Haagerup inequality and contractivity properties of Ornstein-Uhlenbeck semigroup for general commutation relations

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Abstract: We study a certain class of von Neumann algebras generated by selfadjoint elements $\omega_i = a_i + a_i^+$, for a_i , a_i^+ satisfying the general commutation relations:

$$a_i a_j^+ = \sum_{r,s} t_{j\,s}^{i\,r} \; a_r^+ a_s + \delta_{ij} Id.$$

We assume that operator T for which the constants $t_{j\,s}^{i\,r}$ are matrix coefficients satisfies the braid relation. Such algebras were investigated in [BSp] and [K] where the positivity of the Fock representation and factoriality was shown. In this paper we prove that T-Ornstein-Uhlenbeck semigroup $\Gamma_T(e^{-t})$, $t > 0$ arising from the second quantization procedure is hyper- and ultracontractive. The optimal bounds for hypercontractivity are also discussed.

Key words: the general commutation relations, Fock space, von Neumann algebras

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0. Introduction

Let $\mathcal{H}_R$ be a real Hilbert space and $\mathcal{H} = \mathcal{H}_{\mathcal{R}} + \mathrm{i}\mathcal{H}_{\mathcal{R}}$ its complexification. Denote by $\overline{\mathcal{H}}$ the complex conjugate of $\mathcal{H}$. Let $\overline{T} : \overline{\mathcal{H}} \otimes \mathcal{H} \to \mathcal{H} \otimes \overline{\mathcal{H}}$ be linear.
Consider the algebra of all tensors over H and $\overline{\mathcal{H}}$ and Id with tensor multiplication as a product denoted by $\Upsilon(\mathcal{H}, \overline{\mathcal{H}})$. Define the Wick algebra as the following quotient algebra:

$$(\star) \qquad W(T) \simeq \Upsilon(\mathcal{H}, \overline{\mathcal{H}}) \big/ \langle \overline{f} \otimes g - \overline{T}(\overline{f} \otimes g) - \delta_{ij} Id \rangle .$$

This means that we identify tensors from the ideal with zero. For $f, g \in \mathcal{H}$ we set $a(f) = \overline{f}$, $a^+(g) = g$, where the action of the operators $a(f)$ and $a^+(g)$ is given by multiplication by $\overline{f}$ and g respectively. We are interested in positive representations of $W(T)$ that is the representations of the $a(f)$'s as operators on a Hilbert space such that $a^+(f)$ is the restriction of the operator adjoint to $a(f)$.

In this paper we will deal with the Fock representation. It is constructed from a cyclic vector Ω with the property $a(f)\Omega = 0$ for all $f \in \mathcal{H}$. Such a representation is unique and carries a unique hermitian form. There are various criteria implying that this form is positive definite and gives the appropriate scalar product. Most of them are written in the language of the properties of the operator $T \in L(\mathcal{H} \otimes \mathcal{H})$ defined as $\langle T(f \otimes g) | h \otimes w \rangle = \langle \overline{T}(\overline{h} \otimes f) | w \otimes \overline{g} \rangle$.
It was proved (see [BSp]) that if T is a selfadjoint contraction and satisfies the braid relation (or Yang-Baxter relation)

$$\textbf{(BR)} \qquad [(1 \otimes T)(T \otimes 1)(1 \otimes T) = (T \otimes 1)(1 \otimes T)(T \otimes 1)] \quad \text{on } \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$$

then the form is positive definite.

We study the von Neumann algebra $\Gamma_T(\mathcal{H})$ generated by the operators
$\omega(f) = a(f) + a^+(f)$, $f \in \mathcal{H}_R$.

The choice of the relations $(\star)$ was made since several examples of such structures are investigated in the literature. The case $T(f \otimes g) = (g \otimes f)$ is known as the canonical commutation relations (CCR). The case $T(f \otimes g) = -(g \otimes f)$ is known as the canonical anti-commutation relations (CAR). More generally for $T(f \otimes g) = q(g \otimes f)$, where $q \in [-1, 1]$ we use the name q-canonical commutation relations (q-CCR). The case $T = 0$ plays a fundamental role. In [Vo] Voiculescu proved that $\Gamma_0(\mathcal{H}) = VN(\mathbf{F}_N)$, where $\mathbf{F}_N$ denotes the free group with $N = dim\mathcal{H}$ generators. Other examples as well as those mentioned above were studied in a series of papers
(see e.g. [JSW],[L],[BSp],[BKSp],[PW] and references therein).

For our needs we will use the following assumptions:

1. $T = T^*$, $\|T\| = q < 1$, 2. T satisfies the braid relation,

3. Under the identification $\overline{\mathcal{H}}_R \simeq \mathcal{H}_R$ we have $T = \overline{T}$ on $\mathcal{H}_R$.

If we choose $\{e_i\}$ to be an orthonormal basis of $\mathcal{H}_R$ and define matrix coefficients of T by the formula $\langle T(e_a \otimes e_b)|e_c \otimes e_d\rangle$ than the relations ($\star$) have the form

$$a_i a_j^+ = \sum_{r,s} t_{j\,s}^{i\,r}\ a_r^+ a_s + \delta_{ij} Id.$$

and equation $T = \overline{T}$ on $\mathcal{H}_R$ can be rewritten as $t_{j\,s}^{i\,r} = t_{i\,j}^{r\,s}$.

Under assumptions 1, 2, 3 the algebra $\Gamma_T(\mathcal{H})$ can be continuously embedded into the completion of the space $\mathcal{F}(\mathcal{H}) = \mathbf{C}\ \Omega \bigoplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ with respect to T-deformed scalar product $\langle \cdot \mid \cdot \rangle_T$. The map $D: \Gamma_T(\mathcal{H}) \to \mathcal{F}_T(\mathcal{H})$ defined as $D(X) = X\Omega$ gives the embedding. The operator $\psi(f_1 \otimes \ldots \otimes f_n)$ which corresponds to a vector $f_1 \otimes \ldots \otimes f_n$ is called the Wick product of the operators $\omega(f_1), \ldots, \omega(f_n)$. Such an element of $\Gamma_T(\mathcal{H})$ exists and is unique. In [K] an explicit formula for the Wick product is given. It provides a useful tool for studying $\Gamma_T(\mathcal{H})$ (see [K] and [N] where factoriality and non-injectivity for $\Gamma_T(\mathcal{H})$ was shown).

This paper is devoted to the contractivity properties of Ornstein-Uhlenbeck semigroup. In Section 1 we describe the Fock representation, basic properties of the von Neumann algebra $\Gamma_T(\mathcal{H})$ and give the formula for the Wick product. Section 2 contains information about second quantization functor for q-CCR and its generalization to the case of general commutation relations. Following [BXu] we describe Ornsein-Uhlenbeck semigroup and some of its properties. In Section 3 we prove the main results of the paper. First we show that Ornstein-Uhlenbeck semigroup is ultracontractive i.e. for $X \in \Gamma_T(\mathcal{H})$ we have:

$$||U_t X||_\infty \leq C_q^{-\frac{3}{2}} t^{-\frac{3}{2}} ||X||_2 \quad , \quad 0 < t < 1,$$

where $||\cdot||_\infty$ stands for the operator norm and $||\cdot||_2 = \langle \cdot\Omega| \cdot \Omega\rangle_T$. In the second part we focus on hypercontractivity.

In his famous paper E.Nelson,(see [N]), proved that in case of CCR U_t is a contraction of L^p into L^r for some r. He gave an optimal condition for r.

Theorem N. *Let $1 \leq p, r < \infty$. Then U_t extends to a contraction from $L^p(\mathcal{H})$ to $L^r(\mathcal{H})$ if and only if $e^{-2t} \leq \frac{p-1}{r-1}$.*

This theorem has many generalizations. In particular E.Carlen and E.Lieb in [CL] extended it to the case of CAR algebras.In [Bia] Philippe Biane showed that Theorem N. holds also with Γ_q instead Γ_1 and with respect to the non-commutative L^p norms. We are able to prove that this also holds in general case for $p = 2$ and $r \in \mathbf{N}$, $r \geq 2$.

1. The Fock representation

This section is based on [BSp] and [Bo]. It contains all the definitions and theorems which we will use in this paper and a description of the Fock representation of the general commutation relations. Unless otherwise stated we assume that $T \in B(\mathcal{H} \otimes \mathcal{H})$ is a selfadjoint, strict contraction ($\|T\| = q < 1$) which satisfies the braid relation.
The following lemma is true:

Lemma 1. *For fixed $m \in \mathbf{N}$ consider the permutation group S_m and denote by π_i the transposition between i and $(i+1)$. Let $T = T^* \in B(\mathcal{H} \otimes \mathcal{H})$ and let T satisfy* **(BR)**. *Define now a function $\varphi : S_m \to B(\mathcal{F}(\mathcal{H}))$ by quasi multiplicative extension of*

$$\varphi(e) = 1 \quad , \quad \varphi(\pi_i) = T_i,$$

where

$$T_i(f_1 \otimes \ldots \otimes f_s) = f_1 \otimes \ldots \otimes f_{i-1} \otimes T(f_i \otimes f_{i+1}) \otimes \ldots \otimes f_s.$$

This means that for a reduced word $\sigma = \pi_{i_1} \cdots \pi_{i_k}$ we put $\varphi(\sigma) = T_{i_1} \cdots T_{i_k}$.
Then φ is well defined, i.e. $\varphi(\sigma)$ depends only on the permutation σ.

We will work with the following assumptions on T:

I. $T = T^*$, $\|T\| = q < 1$,

II. For $f, g \in \mathcal{H}_R$ we have $\langle T(f \otimes g) | h \otimes w\rangle = \langle T(h \otimes f) | w \otimes g\rangle$.

III. T satisfies the braid relation i.e. one has $T_1T_2T_1 = T_2T_1T_2$ on $\mathcal{H}^{\otimes 3}$,

Now we define for each $f \in \mathcal{H}$ a creation operator $a^+(f)$ and an annihilation operator $a(f)$ on the Fock space $\mathcal{F}(\mathcal{H}) = \mathbf{C}\Omega \bigoplus \mathcal{H}^{\otimes n}$.

Let $\quad l^+(f)\Omega = f \quad , \quad l^+(f)g_n = f \otimes g_n,\ g_n \in \mathcal{H}^{\otimes n}$.
and $\quad l(f)\Omega = 0 \quad , \quad l(f)f_1 \otimes \ldots \otimes f_n = \langle f_1 \mid f\rangle f_2 \otimes \ldots \otimes f_n$ and extends lineary.
We set $\quad a^+(f) = l^+(f) \quad , \quad a(f) = l(f)(1 + T_1 + T_1T_2 + \ldots + T_1T_2T_3 \cdots T_{n-1})$ on $\mathcal{H}^{\otimes n}$.

Remark 2. *$a(f)$, $a^+(f)$ for $f \in \mathcal{H}_R$ are bounded and adjoint to each other with respect to the scalar product $\langle \xi \mid \eta\rangle_T = \langle P^{(n)}\xi \mid \eta\rangle$ for $\xi \in \mathcal{H}^{\otimes n}, \eta \in \mathcal{H}^{\otimes m}$, where $P^{(n)} = \sum_{\sigma \in S_n} \varphi(\sigma)$.*

According to the main theorem in [BSp] this scalar product is well defined, i.e. $P^{(n)}$ is nonnegative for every n (even strictly positive in case $\|T\| < 1$). One can check that the operators $a(f)$ and $a^+(f)$ satisfy the relations $(\star)$.

Now let recall some notation and definitions.

Notation 3. *For $g = g_1 \otimes \ldots \otimes g_n$ we write*

$$a^+(g) = a^+(g_1) \cdots a^+(g_n) \quad \text{and} \quad a(g) = a(g_1) \cdots a(g_n).$$

We extend $a^+(g)$ lineary and $a(g)$ antilineary as a function on g.

Definition 4. *$\mathcal{F}_T(\mathcal{H})$ is the completion of the Fock space with respect to the scalar product $\langle \cdot \mid \cdot \rangle_T$. In this notation $\Gamma_T(\mathcal{H})$ is the von Neumann algebra of operators on $\mathcal{F}_T(\mathcal{H})$ generated by selfadjoint operators $\omega(f) = a(f) + a^+(f)$ for $f \in \mathcal{H}_R$.*

Definition 5. *Normal ordered term is an expression of the form*

$$a^+(f_1)\cdots a^+(f_l)a(f_{l+1})\cdots a(f_m).$$

By the Diamond Lemma each element of $W(T)$ can be written as a sum of normal ordered terms.

Proposition 6. *[BSp]*
For the algebra $\Gamma_T(\mathcal{H})$, Ω is a cyclic and separating vector. Moreover the functional $\tau_T(X) = \langle X\Omega \,|\Omega\rangle_T$ is a faithfull normal state which is a trace if and only if for $f, g \in \mathcal{H}_R$ we have $\langle T(f \otimes g)|h \otimes w\rangle = \langle T(h \otimes f)|w \otimes g\rangle$.

The following result holds:

Lemma 7. *The map $D: \Gamma_T(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ defined as $D(X) = X(\Omega)$ is a continuous embedding of $\Gamma_T(\mathcal{H})$ into $\mathcal{F}_T(\mathcal{H})$ which extends to a unitary isomorphism of $L^2(\Gamma_T(\mathcal{H}), \tau_T)$ onto $\mathcal{F}_T(\mathcal{H})$.*

Definition 8. *For $\xi \in \mathcal{F}(\mathcal{H})$ we denote by $\psi(\xi)$ a unique element of $\Gamma_T(\mathcal{H})$ such that $\psi(\xi)\Omega = \xi$. The operator $\psi(f_1 \otimes \ldots \otimes f_n)$ is called the T–Wick product of the operators $\omega(f_1), \ldots, \omega(f_n)$. By the previous lemma such an operator exists and is unique.*

Remark 9. *Conditions I,II and III hold in q-commutation relations case, where $T(f \otimes g) = q\, g \otimes f$, for $q \in [-1, 1]$. The operator P_q is defined by the linear extension of*

$$P_q(\Omega) = \Omega, \quad P_q(f_1 \otimes \ldots \otimes f_n) = \sum_{\pi \in S_n} q^{Inv(\pi)} f_{\pi(1)} \otimes \ldots \otimes f_{\pi(n)},$$

where S_n denotes the symetric group of permutations of n elements and $Inv(\pi)$ is the number of inversions of π i.e.

$$Inv(\pi) := \sharp\{(i, j), 1 \leq i < j \leq n, \pi(i) > \pi(j)\}.$$

It was proved that P_q is positive for $q \in (-1, 1)$. In this case the annihilation operator a_k acts by the following procedure:

$$a_k\Omega = 0$$
$$a_k(f_1 \otimes \ldots \otimes f_n) = \sum_{p=1}^{n} q^{p-1} \langle f_p|e_k\rangle f_1 \otimes \ldots \otimes \breve{f}_p \otimes \ldots \otimes f_n,$$

where $(\breve{f}_i)$ means that f_i has to be deleted in the tensor.
The formula for the Wick product in this case was proved in [BKSp].

Proposition 10. *Let $\psi(f_1 \otimes \ldots \otimes f_n)$ be the q–Wick product of $\omega(f_1), \ldots, \omega(f_n)$. Then*

$$\psi(f_1 \otimes \ldots \otimes f_n) =$$
$$= \sum_{k=0}^{n} \sum_{\sigma \in S_n/(S_k \times S_{n-k})} q^{Inv(\sigma)} a^+(f_{\sigma^{-1}(1)}) \ldots a^+(f_{\sigma^{-1}(k)}) \ldots a(f_{\sigma^{-1}(k+1)}) \ldots a(f_{\sigma^{-1}(n)}),$$

where σ runs over the representatives of the right cosets of $S_k \times S_{n-k}$ in S_n with minimal number of inversions. .

The representatives are well defined since by the Coxeter theorem:
$\forall \hat{\sigma} \in S_n \; \exists! \sigma \in S_n$, $\tilde{\sigma} \in S_k \times S_{n-k}$ s.t. $\hat{\sigma} = \tilde{\sigma}\sigma$ and σ has the minimal number of inversions.

In case of general case we have (see [K] for details)

Definition 11. *Let $W(T)$ be an involutive algebra generated by $\{a(f)\}$. U_k is a linear operator U_k: $\bigoplus_{m=k}^{\infty} \mathcal{H}^{\otimes m} \to W(T)$ defined as follows:*

$$U_k(f_1 \otimes \ldots \otimes f_n) = a^+(f_1) \ldots a^+(f_k) a(f_{k+1}) \ldots a(f_n) \quad \text{for } f_i \in \mathcal{H}_R \text{ and } n \geq k.$$

Theorem 12. *Let $T \in B(\mathcal{H} \otimes \mathcal{H})$ satisfy conditions I, II, III.*
Let $\psi(f_1 \otimes \ldots \otimes f_n) \in \Gamma_T(\mathcal{H})$ be the Wick product of $\omega(f_1), \ldots, \omega(f_n)$. Then

$$\psi(f_1 \otimes \ldots \otimes f_n) = \sum_{k=0}^{n} \sum_{\sigma \in S_n/(S_k \times S_{n-k})} U_k\big[\varphi(\sigma)(f_1 \otimes \ldots \otimes f_n)\big].$$

This formula is very useful for studying $\Gamma_T(\mathcal{H})$.

2. Ornstein-Uhlenbeck semigroup

This section is devoted to the second quantization. In [BKSp] the authors introduced the second quantization functor associated with $q-CCR$ algebra. It assigns to each real Hilbert space $\mathcal{H}_{\mathbf{R}}$ with complexification $\mathcal{H}$, a von Neumann algebra with specified trace-state $(\Gamma_q(\mathcal{H}), \tau_q)$ and to every contraction $A : \mathcal{H}_{1\mathbf{R}} \to \mathcal{H}_{2\mathbf{R}}$ between real Hilbert spaces a unital completely positive trace preserving map: $\Gamma_q(A) : \Gamma_q(\mathcal{H}_1) \to \Gamma_q(\mathcal{H}_2)$ defined as follows:

$$\Gamma_q(A)\psi(f_1 \otimes \ldots \otimes f_n) = \psi(Af_1 \otimes \cdots \otimes Af_n).$$

Following these ideas we can introduce T-second quatization. We refer to [BXu] for details of this theory. Let $A_{\mathbf{R}} : \mathcal{H}_{1\mathbf{R}} \to \mathcal{H}_{2\mathbf{R}}$ be a contraction between real Hilbert spaces $\mathcal{H}_{1\mathbf{R}}$, $\mathcal{H}_{2\mathbf{R}}$ with complexification A and T, S two Yang-Baxter operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Additionally we assume that

$$(A \otimes A)T = S(A \otimes A). \tag{$**$}$$

We define linear map $F_{T,S} : \mathcal{F}_T(\mathcal{H}_1) \to \mathcal{F}_S(\mathcal{H}_2)$ on elementary tensors by:

$$F_{T,S}(A)(f_1 \otimes \ldots \otimes f_n) = (Af_1) \otimes \ldots \otimes (Af_n).$$

By $(**)$ we have $F_{T,S}(A)P_T = P_S F_{T,S}(A)$, where P_T and P_S are the operators which give the scalar products on $\mathcal{F}_T(\mathcal{H}_1)$ and $\mathcal{F}_S(\mathcal{H}_2)$. As a consequence we obtain

$$\begin{gathered}\langle F_{T,S}(A)(\xi)|F_{T,S}(A)(\xi)\rangle_S = \langle P_S F_{T,S}(A)\xi | F_{T,S}(A)\xi\rangle \\ = \langle (P_S)^{\frac{1}{2}} F_{T,S}(A)\xi | (P_S)^{\frac{1}{2}} F_{T,S}(A)\xi\rangle = \langle F_{T,S}(A)(P_T)^{\frac{1}{2}}\xi | F_{T,S}(A)(P_T)^{\frac{1}{2}}\xi\rangle \\ \leq \langle (P_T)^{\frac{1}{2}}\xi | (P_T)^{\frac{1}{2}}\xi\rangle = \langle \xi|\xi\rangle_T = \|\xi\|_T^2.\end{gathered}$$

This implies that $F_{T,S}(A)$ extends to a contraction

$$F_T(A) : \mathcal{F}_T(\mathcal{H}_1) \to \mathcal{F}_S(\mathcal{H}_2).$$

Definition 13. *The map $\Gamma_{T,S}(A) : \Gamma_T(\mathcal{H}_1) \to \Gamma_S(\mathcal{H}_2)$ is defined by the linear extension of the following formula:*

$$\Gamma_{T,S}(A)\psi(f_1 \otimes \ldots \otimes f_n) = \psi(A(f_1) \otimes \ldots \otimes A(f_n)) = \psi(F_{T,S}(A)(f_1 \otimes \ldots \otimes f_n)).$$

Lemma 14.([BXu]). *Let T, S satisfy I, II, III. Assume that*

$$(A \otimes A)T = S(A \otimes A).$$

If A is an isometry, an orthogonal projection or is orthogonal then $\Gamma_{T,S}(A)$ is an isometry, a faithful, normal conditional expectation or an isomorphism respectively.

The proof can be found in [BXu].

Let $\mathcal{H}_1 = \mathcal{H}_2$ and $T = S$. The following result can be found in [BXu].

Theorem 15.([BXu]). *Let $\mathcal{H}_R$ be a real Hilbert space and let T satisfy I, II, III. Suppose that V is a contraction on $\mathcal{H}_R$ and*

$$(Id_{\mathcal{H}} \otimes V)\, T = T\,(V \otimes Id_{\mathcal{H}})\,, \tag{2.1}$$

$$(V \otimes Id_{\mathcal{H}})\, T = T\,(Id_{\mathcal{H}} \otimes V)\,. \tag{2.2}$$

Then $\Gamma_T(V)$ as a map from $\Gamma_T(\mathcal{H})$ into $\Gamma_T(\mathcal{H})$ is unital, completely positive and preserves the trace τ_T.

For the convenience of the reader we give below the sketch of the proof from [BXu]. Set

$$K = \mathcal{H} \oplus \mathcal{H}, \quad D_V = \sqrt{1 - V^*V}, \quad D_{V^*} = \sqrt{1 - VV^*}.$$

The V can be written in the form $V = PUI$, where I is the natural embedding of K into $\mathcal{H}$, P is the natural projection of K onto $\mathcal{H}$ and $U : K \to K$ is an orthogonal map, which is defined by

$$U = \begin{pmatrix} V & -D_{V^*} \\ D_V & V^* \end{pmatrix}.$$

Further we identify K with $H \otimes \mathbf{C}^2$ in the natural way. Then

$$K \otimes K = \mathcal{H} \otimes \mathcal{H} \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$$

and we define $\widetilde{T}$ as

$$\widetilde{T} = T \otimes Q,$$

where $Q : \mathbf{C}^2 \otimes \mathbf{C}^2$ is the following operator:

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The operator $\widetilde{T}$ satisfies the braid relation because both T and Q do. Its matrix coefficients satisfy the assumption II because both T and Q do. Also $\|\widetilde{T}\| = q < 1$.

Additionally $\widetilde{T}$ has the following properties:

(i) $(I \otimes I)T = \widetilde{T}(I \otimes I)$,

(ii) $T(P \otimes P) = (P \otimes P)\widetilde{T}$,

(iii) $(U \otimes U)\widetilde{T} = \widetilde{T}(U \otimes U)$.

Therefore we can apply the previous lemma to each operator in the decomposition of V and get the required result.

Remarks 16. *(i) In the q-case all contraction satisfy (2.1) and (2.2).*

(ii) The above Theorem applies trivially to $V = e^{-t}Id$, $t > 0$.

(iii) In the q_{ij}-case (i.e. when $T(e_i \otimes e_j) = q_{ij}(e_j \otimes e_i)$ for a symmetric real matrix $(q_{ij})_{i,j \in \mathbf{I}}$), the second quantization $\Gamma_T(e^{-t}Id)$, $t > 0$ was also obtained in [L].

Now we can define the Ornstein-Uhlenbeck semigroup.

Definition 17. *Let $\mathcal{H}$ be a real Hilbert space and $A_t = e^{-t} Id_{\mathcal{H}}$, $t \geq 0$. The maps $U_t = \Gamma_T(A_t)$ form a semigroup called T-Ornstein-Uhlenbeck semigroup.*

Corollary 18. *By Theorem 15. the maps $U_t = \Gamma_T(A_t)$ are bounded, unital, completely positive and preserve the trace τ_T.*

Applying non-commutative version of the Riesz-Thorin Marcinkiewicz interpolation theorem (see [Ku] and [Y]) we obtain the following corollary:

Definition 19. *$L^p = L^p(\Gamma_T(\mathcal{H}), \tau_T)$ for $1 \leq p \leq \infty$ is the completion of $\Gamma_T(\mathcal{H})$ with respect to the norm:*

$$\|X\|_p^p = \tau_T[(X^*X)^{\frac{p}{2}}], \quad 1 \leq p < \infty.$$

We set $L^\infty = \Gamma_T(\mathcal{H})$ with the operator norm.

Corollary 20. *U_t extends to a semigroup of contractions on non-commutative L^p spaces.*

Since the functional τ_T is normalized then for $p > r$ $L^p(\tau_T)$ is a proper subspace of $L^r(\tau_T)$. However U_t is still a contraction from L^p to L^r for some t. This property is called hypercontractivity. The map U_t is also bounded as a map from L^2 to $L^\infty = \Gamma_T(\mathcal{H})$. In the next section we prove these two results.

3. Hypercontractivity and ultracontractivity for Ornstein-Uhlenbeck semigroup. This section contains the main results of this paper. We show that U_t is ultracontractive and that it is a contraction from L^2 to L^p, $p \in \mathbf{N}$, $p \geq 2$ for $e^{-2t} \leq \frac{1}{p-1}$. This is a generalization of the theorem proved by P.Biane in [Bia] which stands that in q-commutation case we have:

Theorem 21. $||\Gamma_T(A)||_{L^r \to L^p} \leq 1$ *iff* $||A||^2 \leq \frac{r-1}{p-1}$.

For the proofs we will need estimation for L^p norms of the Wick product $\psi(f)$.

Theorem 22. *Let T satisfy the conditions I, II, III. Let $f \in \mathcal{H}_R^{\otimes k+s}$, $g \in \mathcal{H}_R^{\otimes l+s}$. Denote by Π_m The orthogonal projection on $\mathcal{H}^{\otimes m}$ in $\mathcal{F}_T(\mathcal{H})$. Then*

$$||\Pi_{k+l}(\psi(f)g)||_T^2 \leq \binom{l+k}{k}_q \binom{l+s}{s}_q \binom{k+s}{k}_q ||f||_T^2 ||g||_T^2,$$

where

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{[n]_q \ldots [n-k+1]_q}{[k]_q!} = \frac{(1-q^n)\ldots(1-q^{n-k+1})}{(1-q^k)\ldots(1-q)}.$$

proof: By Theorem 12 one can write $\psi(f) = \sum_{m=0}^{k+s} F^{(m)}(f)$, where

$$F^{(m)}(f) = U_m \left[\sum_{\sigma \in G(k+s,m)} \phi(\sigma) f \right].$$

Then

$$\Pi_{k+l}(\psi(f)g) = \left[U_k \sum_{\sigma \in G(k+s,k)} \phi(\sigma) f \right] g.$$

Let $\{e_i\}$ be an orthonormal basis for $\mathcal{H}_R$. Then

$$U_k \left[\sum_{\sigma \in G(k+s,k)} \phi(\sigma) f \right] = \sum_{\substack{|\underline{i}|=k \\ |\underline{j}|=s}} \alpha(\underline{i}, \underline{j}) a^+(e_{\underline{i}}) a(e_{\underline{j}}).$$

From the definition of $a(e_j)$ we have

$$a(e_{\underline{j}})g = l(e_{j_1}) \ldots l(e_{j_s}) \prod_{m=1}^{s} (1 + T_m + T_m T_{m+1} + \cdots + T_m T_{m+1} \ldots T_{s+l-1}),$$

where $\prod_{m=1}^{s} A_m$ should be read as $A_s \cdots A_2 A_1$ and $l(f) g_1 \otimes \ldots \otimes g_n = \langle g_1 \,|\, f \rangle \, g_2 \otimes \ldots \otimes g_n$. To see this it is enough to observe that $l(e_j)T_{m+1} = T_m l(j_i)$ for $1 \leq m < s+l-2$ and apply the definition of $a(e_j)$.

Let us define

$$H(l+s,s)=\left\{\pi\in S_{l+s}\,,\,\sigma^{-1}=\begin{pmatrix}1 & \dots & s & \dots & l+s\\ i_1 & \dots & i_s & \dots & i_{l+s}\end{pmatrix}\,,\,i_{s+1}<i_{s+2}<\ldots<i_{l+s}\right\}.$$

Notice that number of elements of this set is equal to $(l+1)(l+2)\cdots(l+s)$. Further define

$$\begin{aligned}\widetilde{H}(l+s,s)=&\{\pi\in S_{l+s}\,,\,\pi=(\pi_s\pi_{s+1}\ldots\pi_{r_s})\ldots(\pi_2\pi_3\ldots\pi_{r_2})(\pi_1\pi_2\ldots\pi_{r_1})\,,\\ &r_1>0,r_2>1,\ldots,r_s>s-1,r_i<l+s\},\end{aligned}$$

where we set $\pi_m\pi_{m+1}\ldots\pi_{r_m}=1$ if $r_m=m-1$. Notice that sequence $r_1,\ldots,r_s$ uniquely determines the permutation π.
Each element of $\widetilde{H}(l+s,s)$ is an element of $H(l+s,s)$. Also the sets $\widetilde{H}(l+s,s)$ and $H(l+s,s)$ have the same number of elements. This implies that $\widetilde{H}(l+s,s)=H(l+s,s)$. Additionally for $r_1>0,r_2>1,\ldots,r_s>s-1$, $r_i<l+s$, $(\pi_s\pi_{s+1}\ldots\pi_{r_s})\ldots(\pi_1\pi_2\ldots\pi_{r_1})$ is a reduced word in S_{l+s}. Therefore

$$a(e_{\underline{j}})=l(e_{\underline{j}})\sum_{\pi\in H(l+s,s)}\varphi(\pi)g.$$

This implies

$$\begin{aligned}\Pi_{(k+l)}(\psi(f)g)&=\sum_{\substack{|\underline{i}|=k\\|\underline{j}|=s}}\alpha(\underline{i},\underline{j})a^+(e_{\underline{i}})l(e_{\underline{j}})\sum_{\pi\in H(l+s,s)}\varphi(\pi)g\\ &=\left[U_k^0\sum_{\sigma\in G(k+s,k)}\phi(\sigma)f\right]\left(\sum_{\pi\in H(l+s,s)}\varphi(\pi)g\right),\end{aligned}$$

where $U_k^0(f_1\otimes\ldots\otimes f_n)=a^+(f_1)\ldots a^+(f_k)l(f_{k+1})\ldots l(f_n)$.
Notice that each element of $H(l+s,s)$ can be uniquely written in the form $\sigma=\sigma_1\sigma_2$ where $\sigma_1\in S_s\subset S_{l+s}$ (first s coordinates) and $\sigma_2\in G(l+s,s)$. Moreover $Inv(\sigma)=Inv(\sigma_1)+Inv(\sigma_2)$. As a consequence we get the following decomposition:

$$\sum_{\pi\in H(l+s,s)}\varphi(\pi)=\left(P_T^{(s)}\otimes Id^{(l)}\right)\left(\sum_{\pi\in G(l+s,s)}\varphi(\pi)\right)=\left(P_T^{(s)}\otimes Id^{(l)}\right)R_{l+s,s}\,,$$

where $Id^{(m)}=\underbrace{Id\otimes\ldots\otimes Id}_{\text{m times}}$ and $R_{n,m}=\sum_{\sigma\in G(n,m)}\varphi(\sigma)$. Now since $T=T^*$ we have

$$l(h_2)l(h_1)T(h_3\otimes h_4)=\langle T(h_3\otimes h_4\,|\,h_1\otimes h_2)\rangle=\langle h_3\otimes h_4\,|\,T(h_1\otimes h_2)\rangle=l(T(h_1\otimes h_2))h_3\otimes h_4$$

for $h_1,h_2,h_3,h_4\in\mathcal{H}_R$, where $l(g_1\otimes g_2)=l(g_1)l(g_2)$ and extends antilineary.

From this for $m \leq s$ we get

$U_k^0(f_1 \otimes \ldots \otimes f_{k+s})(T_m(g_1 \otimes \ldots \otimes g_{l+s})) = U_k^0\big((Id^{(k)} \otimes T_{s-m})(f_1 \otimes \ldots \otimes f_{k+s})\big)(g_1 \otimes \ldots \otimes g_{l+s})$.

We can iterate this formula and this way generalize it to a permutation $\pi_{i_j} \ldots \pi_{i_1}$. This leads to the following transformation

$$\pi = (\pi_{i_j} \ldots \pi_{i_1}) \to \pi_{s-i_1} \ldots \pi_{s-i_j} = w(\pi)^{-1} w = (w \pi w)^{-1},$$

where $w(1, 2, \ldots n) = (n, n-1, \ldots 1)$. Since the map w is an automorphism of the permutation group S_s we can write $P_T^{(s)}$ instead of T_m and we finally have:

$$\begin{aligned} \Pi_{(k+l)}(\psi(f)g) &= \Big(U_k^0 \sum_{\sigma \in G(k+s,k)} \varphi(\sigma) f\Big)\big(P_T^{(s)} \otimes Id^{(l)}\big) \sum_{\pi \in G(l+s,s)} \varphi(\pi) g \\ &= \Big(U_k^0 R_{k+s,k}(f)\Big)\big(P_T^{(s)} \otimes Id^{(l)}\big) R_{l+s,s}(g) \\ &= \Big(U_k^0 \big(Id^{(k)} \otimes P_T^{(s)}\big) R_{k+s,k}(f)\Big) R_{l+s,s}(g). \end{aligned}$$

To finish the proof of Theorem 22. we need additional lemmas:

Lemma 23. *Let T satisfy I, II, III. Then for* $1 \leq k \leq n$

$$P_T^{(n)} \leq \binom{n}{k}_q (P_T^{(k)} \otimes P_T^{(n-k)})$$

proof: As it was mentioned above

$$P_T^{(n)} = (P^{(k)} \otimes P^{(n-k)}) R_{n,k}, \text{ where } R_{n,k} = \sum_{\sigma \in G(n,k)} \varphi(\sigma).$$

Additionally

$$\begin{aligned} \|R_{n,k}\| &\leq \sum_{\sigma \in G(n,k)} \|\varphi_T(\sigma)\| \leq \sum_{\sigma \in G(n,k)} \|T\|^{i(\sigma)} \\ &\leq \sum_{\sigma \in G(n,k)} q^{i(\sigma)} = \frac{[n]_q!}{[k]_q![n-k]_q!} = \binom{n}{k}_q \end{aligned} \qquad (\star\star\star)$$

We know that $P_T^{(n)} = [P_T^{(n)}]^* > 0$ and therefore

$$P_T^{(n)}[P_T^{(n)}]^* = (P_T^{(k)} \otimes P_T^{(n-k)}) R_{n,k} R_{n,k}^* (P_T^{(k)} \otimes P_T^{(n-k)}).$$

But by $(\star\star\star)$ we have $\|R_{n,k}\| \leq \binom{n}{k}_q$ and thus

$$(P_T^{(n)})^2 \leq \left[\binom{n}{k}_q\right]^2 (P_T^{(k)} \otimes P_T^{(n-k)})^2.$$

Since the function $\{z \in \mathbf{C};\, Im(z) > 0\} = Im(\mathbf{C}) \ni z \to z^{\frac{1}{2}} \in Im(\mathbf{C})$ is a Newalina type function [see [AG]] we get that if $A \geq B \geq 0$ then $A^{\frac{1}{2}} \geq B^{\frac{1}{2}} \geq 0$ for A, B-bounded operators on some Hilbert space. The proof is finished. ∎

The next lemma gives us the generalization of Schwartz inequality.

Lemma 24. *Let* $C > 0$, $C \in B(\mathcal{H}^{\otimes s})$, $x \in \mathcal{H}_R^{\otimes k+s}$, $y \in \mathcal{H}_R^{\otimes s+l}$.
If

$$U_k^0(x)\big(C \otimes Id^{(l)}\big)y = \big(U_k^0(Id^{(k)} \otimes C)x\big)y$$

then

$$\Big\|(U_k^0(Id^{(k)} \otimes C)x)y\Big\|^2 \leq \Big\langle \big(Id^{(k)} \otimes C\big)x \,|\, x \Big\rangle \, \Big\langle \big(C \otimes Id^{(l)}\big)y \,|\, y \Big\rangle .$$

proof: By Schwartz inequality

$$\begin{aligned}
\Big\|\big(U_k^0(Id^{(k)} \otimes C)x\big)y\Big\|^2 &= \Big\|\big(U_k^0(Id^{(k)} \otimes \sqrt{C})x\big)(\sqrt{C} \otimes Id^{(l)})y\Big\|^2 \\
&\leq \Big\|\big(Id^{(k)} \otimes \sqrt{C}\big)x\Big\|^2 \, \Big\|\big(\sqrt{C} \otimes Id^{(l)}\big)y\Big\|^2 \\
&= \Big\langle \big(Id^{(k)} \otimes C\big)x \,|\, x \Big\rangle \, \Big\langle \big(C \otimes Id^{(l)}\big)y \,|\, y \Big\rangle \quad \blacksquare.
\end{aligned}$$

Now we can finish the proof of Theorem 22. Set $x = \left(\sqrt{P_T^{(k)}} \otimes Id^{(s)}\right) R_{k+s,k}(f)$, $y = \left(Id^{(s)} \otimes \sqrt{P_T^{(l)}}\right) R_{l+s,s}(g)$ and $C = P_T^{(s)}$.
Now we can apply Lemma 23 with and deduce that:

$$\begin{aligned}
\|\Pi_{(k+l)}(\psi(f)g)\|_T^2 &= \left\|\sqrt{P_T^{(k+l)}}\big(U_k^0 R_{k+s,k}(f)\big)(P_T^{(s)} \otimes Id^{(l)})R_{l+s,s}(g)\right\|^2 \\
&\leq \binom{k+l}{k}_q \left\|\sqrt{P_T^{(k)} \otimes P_T^{(l)}}\big(U_k^0 R_{k+s,k}(f)\big)(P_T^{(s)} \otimes Id^{(l)})R_{l+s,s}(g)\right\|^2 \\
&= \binom{k+l}{k}_q \left\|\left(\sqrt{P_T^{(k)}} \otimes Id^{(s)}\right) \big(U_k^0 R_{k+s,k}(f)\big)\big(P_T^{(s)} \otimes Id^{(l)}\big) \times \right. \\
&\qquad \left. \times \left(Id^{(k)} \otimes \sqrt{P_T^{(l)}}\right) R_{l+s,s}(g)\right\|^2
\end{aligned}$$

.

Finally Lemma 24 implies :

$$\begin{aligned}
\|\Pi_{(k+l)}(\psi(f)g)\|_T^2 &\leq \binom{k+l}{k}_q \Big\langle (Id^{(k)} \otimes P_T^{(s)})(P_T^{(k)} \otimes Id^{(s)})R_{k+s,k}(f) \,|\, R_{k+s,k}(f)\Big\rangle \times \\
&\qquad \times \Big\langle (P_T^{(s)} \otimes Id^{(l)})(Id^{(s)} \otimes P_T^{(l)})R_{l+s,s}(g) \,|\, R_{l+s,s}(g)\Big\rangle \\
&= \binom{k+l}{k}_q \langle P_T^{(k+s)}(f) \,|\, R_{k+s,k}(f)\rangle \langle P_T^{(l+s)}(g) \,|\, R_{l+s,s}(g)\rangle \\
&\leq \binom{k+l}{k}_q \binom{k+s}{k}_q \binom{l+s}{s}_q \|f\|_T^2 \|g\|_T^2.
\end{aligned}$$

The last inequality is a consequence of Schwartz inequality for $\langle\cdot\,|\,\cdot\rangle_T$ and the following estimation:

$$\begin{aligned}\|R_{n,m}f\|_T^2 &= \langle P^{(n)}R_{n,m}f|R_{n,m}f\rangle\\ &\le \binom{n}{m}_q \langle (P^{(m)}\otimes P^{(n-m)})R_{n,m}f|R_{n,m}f\rangle\\ = \binom{n}{m}_q \langle f|R_{n,m}f\rangle_T &\le \binom{n}{m}_q \|f\|_T\|R_{n,m}f\|_T \qquad \blacksquare.\end{aligned}$$

Now since

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{[n]_q\dots[n-k+1]_q}{[k]_q!} =$$
$$\frac{(1-q^n)\dots(1-q^{n-k+1})}{(1-q^k)\dots(1-q)} \le \left(\prod_{m=1}^{\infty}(1-q^m)\right)^{-1} = C_q$$

we have

Corollary 25. *Under assumptions of Theorem 22*

$$\|\Pi_{k+l}(\psi(f)g)\|_T^2 \le C_q^3\|f\|_T^2\|g\|_T^2.$$

The above result implies the following theorem:

Theorem 26. *Let T satisfy the conditions I, II, III.*
Let $f\in\mathcal{H}^{\otimes n}$ and $\|T\| = q < 1$.
Then

$$\|f\|_T \le \|\psi(f)\|_\infty \le (n+1)C_q^{\frac{3}{2}}\|f\|_T,$$

where $C_q^{-1} = \prod_{m=1}^{\infty}(1-q^m)$ and $\|f\|_T^2 = \langle P_T f \mid f\rangle$.

proof: Let $f\in\mathcal{H}_R^{\otimes n}$ and $g\in\mathcal{H}_R^{\otimes m}$. By Theorem 12 $\psi(f) = \sum_{k=0}^{n} F^{(k)}(f)$, where

$$F^{(k)}(f) = U_k\left[\sum_{\sigma\in G(n,k)}\varphi(\sigma)f\right].$$

Therefore $F^{(k)}:\mathcal{H}^{\otimes m}\to\mathcal{H}^{\otimes m-(n-k)+k}$ if $m\ge n-k$ and $F^{(k)}:\mathcal{H}^{\otimes m}\to\{0\}$ if $m<n-k$. Further

$$\|F^{(k)}(f)g\|_T^2 = \|\Pi_{m-n+2k}(\psi(f)g)\|_T^2 \le C_q^3\|f\|_T^2\|g\|_T^2.$$

Fix k and let $g = \sum\limits_{m=0}^{\infty} g_m$ for $g_m \in \mathcal{H}^{\otimes m}$. Then

$$\|F^{(k)}g\|_T^2 = \sum_{m=0}^{\infty} \|F^{(k)}g_m\|_T^2 = \sum_{m=0}^{\infty} C_q^3 \|f\|_T^2 \|g_m\|_T^2 = C_q^3 \|f\|_T^2 \|g\|_T^2.$$

Finally

$$\|\psi(f)g\|_T^2 \leq \|\sum_{k=0}^{n} F^{(k)}g\|_T^2 \leq (n+1)\|F^{(k)}g\|_T^2 \leq (n+1)C_q^3 \|f\|_T^2 \|g\|_T^2. \qquad \blacksquare$$

This inequality for qCCR was found by M.Bożejko in [Bo]. It resembles the Haagerup inequality for free group (see [Ha]). It was used to obtain ultracontractivity for U_t.

Theorem 27. *Let T satisfy I, II, III and $X \in \Gamma_T(\mathcal{H})$. If $0 < t \leq 1$, then*

$$\|U_t X\|_\infty \leq C_q^{-\frac{3}{2}} t^{-\frac{3}{2}} \|X\|_2.$$

where $C_q^{-1} = \prod\limits_{m=1}^{\infty}(1-q^m)$ and $\|\cdot\|_\infty$ stands for the operator norm.

The proof is identical as in [Bo].

Take $X \in L^\infty$. Then we can write $X = \sum\limits_{n=0}^{\infty} X_n$, where $X_n\Omega \in \mathcal{H}^{\otimes n}$. Moreover

$$U_t X = \sum_{n=0}^{\infty} e^{-tn} X_n.$$

Hence for $0 < t < 1$

$$\begin{aligned}
\|U_t X\|_\infty &\leq \sum_n e^{-tn} C_q^{\frac{3}{2}} \|X_n\|_\infty \\
&\leq \sum_n e^{-tn} C_q^{\frac{3}{2}} (n+1) \|X_n\Omega\|_T \\
&\leq C_q^{\frac{3}{2}} \left(\sum_n e^{-2tn}(n+1)^2\right)^{\frac{1}{2}} \left(\sum_n \|X_n\Omega\|_T^2\right)^{\frac{1}{2}} \\
&\leq \widetilde{C_q} t^{-\frac{3}{2}} \|X\|_2.
\end{aligned} \qquad \blacksquare$$

As a corollary we obtain a generalization of the Poincare-Sobolev inequality:

Corollary 28. *There exists a constant $C(q) > 0$ such that for all X in the domain of the quadratic form $Q_T(X) = \tau_T(X^* N^T(X))$, for $N^T(\psi(f_1 \otimes \ldots \otimes f_n)) = n\psi(f_1 \otimes \ldots \otimes f_n)$ (N^T is linear) we have*

$$\|X\|_{L^3}^2 \leq C(q)(|\tau_T(X)|^2 + Q_T(X)).$$

The proof can be found in [Bia]. One needs to replace the functional τ_0 by τ_T.

Consider now the q-commutation relations for $q \in (-1,1)$ i.e. $a_i a_j^+ = q a_j^+ a_i + \delta_{ij} Id$ and the corresponding Wick product ψ_q in $\Gamma_q(\mathcal{H})$. Denote by e_1 a normalized real vector in $\mathcal{H}$ which can be treated as the first element of a basis. Since $\|e_1^{\otimes m}\|_q^2 = [m]_q!$ and $\Pi_{k+l}(\psi(e_1^{\otimes k+s})e_1^{\otimes l+s}) = \binom{k+s}{k}_q [l+s]_q[l+s-1]_q \cdots [l+1]_q e_1^{\otimes k+l}$ the following equality holds

$$\frac{\|\Pi_{k+l}(\psi(e_1^{\otimes k+s})e_1^{\otimes l+s})\|_q}{\|e_1^{\otimes k+s}\|_q \|e_1^{\otimes l+s}\|_q} = \binom{k+l}{k}_q \binom{k+s}{k}_q \binom{l+s}{s}_q .$$

From this we obtain

Corollary 29. *Under assumptions of Theorem 22*

$$\|\Pi_{k+l}(\psi(f)g)\|_T \leq \|\Pi_{k+l}(\psi(e_1^{\otimes k+s})e_1^{\otimes l+s})\|_q \frac{\|f\|_T}{\|e_1^{\otimes k+s}\|_q} \frac{\|g\|_T}{\|e_1^{\otimes l+s}\|_q}.$$

We may generalize the above inequality

Lemma 30. *Assume that T has properties I, II, III. Fix $p \in \mathbf{N}$, , $p \geq 2$. Let $X \in L^p(\Gamma_T(\mathcal{H}))$ and $X = \sum_{n=0}^{\infty} X_n$. Set $f_n = X_n \Omega$. Then*

$$|\tau_T(X^p)| \leq \tau_q \left(\left(\sum_{n=0}^{\infty} \frac{\|f_n\|_T}{\|e_1^{\otimes n}\|_q} \psi_q(e_1^{\otimes n}) \right)^p \right),$$

where ψ_q denotes the Wick product in $\Gamma_q(\mathcal{H})$. The equality holds for $p=2$ and $X = X^\star$.

proof: By Corollary 29

$$\|\Pi_{j_1}(X_{n_1}\Pi_{j_2}(Y))\|_T \leq \|\Pi_{j_1}(\psi_q(e_1^{\otimes n_1})e_1^{\otimes j_2})\|_q \frac{\|X_{n_1}\|_T}{\|e_1^{\otimes n_1}\|_q} \frac{\|\Pi_{j_2}(Y)\|_T}{\|e_1^{\otimes j_2}\|_q}.$$

Set $j_p = n_p$. Iterating the above inequality one can prove that

$$\begin{aligned}
&\left\|\Pi_{j_1}\left(X_{n_1}\left(\Pi_{j_2}\left(X_{n_2}\cdots\Pi_{j_{p-1}}(X_{n_{p-1}}X_{n_p}\Omega)\ldots\right)\right)\right)\right\|_T \leq \\
&\leq \prod_{k=1}^{p} \frac{\|X_{n_k}\|_T}{\|e_1^{\otimes n_k}\|_q} \prod_{i=2}^{p-1} \|e_1^{\otimes j_i}\|_q^{-1} \prod_{r=1}^{p-1} \left\|\Pi_{j_r}\left(\psi(e_1^{\otimes n_r})e_1^{\otimes j_{r+1}}\right)\right\|_q = \\
&= \prod_{k=1}^{p} \frac{\|X_{n_k}\|_T}{\|e_1^{\otimes n_k}\|_q} \left\|\Pi_{j_1}\left(e_1^{\otimes n_1}\left(\Pi_{j_2}(e_1^{\otimes n_2}\ldots\Pi_{j_{p-1}}(e_1^{\otimes n_{p-1}}e_1^{\otimes n_p})\ldots)\right)\right)\right\|_T .
\end{aligned}$$

Consider now $j_1 = 0$. Then

$$\begin{aligned}
&\left\|\Pi_{j_1}\left(X_{n_1}\left(\Pi_{j_2}(X_{n_2}\ldots\Pi_{j_{p-1}}(X_{n_{p-1}}X_{n_p}\Omega)\ldots)\right)\right)\right\|_T \\
&= \left|\tau_T\left(X_{n_1}\left(\Pi_{j_2}(X_{n_2}\ldots\Pi_{j_{p-1}}(X_{n_{p-1}}X_{n_p}\Omega)\ldots)\right)\right)\right|
\end{aligned}$$

As a conclusion we get

$$\left|\tau_T\left(X_{n_1}\left(\Pi_{j_2}(X_{n_2}\ldots\Pi_{j_{p-1}}(X_{n_{p-1}}X_{n_p}\Omega)\ldots)\right)\right)\right| \le$$

$$\le \prod_{k=1}^{p}\frac{\|X_{n_k}\|_T}{\|e_1^{\otimes n_k}\|_q}\,\tau_q\left(e_1^{\otimes n_1}\left(\Pi_{j_2}(e_1^{\otimes n_2}\ldots\Pi_{j_{p-1}}(e_1^{\otimes n_{p-1}}e_1^{\otimes n_p})\ldots)\right)\right).$$

Now it is enough to observe that the left hand side as well as the right hand side of the required inequality are sums of expressions like the expression above. This concludes the proof. ■

Observe that if $X \in L_p(\Gamma_T(\mathcal{H}))$, $p \ge 2$ then $X \in L_2(\Gamma_T(\mathcal{H}))$. We can see also that all terms on the right hand side of the expression in Lemma 30 are positive and therefore it is well defined.

Our goal is to prove that the optimal hypercontractive bounds for U_t are the same as in q-case. We shall show first that $sup\{\|U_tX\|_p\,,\,\|X\|_2=1\}$ can be restricted to the positive operators X.

Proposition 31. *Suppose that T satisfies assumptions I, II, III. Then*

$$\|U_t\|_{L_r\to L_p} = sup\{\|U_tX\|_p\,,\,\|X\|_r=1\,,\,X\ge 0\}.$$

The proof is analogous to that of the conditional expectation instead of U_t.
proof: Let $C = V|C|$ and $X = O|X|$ be the polar decompositions. It was proved that U_t is completely positive for $t>0$. Fix $t>0$. By Stinespring theorem it has the form $U_t(X) = W^*\pi(X)W$, where W is a partial isometry and π a representation. Let $\frac{1}{p}+\frac{1}{r}=1$. Using Schwarz inequality and trace property of τ_T we have

$$\begin{aligned}
\|U_tX\|_p &= \sup_{\|C\|_r=1}\tau_T(CU_t(X)) = \sup\tau_T(V|C|W^*\pi(O|X|)W) = \\
&= \sup\tau_T(\sqrt{|C|}W^*\pi(O)\pi(\sqrt{|X|})\pi(\sqrt{|X|})WV\sqrt{|C|}) \le \\
&\le \tau_T\left(\sqrt{|C|}W^*\pi(O)\pi(\sqrt{|X|}\pi(\sqrt{|X|}\pi(O^*)W\sqrt{|C|}\right)^{\frac{1}{2}}\times \\
&\qquad\qquad \times\tau_T\left(\sqrt{|C|}V^*W^*\pi(\sqrt{|X|}\pi(\sqrt{|X|}WV\sqrt{|C|}\right)^{\frac{1}{2}} = \\
&= \sup\tau_T(|C|U_t(|X^*|))^{\frac{1}{2}}\tau_T(|C^*|U_t(|X|))^{\frac{1}{2}} = \\
&= \|U_t(|X^*|)\|_p^{\frac{1}{2}}\|U_t(|X|)\|_p^{\frac{1}{2}}.
\end{aligned}$$

From this one can easily obtain the result.
Finally we get:

Theorem 32. *Let T satisfy conditions I, II, III. Take $p\in\mathbf{N}$, $p\ge 2$ and $X\in\Gamma_T(\mathcal{H})$. Then*

$$\|U_tX\|_{L^2\to L^p}\le 1 \text{ if and only if } e^{-2t}\le\frac{1}{p-1}.$$

The proof of the necessity condition is the same as in free case (see [Bia]). We will show that the sufficiency condition for the general commutation relations reduces to those of q-commutation case. To this end take $X \geq 0$ and $e^{-2t} \leq \frac{1}{p-1}$. Then

$$\begin{aligned}
\|U_t X\|_{L_p}^p &= \|\sum_n e^{-tn} X_n\|_{L_p}^p = \tau_T\left((\sum_{n=0}^{\infty} e^{-tn} X_n)^p \right) \\
&\leq \tau_q\left(\left(\sum_{n=0}^{\infty} e^{-tn} \frac{\|f_n\|_T}{\|e_1^{\otimes n}\|_q} \psi_q(e_1^{\otimes n}) \right)^p \right) \\
&\leq \left\| \sum_{n=0}^{\infty} e^{-tn} \frac{\|f_n\|_T}{\|e_1^{\otimes n}\|_q} \psi_q(e_1^{\otimes n}) \right\|_{L_p}^p \\
&\leq \left\| \sum_{n=0}^{\infty} \frac{\|f_n\|_T}{\|e_1^{\otimes n}\|_q} \psi_q(e_1^{\otimes n}) \right\|_{L_2}^p \\
&= \left(\sum_{n=0}^{\infty} \frac{\|f_n\|_T^2}{\|e_1^{\otimes n}\|_q^2} \|e_1^{\otimes n}\|_q^2 \right)^{\frac{p}{2}} \\
&= \left(\sum_{n=0}^{\infty} \|f_n\|_{L_2}^2 \right)^{\frac{p}{2}} = \|X\|_{L_2}^p.
\end{aligned}$$

This finishes the proof. ∎

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