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GRAPH C^*-ALGEBRAS AND $\mathbb{Z}_2$-QUOTIENTS OF QUANTUM SPHERES

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We consider two $\mathbb{Z}_2$-actions on the Podleś generic quantum spheres. They yield, as noncommutative quotient spaces, the Klimek-Lesniewski q-disc and the quantum real projective space, respectively. The C^*-algebras of all these quantum spaces are described as graph C^*-algebras. The K-groups of the thus presented C^*-algebras are then easily determined from the general theory of graph C^*-algebras. For the quantum real projective space, we also recall the classification of the classes of irreducible $*$-representations of its algebra.

Keywords: K-theory of C^*-algebras, Galois extensions of noncommutative rings, quantum-group homogeneous spaces

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1. Introduction

In noncommutative geometry, one thinks of quantum spaces as objects dual to noncommutative algebras in the sense of the Gelfand-Neumark correspondence between spaces and function algebras. Analogously to classical topology, where taking quotients of proper group actions is a standard way of obtaining new topological spaces, one is then lead to think of Hopf-algebra coinvariant subalgebras as encoding noncommutative quotient spaces. For Hopf algebras that are function algebras on finite groups, their coactions on algebras can be easily understood as group actions on algebras. Then the coinvariant subalgebras are simply fixed-point subalgebras. In this note, we study two simple examples of such actions, notably actions of $\mathbb{Z}_2$ on the algebra of the Podleś equator quantum sphere $S^2_{q\infty}$. For this noncommutative sphere, there is still an analogue of an equator formed by the classical points (one-dimensional representations). One can quotient $S^2_{q\infty}$ by the reflection with respect to the equator plane, and verify that the quotient coincides with a well-known quantum disc of Klimek and Lesniewski [KL93]. On the other hand, one can also define the antipodal action. The quotient under this action is the quantum real projective space [H-PM96]. Here we review results from [HMS] concerning the aforementioned noncommutative quotient spaces, recast in the more recent language of graph C^*-algebras [HS].

2. Graph C^*-algebras

For the convenience of the reader, we briefly recall the definition of a graph C^*-algebra (e.g., see [FLR00]). Let G be a countable graph with the set of vertices G^0 and the set of directed edges G^1. (If $e \in G^1$ then $s(e)$ and $r(e)$ are the source and range of e, respectively.) For the sake of simplicity, we assume that every vertex in G emits only finitely many edges. Then $C^*(G)$ is, by definition, the universal C^*-algebra generated by partial isometries $\{S_e | e \in G^1\}$ with mutually orthogonal ranges and by mutually orthogonal projections $\{P_v | v \in G^0\}$ such that

1. $S_e^* S_e = P_{r(e)}$ for any $e \in G^1$,
2. $P_v = \sum_{s(f)=v} S_f S_f^*$, for any $v \in G^0$ emitting at least one edge.

These so-called graph C^*-algebras generalize the classical Cuntz-Krieger algebras [CK80].

According to the general Cuntz formula valid for graph C^*-algebras [RS, Theorem 3.2], the K-theory of $C^*(G)$ can be calculated as follows. Let G^0_+ be the set of those vertices of a graph G that emit at least one edge, and let $\mathbb{Z}G^0$ and $\mathbb{Z}G^0_+$ be the free abelian groups with generators G^0 and G^0_+, respectively. Let $A_G : \mathbb{Z}G^0_+ \to \mathbb{Z}G^0$ be the map defined by

$$A_G(v) = \Big(\textstyle\sum_{s(e)=v, e\in G^1} r(e)\Big) - v. \tag{1}$$

Then

$$K_0(C^*(G)) \simeq \mathrm{Coker}(A_G), \quad K_1(C^*(G)) \simeq \mathrm{Ker}(A_G). \tag{2}$$

3. Quantum spheres

The quantum spheres S^2_{qc} were discovered by Podleś as $SU_q(2)$-homogeneous spaces [P-P87]. Their algebras can be defined in the following way. The polynomial algebra $\mathcal{O}(S^2_{qc})$, $q\in\mathbb{R}$, $0<|q|<1$, $c\in[0,\infty]$ is the universal $*$-algebra generated by elements A, B subject to

$$A^*=A,\quad BA=q^2AB, \tag{3}$$

$$B^*B=-A^2+\frac{A}{1+\sqrt{c}}+\frac{c}{(1+\sqrt{c})^2}I,\quad BB^*=-q^4A^2+\frac{q^2A}{1+\sqrt{c}}+\frac{c}{(1+\sqrt{c})^2}I. \tag{4}$$

The C^*-algebra $C(S^2_{qc})$ can be given as the norm closure of $\mathcal{O}(S^2_{qc})$. Note that we rescaled the generators used by Podleś, so as to write the relations (4) in a uniform way for all $c\in[0,\infty]$, see [HMS]. For $c=\infty$, (4) reduce to

$$B^*B=-A^2+I,\quad BB^*=-q^4A^2+I. \tag{5}$$

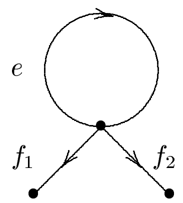
It seems interesting that the algebra $\mathcal{O}(S^2_{q\infty})$ is isomorphic with $\mathcal{O}(SU_{q^2}(2))/\langle b-b^*\rangle$, where $\mathcal{O}(SU_{q^2}(2))$ is the polynomial algebra of $SU_{q^2}(2)$ and b is the upper-off-diagonal generator in the fundamental representation $\begin{pmatrix} a & b\\ -q^{-2}b^* & a^*\end{pmatrix}$ of $SU_{q^2}(2)$. (Cf. the paragraph above Proposition 3.2 in [HS] and the relevant considerations in [HL].) Indeed, it follows immediately from the defining relations of $\mathcal{O}(S^2_{q\infty})$ and $\mathcal{O}(SU_{q^2}(2))$ (e.g., see [W-SL87] and put $\nu=q^2,\alpha=a,\gamma=q^{-2}b^*$) that the assignment $A\mapsto q^{-2}b$, $B\mapsto a$ defines an algebra epimorphism $F:\mathcal{O}(S^2_{q\infty})\to\mathcal{O}(SU_{q^2}(2))/\langle b-b^*\rangle$. The injectivity of F follows from the representation theory of these algebras. The representations $\rho_\pm$ of $\mathcal{O}(SU_{q^2}(2))$ [VS88]

$$\rho_\pm(b)e_k=\pm q^{2(k+1)}e_k,\quad \rho_\pm(a)e_k=\sqrt{1-q^{4k}}\,e_{k-1},\quad \langle e_k|e_l\rangle=\delta_{kl},\quad k,l\in\mathbb{N}, \tag{6}$$

annihilate the ideal $\langle b-b^*\rangle$ and composed with F agree with Podleś representations [P-P87]: $\pi_\pm=\rho_\pm\circ F$. Since $\pi_+\oplus\pi_-$ is faithful, one can conclude that F is injective. Hence it is an isomorphism. The geometric meaning of this isomorphism is that $S^2_{q\infty}$ is embedded as an equator in $SU_{q^2}(2)$ thought of as a quantum 3-sphere. The other extreme value of c, i.e., $c=0$, also simplifies the relations. In this case, it turns out that $\mathcal{O}(S^2_{q0})$ is isomorphic with the fixed-point subalgebra $\mathcal{O}(SU_q(2))^{U(1)}$, so that we can interpret S^2_{q0} as the quotient sphere $SU_q(2)/U(1)$ in the spirit of the Hopf fibration. (Here the action of $U(1)$ is given by rescaling the generator a by $e^{i\phi}$ and b by $-e^{i\phi}$.) This way we can view the family of Podleś spheres S^2_{qc} as an approximation between the quotient sphere $S^2_{q0}\simeq SU_q(2)/U(1)$ and the embedded sphere $S^2_{q\infty}\subset SU_{q^2}(2)$. Although the desired $\mathbb{Z}_2$-actions can be defined on the C^*-level for any $c\in]0,\infty]$ (for $c>0$, $C(S^2_{qc})\simeq C(S^2_{q\infty})$), in what follows, we restrict our attention to $S^2_{q\infty}$ because this is where we can define these actions on the algebraic level.

It is shown in [HS, Proposition 3.1] that, for any $q\in]0,1[$, there exists a C^*-algebra isomorphism ϕ_q from $C(S^2_{q\infty})$ to the graph C^*-algebra $C^*(G_1)$ corresponding to the following graph:

G_1



(Note that $C^*(G_1)$ is generated by partial isometries S_e, S_{f_1} and S_{f_2}, corresponding to the edges of G_1.) The matrix of the group homomorphism A_{G_1} (see (1)) is

$$A_{G_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : \mathbb{Z} \longrightarrow \mathbb{Z}^3. \tag{7}$$

Hence the formulas (2) yield $K_0(C(S^2_{q\infty})) \simeq \mathbb{Z}^2$, $K_1(C(S^2_{q\infty})) \simeq 0$, which agrees with the computation of [MNW91] and with the classical situation.

4. Quantum disc

The $*$-algebra of the quantum disc D_q is defined as follows:

$$\mathcal{O}(D_q) := \mathbb{C}\langle x, x^* \rangle / \langle x^*x - qxx^* - (1-q) \rangle, \quad 0 < q < 1. \tag{8}$$

This is a one-parameter subfamily of a two-parameter family of quantum discs introduced by Klimek and Lesniewski [KL93] as homogeneous spaces of $SU_q(1,1)$. It was shown in [KL93] that $\|\pi(x)\| = 1$ in every bounded $*$-representation of $\mathcal{O}(D_q)$. Thus the C^*-algebra $C(D_q)$ can be defined using the supremum over the norms in bounded $*$-representations. (Note here that there are unbounded representations – the disc algebra is easily transformed into the q-oscillator algebra.) As also shown in [KL93], $C(D_q)$ is isomorphic to the C^*-algebra of the one-sided shift (Toeplitz algebra). This isomorphism is provided by a faithful infinite dimensional representation π [KL93, p.14]. The one-dimensional representations of $C(D_q)$ form a circle that can be considered the boundary of the quantum disc D_q. The infinite dimensional representation corresponds to the interior of that disc.

We define now a $\mathbb{Z}_2$-action by sending $1 \in \mathbb{Z}_2$ to $r_1 : C(S^2_{q\infty}) \to C(S^2_{q\infty})$ that "identifies upper and lower hemispheres":

$$r_1(B) = B, \quad r_1(A) = -A. \tag{9}$$

We have shown in [HMS] the following results concerning this action:

- The fixed-point polynomial algebra $\mathcal{O}(S^2_{q\infty}/\mathbb{Z}_2) := \{a \in \mathcal{O}(S^2_{q\infty}) \mid r_1(a) = a\}$ is the $*-$subalgebra generated by B, and can be identified with $\mathcal{O}(D_{q^4})$ by sending the generator x to the generator B. This extends to the polynomial level the fact that the r_1-fixed point subalgebra of $C(S^2_{q\infty})$ coincides with the Toeplitz algebra.

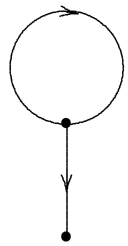
- The $\mathbb{Z}_2$-extension $\mathcal{O}(D_{q^4}) \subset \mathcal{O}(S^2_{q\infty})$ defined in this way is not Galois. (The $\mathbb{Z}_2$-action is not free.)
- All $*$-representations of $\mathcal{O}(D_{q^4})$ are restrictions of $*$-representations of $\mathcal{O}(S^2_{q\infty})$.
- The automorphism r_1 commutes with the $SU_q(2)$-induced $U(1)$-action on $S^2_{q\infty}$.

Under the aforementioned isomorphism $\phi_q : C(S^2_{q\infty}) \to C^*(G_1)$, the order two automorphism r_1 of $C(S^2_{q\infty})$ is transformed into the automorphism of $C^*(G_1)$ determined by

$$S_e \mapsto S_e, \quad S_{f_i} \mapsto S_{f_{3-i}}. \tag{10}$$

This automorphism of $C^*(G_1)$ is induced from the automorphism of the graph G_1 that fixes the edge e and interchanges f_1 with f_2. The fixed-point subalgebra for this $\mathbb{Z}_2$-action coincides with the Toeplitz algebra and corresponds to the following graph:

G_2



The K-theory of the Toeplitz algebra is well-known (e.g. see [W-NE93]). Here, we can straightforwardly determine it from (1) and (2):

$$A_{G_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \mathbb{Z} \longrightarrow \mathbb{Z}^2, \quad K_0(C(D_q)) \simeq \mathbb{Z}, \quad K_1(C(D_q)) \simeq 0. \tag{11}$$

5. Quantum two-dimensional real projective space

We define the antipodal $\mathbb{Z}_2$-action on $S^2_{q\infty}$ by sending $1 \in \mathbb{Z}_2$ to the $*$-automorphism of $C(S^2_{q\infty})$ given by

$$r_2(A) = -A, \quad r_2(B) = -B.$$

The $*$-algebra $\mathcal{O}(\mathbb{R}P^2_q) := \{a \in \mathcal{O}(S^2_{q\infty}) | r_2(a) = a\}$ describes the quantum two-dimensional real projective space $\mathbb{R}P^2_q$. This algebra is generated by $P = A^2, \quad R = B^2, \quad T = AB$. They fulfill the relations

$$P = P^*, \tag{12}$$

$$RP = q^8 PR, \quad RT = q^4 TR, \quad PT = q^{-4} TP, \tag{13}$$

$$T^2 = q^2 PR, \quad RT^* = q^2 T(-q^4 P + I), \quad R^* T = q^{-2} T^*(-P + I), \tag{14}$$

$$RR^* = q^{12} P^2 - q^4(1 + q^4)P + I, \quad R^* R = q^{-4} P^2 - (1 + q^{-4})P + I, \tag{15}$$

$$TT^* = -q^4 P^2 + P, \quad T^* T = q^{-4}(P - P^2). \tag{16}$$

One can show that $\mathcal{O}(\mathbb{R}P^2_q)$ is isomorphic to the free $*$-algebra generated by P, R, T divided by the ideal of relations (12)–(16), see [HMS]. Concerning representations, we have:

Theorem 1. *[HMS] There are no unbounded $*$-representations of the $*$-algebra $\mathcal{O}(\mathbb{R}P_q^2)$. Up to unitary equivalence, all irreducible $*$-representations of this algebra are the following:*

(i) A family of one-dimensional representations $\rho_\theta : \mathcal{O}(\mathbb{R}P_q^2) \to \mathbb{C}$, parameterised by $\theta \in [0, 2\pi)$, given by

$$\rho_\theta(P) = \rho_\theta(T) = 0, \quad \rho_\theta(R) = e^{i\theta}. \tag{17}$$

(ii) An infinite dimensional representation ρ (in a Hilbert space H with an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$) given by

$$\rho(P)e_k = q^{4k}e_k, \tag{18}$$

$$\rho(T)e_k = q^{2(k-1)}(1-q^{4k})^{1/2}e_{k-1}, \tag{19}$$

$$\rho(R)e_k = (1-q^{4k})^{1/2}(1-q^{4(k-1)})^{1/2}e_{k-2}. \tag{20}$$

Proof (sketch): The boundedness of $\rho(P) = \rho(P^*)$ comes from the relation $T^*T = q^{-4}(P - P^2)$. Then it is an easy consequence of the relations (15) and (16) that $\rho(R)$ and $\rho(T)$ are also bounded. Next, the assumption $\rho(P) = 0$ immediately leads to $\rho(T) = 0$, and the only remaining relations are $\rho(R^*R) = 1 = \rho(RR^*)$. This yields the one-dimensional representations. On the other hand, assuming $\rho(P) \neq 0$ implies, by the irreducibility of ρ, that $\mathrm{Ker}\rho(P) = 0$. Using the characterisation of the spectrum by approximate eigenvectors, and taking advantage of the relations, it is possible to identify the spectrum of $\rho(P)$ with $\{0\} \cup \{q^{4k} | k \in \mathbb{N}\}$. Now, one builds a Hilbert space out of the eigenvectors of $\rho(P)$, and identifies $\rho(T)$ and $\rho(R)$ as weighted-shift operators. $\square$

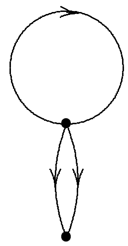
Note that the irreducible $*$-representations of $\mathcal{O}(\mathbb{R}P_q^2)$ are the restrictions of the irreducible $*$-representations $\mathcal{O}(S^2_{q\infty})$. Next, let us observe that, since r_2 and the right coaction of $SU_q(2)$ on $S^2_{q\infty}$ commute [P-P87, HMS], $\mathbb{R}P_q^2$ is a quantum homogeneous space of $SU_q(2)$. One can also show that the $\mathbb{Z}_2$-extension $\mathcal{O}(\mathbb{R}P_q^2) \subset \mathcal{O}(S^2_{q\infty})$ is Galois [H-PM96], i.e., we have an algebraic quantum principal bundle. Moreover, since $\mathcal{O}(S^2_{q\infty}) \otimes \mathrm{Map}(\mathbb{Z}_2, \mathbb{C})$ is norm dense in $C(S^2_{q\infty}) \otimes \mathrm{Map}(\mathbb{Z}_2, \mathbb{C})$, one can see that the antipodal $\mathbb{Z}_2$-action on $C(S^2_{q\infty})$ is principal in the sense of [E-DA00].

Finally, let us look at $C(\mathbb{R}P_q^2)$ as a graph C^*-algebra. The automorphism r_2 of $C(S^2_{q\infty})$ corresponds via the previously-used isomorphism ϕ_q to the $\mathbb{Z}_2$-action on $C^*(G_1)$ determined by

$$S_e \mapsto -S_e, \quad S_{f_i} \mapsto -S_{f_{3-i}}.$$

This is a quasi-free automorphism of $C^*(G_1)$ not induced from a graph automorphism. The fixed-point subalgebra for this action coincides with $C(\mathbb{R}P_q^2)$ and is isomorphic with the C^*-algebra of the following graph [HS]:

G_3



The group homomorphism A_{G_3} (see (1)) has the form

$$A_{G_3} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} : \mathbb{Z} \longrightarrow \mathbb{Z}^2. \tag{21}$$

It follows now from (2) that the K-groups of $C(\mathbb{R}P^2_q)$ agree with their classical counterparts:

THEOREM 2. *[HMS]* $K_0(C(\mathbb{R}P^2_q)) \simeq \mathbb{Z} \oplus \mathbb{Z}_2, \quad K_1(C(\mathbb{R}P^2_q)) \simeq 0.$

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REFERENCES

[CK80] J. Cuntz, W. Krieger: A class of C^*-algebras and topological Markov chains, *Invent. Math.* **56**, 251–268 (1980).

[E-DA00] D. A. Ellwood: A new characterisation of principal actions, *J. Funct. Anal.* **173**, 49–60 (2000).

[FLR00] N. J. Fowler, M. Laca, I. Raeburn: The C^*-algebras of infinite graphs, *Proc. Amer. Math. Soc.* **128**, 2319–2327 (2000).

[H-PM96] P. M. Hajac: Strong connections on quantum principal bundles, *Commun. Math. Phys.* **182**, 579–617 (1996).

[HMS] P. M. Hajac, R. Matthes, W. Szymański: Quantum real projective space, disc and sphere, *Algebr. Represent. Theory*, to appear, math.QA/0009185.

[HL] E. Hawkins, G. Landi: Fredholm modules for quantum euclidean spheres, University of Trieste, in preparation.

[HS] J. H. Hong, W. Szymański. Quantum spheres and projective spaces as graph algebras, *Commun. Math. Phys.*, to appear.

[KL93] S. Klimek, A. Lesniewski: A two-parameter quantum deformation of the unit disc, *J. Funct. Anal.* **115**, 1–23 (1993).

[MNW91] T. Masuda, Y. Nakagami, J. Watanabe: Noncommutative differential geometry on the quantum two sphere of Podleś. I. An algebraic viewpoint, *K-Theory* **5**, 151–175 (1991).

[P-P87] P. Podleś: Quantum spheres, *Lett. Math. Phys.* **14**, 193–202 (1987).

[RS] I. Raeburn, W. Szymański: Cuntz-Krieger algebras of infinite graphs and matrices, preprint, 1999.

[VS88] L. L. Vaksman, Ya. S. Soĭbel'man: An algebra of functions on the quantum group SU(2), (Russian) *Funktsional. Anal. i Prilozhen.* **22**, no. 3, 1–14,(1988); translation in *Funct. Anal. Appl.* **22** (1988), no. 3, 170–181 (1989).

[W-NE93] N. E. Wegge-Olsen: *K-theory and C^*-algebras,* Oxford University Press, Oxford–New York–Tokyo 1994.

[W-SL87] S. L. Woronowicz: Twisted SU(2) group. An example of a noncommutative differential calculus. *Publ. Res. Inst. Math. Sci.* **23**, 117–181 (1987).