \in \mathcal{H}^{\otimes n},$$

$$L_n(f_1 \otimes \ldots \otimes f_n \otimes f_{n+1}) = \langle f_{n+1} \,|\, f_n\rangle f_1 \otimes f_2 \otimes \ldots \otimes f_{n-1}.$$

Remark 15. *In case of the q-commutation relations operator A_N has a simple form i.e.*

$$A_N f = \sum_{n=0}^{\infty} q^n f_n.$$

proof: Observe that

$$\begin{aligned} (L_{n+1}(T_n\ldots T_1)D_i)f_1 \otimes \ldots \otimes f_n &= (L_{n+1}(T_n\ldots T_1))(e_i \otimes f_1 \otimes \ldots \otimes f_n \otimes e_i) \\ &= L_{n+1}\, q^n(f_1 \otimes \ldots \otimes f_n \otimes e_i \otimes e_i) \\ &= q^n\, f_1 \otimes \ldots \otimes f_n. \end{aligned}$$

For the proof of Proposition 14 we will need additional lemma. Before formulating it let us notice that

$$\begin{aligned}\omega(e_i)\omega_r(e_i)f =& e_i \otimes f \otimes e_i + e_i \otimes a_r(e_i)(f) \\ &+ a(e_i)(f \otimes e_i) + a(e_i)a_r(e_i)f.\end{aligned}$$

Furthermore $a(e_i)(f \otimes e_i) = a(e_i)(f) \otimes e_i + \sum_{n=0}^{\infty}(L_{n+1}(T_n \dots T_1)D_i)f_n$.

Lemma 16. *The following estimations hold:*

(a) $\|\sum_{i=1}^{N} e_i \otimes f \otimes e_i\|_T \leq NC_q^2\|f\|_T^2$

(b) $\|\sum_{i=1}^{N} a(e_i)a_r(e_i)f\|_T^2 \leq NC_q^2\|f\|_T^2$

(c) $\|\sum_{i=1}^{N} e_i \otimes a_r(e_i)f\|_T^2 \leq NC_q^2\|f\|_T^2$

(d) $\|\sum_{i=1}^{N} a(e_i)f \otimes e_i\|_T^2 \leq NC_q^2\|f\|_T^2$

Proof: Observe that $e_i \otimes f_n \otimes e_i \in \mathcal{H}^{\otimes n+2}$. Therefore for fixed i we have $\langle e_i \otimes f_n \otimes e_i \,|\, e_i \otimes f_m \otimes e_i\rangle_T = 0$ if $n \neq m$. This yields

$$\begin{aligned}\|\sum_{i=1}^{N} e_i \otimes f \otimes e_i\|_T &= \sum_{n=0}^{\infty}\|\sum_{i=1}^{N} e_i \otimes f_n \otimes e_i\|_T^2 \\ &= \sum_{n=0}^{\infty}\langle\sum_{i=1}^{N} P_T^{(n+2)}(e_i \otimes f_n \otimes e_i) \,|\, \sum_{i=1}^{N} e_i \otimes f_n \otimes e_i\rangle \\ &\leq \sum_{n=0}^{\infty} C_q^2\langle\sum_{i=1}^{N}(I \otimes P_T^{(n)} \otimes I)(e_i \otimes f_n \otimes e_i) \,|\, \sum_{i=1}^{N} e_i \otimes f_n \otimes e_i\rangle\end{aligned}$$

Since $\langle(I \otimes P_T^{(n)} \otimes I)(e_i \otimes f_n \otimes e_i) \,|\, \sum_{i=1}^{N} e_j \otimes f_n \otimes e_j\rangle = 0$ for $i \neq j$ the expression in the line above is equal to the following:

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{i=1}^{N} C_q^2\langle(I \otimes P_T^{(n)} \otimes I)(e_i \otimes f_n \otimes e_i) \,|\, e_i \otimes f_n \otimes e_i\rangle = \\ &= C_q^2\sum_{n=0}^{\infty} N\langle P_T^{(n)} f_n \,|\, f_n\rangle = NC_q^2\sum_{n=0}^{\infty}\|f_n\|_T^2 = NC_q^2\|f\|_T^2.\end{aligned}$$

This gives (a).

For (b) define $B_i(f) = a_r(e_i)a(e_i)f$. By taking the adjoints on both sides we obtain $B_i^* f = a^+(e_i)a_r^+(e_i)f$. Further

$$\left(\sum_{i=1}^{N} a_r(e_i)a(e_i)\right)^* f = \sum_{i=1}^{N} a^+(e_i)a_r^+(e_i)f = \sum_{i=1}^{N} e_i \otimes f \otimes e_i.$$

From (a) we have that $\|\sum\limits_{i=1}^{N} a^+(e_i)a_r^+(e_i)\|_\infty \leq \sqrt{N}C_q$ what implies

$$\|\sum_{i=1}^{N} a(e_i)a_r(e_i)\|_\infty \leq \sqrt{N}C_q.$$

To prove (c) similarly as in (a) we write

$$\begin{aligned}
\|\sum_{i=1}^{N} e_i \otimes a_r(e_i)f\|_T &= \sum_{n=0}^{\infty} \|\sum_{i=1}^{N} e_i \otimes a_r(e_i)f_n\|_T^2 \\
&= \sum_{n=0}^{\infty} \langle \sum_{i=1}^{N} P_T^{(n)}(e_i \otimes a_r(e_i)f_n) \mid \sum_{i=1}^{N} e_i \otimes a_r(e_i)f_n\rangle \\
&\leq \sum_{n=0}^{\infty} C_q \langle \sum_{i=1}^{N} (I \otimes P_T^{(n-1)})(e_i \otimes a_r(e_i)f_n) \mid \sum_{i=1}^{N} e_i \otimes a_r(e_i)f_n\rangle \\
&= \sum_{n=0}^{\infty} \sum_{i=1}^{N} C_q \langle (I \otimes P_T^{(n-1)})(e_i \otimes a_r(e_i)f_n) \mid e_i \otimes a_r(e_i)f_n\rangle \\
&= C_q \sum_{n=0}^{\infty} \sum_{i=1}^{N} \langle P_T^{(n-1)}(a_r(e_i)f_n) \mid a_r(e_i)f_n\rangle \\
&= C_q \sum_{n=0}^{\infty} \sum_{i=1}^{N} \|a_r(e_i)f_n\|_T^2 \quad (\|a_r(e_i)\|_\infty \leq \sqrt{C_q}) \\
&\leq C_q \sum_{n=0}^{\infty} \sum_{i=1}^{N} C_q \|f\|_T^2 = C_q^2 N \|f\|_T^2.
\end{aligned}$$

(d) One can verify that $e_i \otimes a_r(e_i)f = a^+(e_i)a_r(e_i)f$
Also $a(e_i)f \otimes e_i = a_r^+(e_i)a(e_i)f = (a^+(e_i)a_r(e_i))^* f$.
Now (d) can be derived from (c). ■

To prove Theorem 12 we need to approximate the operator P_Ω. Good candidates are powers of operator A_N.

Lemma 17. *Let T and A_N be as in Proposition 14. Then*

$$\|A_N^k - P_\Omega\|_\infty \leq q^k.$$

proof:

$$\|A_N^k f - P_\Omega f\|_T^2 = \|\sum_{n=0}^{\infty} A_N^k f_n - f_0\|_T^2 = \|\sum_{n=1}^{\infty} A_N^k f_n\|_T^2.$$

Moreover $A_N : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$.

$$\|A_N f_n\|_T^2 \leq q^n \|f_n\|_T^2.$$

To deduce that see first $\|L_{n+1}(T_n \cdots T_2 T_1) D_j\| \leq \|(T_n \cdots T_2 T_1)\| \leq q^n$ on $\mathcal{H}^{\otimes n}$ since $L_{n+1}: \mathcal{H}^{\otimes (n+1)} \otimes \{e_j\} \to \mathcal{H}^{\otimes n}$ is a contraction and D_j is an isometry.
Moreover for arbitrary $\sigma \in S_n$ we have

$$\begin{aligned} L_{n+1}(T_n \cdots T_2 T_1) D_j \varphi(\sigma) =& L_{n+1} \varphi(\rho \times Id) \varphi(Id \times \sigma \times Id) D_j \\ =& L_{n+1}(\varphi(\rho) \otimes Id)(\varphi(Id \times \sigma) \otimes Id). \end{aligned}$$

where $\rho = \pi_n \cdots \pi_1$. Further one can verify that $Inv(\rho) + Inv(Id \times \sigma) = Inv(\rho(Id \times \sigma))$. Furthermore $\rho(Id \times \sigma) = (\sigma \times Id)\rho$ and also $Inv(\sigma \times Id) + Inv(\rho) = Inv((\sigma \times Id)\rho)$. This implies that $\varphi(\rho)\varphi(Id \times \sigma) = \varphi(\rho(Id \times \sigma)) = \varphi((\sigma \times Id)\rho) = \varphi(\sigma \times Id)\varphi(\rho)$ and finally gives

$$\begin{aligned} L_{n+1}(T_n \cdots T_1) D_j \varphi(\sigma) =& L_{n+1}(\varphi(\sigma \times Id) \otimes Id)(\varphi(\rho) \otimes Id) D_j \\ =& \varphi(\sigma) L_{n+1}(\varphi(\rho) \otimes Id) D_j \\ =& \varphi(\sigma) L_{n+1}(T_n \cdots T_2 T_1) D_j. \end{aligned}$$

As a consequence we obtain that the operator $L_{n+1}(T_n \cdots T_2 T_1) D_j$ commutes with the operator $P^{(n)} = \sum_{\sigma \in S_n} \varphi(\sigma)$. which gives the scalar product.

Therefore A_N has the same norm with respect to non-deformed and T-deformed norm on $\mathcal{F}(\mathcal{H})$, which is a dense subset of $\mathcal{F}_T(\mathcal{H})$.

It means

$$\|\sum_{n=1}^{\infty} A_N^k f_n\|_T^2 = \sum_{n=1}^{\infty} \|A_N^k f_n\|_T^2 \leq \sum_{n=1}^{\infty} q^{kn} \|f_n\|_T^2 \leq q^{2k} \|f\|_T^2. \qquad \blacksquare$$

proof of Theorem 12

Define $B_N = \frac{1}{N}\sum_{i=1}^{N} \omega(e_i)\omega_r(e_i) \in C^\star(\Gamma_T(\mathcal{H}), \Gamma_T(\mathcal{H})')$. We can approximate A_N^k by B_N^k. Observe that from the above considerations we have $\|A_N\|_\infty = 1$.
We can also estimate the norm of B_N,

$$\|B_N\|_\infty \leq \|\omega(e_i)\|_\infty \|\omega_r(e_i)\|_\infty \leq 4C_q.$$

Using the formula $B^k - A^k = B(B^{k-1} - A^{k-1}) + (B - A)A^{k-1}$ one can inductively prove that

$$\|B_N^k - A_N^k\|_\infty \leq \|B_N - A_N\|_\infty \frac{C_q^{k+1} - 1}{C_q - 1}.$$

Now we using Lemma 16 we get

$$\|B_N^k - A_N^k\|_\infty \leq \frac{C_q}{\sqrt{N}} \frac{C_q^{k+1} - 1}{C_q - 1}$$

Combining this with Lemma 17 we obtain

$$\|B_N^k - P_\Omega\|_\infty \leq \|B_N^k - A_N^k\|_\infty + \|A_N^k - P_\Omega\|_\infty \leq \frac{C_q}{\sqrt{N}} \frac{C_q^{k+1} - 1}{C_q - 1} + q^k.$$

It is now obvious that for given $\epsilon > 0$ we can choose a suitable k and N such that $\|B_N^k - P_\Omega\|_\infty < \epsilon$ what implies that $P_\Omega \in C^\star(\Gamma_T(\mathcal{H}), (\Gamma_T(\mathcal{H}))')$. ■

Since M is a factor then $C^\star(M, M')$ is irreducible and we can use the following corollary from [Di] (Corollary 4.1.10):

Corollary 18. *If M is a II_1 factor in $B(\mathcal{H})$ then*

$$C^\star(M, M') \cap \mathcal{K}(\mathcal{H}) \neq 0 \Leftrightarrow \mathcal{K}(\mathcal{H}) \subset C^\star(M, M'),$$

where $\mathcal{K}(\mathcal{H})$ is the set of all compact operators on $\mathcal{H}$.

From this we have:

Corollary 19. $\mathcal{K}(\mathcal{F}_T(\mathcal{H})) \subset C^\star(\Gamma_T(\mathcal{H}), (\Gamma_T(\mathcal{H}))')$.

We can prove this corollary directly. To this end define a family of operators $T_{z,y}$ by the formula: $T_{z,y}x = \langle x \mid z\rangle_T\, y$. The closure of the linear span of $\{T_{z,y}\,,\ z, y \in \mathcal{F}_{fin}(\mathcal{H})\}$ is equal to the set of all compact operators in $B(\mathcal{F}_T(\mathcal{H}))$. One can obtain the operator $T_{z,y}$ as an operator norm limit of :

$$\left(\sum_{i=1}^{N} \frac{1}{N} \psi(y)\omega(e_i)\psi^\star(z)\omega_r(e_i)\right)^k \quad \text{as } N, k \to \infty.$$

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