NONANALYTICITY IN TIME OF SOLUTIONS TO THE KDV EQUATION

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Abstract. It is proved that the formal power series solutions to the initial value problem $\partial_t u = \partial_x^3 u + \partial_x(u^2),\ u(0,x) = \varphi(x)$, where φ is analytic belong to the Gevrey class G^2 in time. However, if $\varphi(x) = 1/(1+x^2)$, the solution does not belong to the Gevrey class G^s in time for $0 \le s < 2$. The proof is based on the estimation of a double sum of products of binomial coefficients.

1. Introduction. We consider the characteristic Cauchy problem for the Korteweg-de Vries equation

$$\begin{aligned} \partial_t u &= \partial_x^3 u + \partial_x(u^2), \\ u(0,x) &= \varphi(x). \end{aligned} \tag{1}$$

The equation appears in the study of a number of different physical systems e.g. it describes the long time evolution of small amplitude dispersive waves. Since its first derivation in the paper by D. J. Korteweg and G. de Vries in 1895 [KdV] it was extensively studied and numerous results has been obtained. The reader interested in different aspects of its theory is referred to the papers by A. Jeffrey and T. Kakutani [JK], D. M. Kruskal [K], P. D. Lax [L], R. M. Miura [M], J. Bourgain [Bou], C. E. Kenig, G. Ponce and L. Vega [KPV], N. Hayashi [H], P. E. Zhidkov [Z] and the references given there.

Here we are interested in the analyticity properties of solutions to (1). The first result in this direction was obtained by E. Trubowitz who showed that solutions with periodic real analytic data remain spatially real analytic for all time ([T], Sec. 3, Amplification 2, [Bou], Remark (iv)). Next, T. Kato and K. Masuda proved that if the initial data φ is analytic and L^2 in a strip along $\mathbb{R}$ then the solution $u(t,\cdot)$ has the same property for all

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time ([KM], Remark 2.1). An analytic smoothing effect for Gevrey data was established by A. De Bouard, N. Hayashi and K. Kato. If φ belongs to the Gevrey class of order 3 then there exists $T > 0$ such that for $0 < t < T$ the solution $u(t, \cdot)$ has an analytic continuation to the complex domain $\{z = x + iy \in \mathbb{C} : |x| < R, |y| < At^{1/3}\}$ with some $A = A(R) > 0$ ([DBHK], Theorem 1.1, Remark 1.1 (III)). The result was obtained by using operators which commute or almost commute with the linear part of the KdV equation. A remarkable result was obtained by K. Kato and T. Ogawa. Under the assumption that $\varphi \in H^s(\mathbb{R})$, $s > -3/4$, satisfy with some positive A

$$\sum_{k=0}^{\infty} \frac{A^k}{(k!)^3} \|(x\partial_x)^k \varphi\|_{H^s} < \infty$$

they proved analyticity of $u(t, \cdot)$ for any $0 < t < T$ and Gevrey regularity of order 3 of $u(\cdot, x)$ for any $x \in \mathbb{R}$. Moreover, under a stronger condition

$$\sum_{k=0}^{\infty} \frac{A^k}{k!} \|(x\partial_x)^k \varphi\|_{H^s} < \infty$$

(which implies analyticity except at the origin) the solution is analytic in both variables at any point of $(0, T) \times \mathbb{R}$ (see [KO], Theorem 1.1 and Corollary 1.2).

However, the above result does not guarantee analyticity of solution in time at $t = 0$ even if the initial data is analytic. Indeed, if φ is analytic then (1) has a unique formal power series solution

$$u(t, x) = \sum_{n=0}^{\infty} \varphi_n(x) t^n, \tag{2}$$

where φ_n are given by the recurrence relations

$$\begin{cases} \varphi_0 &= \varphi, \\ \varphi_{n+1} &= \dfrac{1}{n+1}\Big(\partial_x^3 \varphi_n + \displaystyle\sum_{i=0}^{n} \partial_x(\varphi_i \varphi_{n-i})\Big), \qquad n \in \mathbb{N}_0. \end{cases} \tag{3}$$

We shall prove that this formal solution belongs to the Gevrey class G^2 in time. (Our definition of Gevrey order differs by one from that used in [DBHK] and [KO], but it is consistent with the one used in the summability theory, see [Ba].) Next, we show that the formal solution (2) is divergent if φ does not extend to an entire function of exponential order 3/2 and has nonnegative Taylor coefficients. Note here that the condition that φ is entire of exponential order at most 3/2 is necessary and sufficient for the existence of analytic solutions to the linear counterpart of (1), $\partial_t u = \partial_x^3 u, u(0, x) = \varphi(x)$. Hence, one could expect that the same holds for (1), but it appears that this condition is not necessary since the soliton solutions of (1), $u(t, x) = 6a^2 \cosh^{-2}(ax + 4a^2 t)$, $a \neq 0$, are analytic in both variables at the origin.

The main aim of our paper is to show the divergence of a formal solution in the case of $\varphi(x) = 1/(1+x^2)$. This function is analytic in a strip along $\mathbb{R}$ and it satisfies the conditions of Kato and Ogawa with $s = 0$. Our main result reads as follows:

THEOREM 1. *Let* $\varphi(x) = c/(1+x^2)$ *with* $c < 0$ *or* $0 < c < 5\frac{305}{359}$. *Then the formal solution* (2) *to the initial value problem* (1) *does not belong to the Gevrey class* G^s *in time for* $0 \le s < 2$. *Thus, the solution of* (1) *is not analytic in time at* $t = 0$.

The elementary proof of Theorem 1 follows the method of the proof of an analogous result for the u^2-heat equation [Ly]. In the case of positive c it is based on the following lemma which combinatorial proof seems to be of independent interest.

THE MAIN LEMMA. *For* $k, n \in \mathbb{N}_0$ *put*

$$C(k,n) = \begin{cases} \displaystyle\sum_{i=0}^{n} \sum_{l=0}^{k+1-i \bmod 2} \binom{n}{i} \binom{2k+2}{2l+i \bmod 2} \Big/ \binom{2k+3n+2}{2l+3i+i \bmod 2} & \text{if } n \text{ is even,} \\ \displaystyle\sum_{i=0}^{n} \sum_{l=0}^{k} \binom{n}{i} \binom{2k+1}{2l+i \bmod 2} \Big/ \binom{2k+3n+1}{2l+3i+i \bmod 2} & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

Then

$$C(k,n) \le \begin{cases} 2\frac{9}{70} & \text{for } n \in \mathbb{N}_0 \text{ if } k = 0, \\ k+2 & \text{for } n \in \mathbb{N}_0 \text{ if } k \ge 1. \end{cases} \tag{5}$$

In order to prove the Main Lemma we represent $C(k,n)$ as a finite sum of sequences of the form

$${}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n) = \sum_{l=0}^{n} \binom{2n+\alpha}{2l+\beta} \Big/ \binom{6n+\gamma}{6l+\delta} \tag{6}$$

with some $\alpha, \beta, \gamma, \delta \in \mathbb{N}_0$ and prove that for $k \in \mathbb{N}$, sequences ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}$ appearing in the sum are decreasing. So, $C(k,n)$ is bounded by $C(k,0) = k+2$.

Recently, we have learned that P. Byers and A. Himonas have constructed, by another method, a nonanalytic solution to the KdV equation for a globally analytic initial data ([ByH]).

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2. Gevrey estimates.

Definition 1. We say that the formal power series (2) is in the *Gevrey class* $G^s(\Omega)$ *in time,* $s \geq 0, \Omega \subset \mathbb{R}$, if for any compact set $K \subset\subset \Omega$ one can find $L < \infty$ such that

$$\sup_{n\in\mathbb{N}_0} \sup_{x\in K} \frac{|\varphi_n(x)|}{L^n(n!)^s} < \infty. \tag{7}$$

In the proof of Theorem 2 we shall need

Lemma 1. *Let* $\nu, \mu, m \in \mathbb{N}_0$. *Then*

$$\sum_{k=0}^{m} \frac{(k+\nu)!}{k!} \frac{(m-k+\mu)!}{(m-k)!} = \frac{\nu!\mu!(m+\nu+\mu+1)!}{(\nu+\mu+1)!\; m!}. \tag{8}$$

Proof. The formula (8) is equivalent to

$$\sum_{k=0}^{m} \binom{k+\nu}{\nu}\binom{m-k+\mu}{\mu} = \binom{m+\nu+\mu+1}{\nu+\mu+1}$$

which can be proved by combinatorial methods (see [PBM], Form. 4.2.5.36).

Theorem 2. *Let* φ *be analytic in* $\Omega \subset \mathbb{R}$. *Then the formal solution* (2) *to the initial value problem* (1) *belongs to* $G^2(\Omega)$ *in time.*

Proof. Let K be compact in Ω. Since $\varphi_0 = \varphi$ is analytic in Ω we can find $1 \leq D < \infty$ such that for $m \in \mathbb{N}_0$,

$$\sup_{x\in K} |\partial^m \varphi_0(x)| \leq D^{m+1} m! \quad \text{and} \quad \sup_{x\in K} |\partial^m \varphi_0^2(x)| \leq D^{m+2}(m+1)!. \tag{9}$$

We shall prove that for $n \in \mathbb{N}, m \in \mathbb{N}_0$

$$\sup_{x\in K} |\partial^m \varphi_n(x)| \leq 4/3 \cdot D^{m+3n+1} \frac{(m+3n)!}{n!} \tag{10}$$

which implies (7) with $s = 2$ and $L = 8D^3$. For $n = 1$ we have $\varphi_1 = \partial^3\varphi_0 + \partial(\varphi_0^2)$. Hence by (9) we get $\sup_{x\in K} |\partial^m\varphi_1(x)| \leq 4/3 D^{m+4}(m+3)!$. To prove (10) for $n \geq 2$ note that the recurrence relations (3) imply

$$\varphi_n = \frac{1}{n!}\left(\partial^{3n}\varphi_0 + \sum_{j=0}^{n-1} j!\partial^{3n-3j-2} \sum_{i=0}^{j} \varphi_i\varphi_{j-i}\right), \quad n \in \mathbb{N}. \tag{11}$$

Next by the Leibniz rule, the inductive assumption and Lemma 1 we derive for $j \geq 1$

$$\begin{aligned}\sup_{x\in K} |\partial^m(\varphi_0\varphi_j)(x)| &\leq \sum_{k=0}^{m} \binom{m}{k} D^{k+1}k! \cdot \frac{4}{3} D^{m-k+3j+1} \frac{(m-k+3j)!}{j!} \\ &\leq \frac{4}{3}\frac{1}{j!(3j+1)} D^{m+3j+2}(m+3j+1)!\end{aligned}$$

and for $j \geq 2, 1 \leq i \leq j-1$

$$
\begin{aligned}
\sup_{x \in K} |\partial^m \bigl(\varphi_i \varphi_{j-i}\bigr)(x)| &\leq \sum_{k=0}^{m} \frac{4}{3} \binom{m}{k} D^{k+3i+1} \frac{(k+3i)!}{i!} \\
&\quad \times \frac{4}{3} D^{m-k+3j-3i+1} \frac{(m-k+3j-3i)!}{(j-i)!} \\
&\leq \frac{16}{9} \frac{1}{i!(j-i)!} \frac{(3i)!(3j-3i)!}{(3j+1)!} D^{m+3j+2} (m+3j+1)!.
\end{aligned}
$$

Hence, by (11) and (9) we get

$$
\begin{aligned}
&\sup_{x \in K} |\partial^m \varphi_n(x)| \leq \frac{1}{n!} D^{m+3n+1} (m+3n)! \\
&\quad \times \left\{ 1 + \frac{1}{D(m+3n)} \left[1 + \sum_{j=1}^{n-1} \frac{8}{3} \frac{1}{3j+1} \left(1 + \frac{2}{3} \sum_{i=1}^{j-1} \binom{j}{i} \Big/ \binom{3j}{3i} \right) \right] \right\} \\
&\quad \leq \frac{4}{3} D^{m+3n+1} \frac{(m+3n)!}{n!}
\end{aligned}
$$

since $D \geq 1$ and for $m \geq 0, n \geq 1$

$$
\begin{aligned}
&\frac{1}{m+3n} \left[1 + \sum_{j=1}^{n-1} \frac{8}{3} \frac{1}{3j+1} \left(1 + \frac{2}{3} \sum_{i=1}^{j-1} \binom{j}{i} \Big/ \binom{3j}{3i} \right) \right] \\
&\leq \frac{1}{3n} \left(1 + \sum_{j=1}^{n-1} \frac{8}{9} \frac{2j+1}{3j+1} \right) \leq \frac{1}{3n} \left(1 + \frac{8}{9}(n-1) \right) = \frac{8n+1}{27n} \leq \frac{1}{3}.
\end{aligned}
$$

THEOREM 3. *Fix $\rho \geq 3/2$. Let φ be analytic in $\Omega \subset \mathbb{R}$ and assume that at a point $\mathring{x} \in \Omega$ the Taylor coefficients of φ are nonnegative. If φ does not extend to an entire function of exponential order ρ then the formal solution* (2) *of* (1) *does not belong to $G^s(\Omega)$ in time for any $0 \leq s \leq 2 - 3/\rho$. In particular, it is divergent.*

Proof. Since φ_n are given by (11) the assumption about nonnegativity of Taylor coefficients of φ imply

$$
\varphi_n(\mathring{x}) \geq \frac{1}{n!} \partial^{3n} \varphi(\mathring{x}). \tag{12}
$$

Next the condition that φ is not an entire function of exponential order ρ is equivalent to (see [Bo], Sec. 2.2)

$$
\overline{\lim} \sqrt[n]{\partial^{3n} \varphi(\mathring{x}) \bigl((3n)!\bigr)^{1/\rho - 1}} = \infty,
$$

which together with (12) contradicts (7) for $s \leq 2 - 3/\rho$.

3. Proof of Theorem 1. Assuming that (2) is a formal power series solution of (1) we easily get the recurrence relations (3) for φ_n. Next note that φ_n can be written in the form

$$\varphi_n(x) = \begin{cases} \frac{1}{n!}\sum_{k=0}^{\infty}(-1)^{k+n/2}A(n,2k)x^{2k}, & \text{if } n \text{ is even,} \\ \frac{1}{n!}\sum_{k=0}^{\infty}(-1)^{k+(n-1)/2}A(n,2k+1)x^{2k+1}, & \text{if } n \text{ is odd,} \end{cases} \tag{13}$$

where the coefficients $A(n,2k), A(n,2k+1)$ satisfy

$$\begin{aligned} A(0,2k) &= c, \\ A(n+1,2k+1) &= (2k+2)(2k+3)(2k+4)A(n,2k+4) \\ &\quad - (2k+2)\sum_{i=0}^{n}\binom{n}{i}B(i,n-i,2k+2) \quad \text{for } n \text{ even,} \\ A(n+1,2k) &= (2k+1)(2k+2)(2k+3)A(n,2k+3) \\ &\quad - (2k+1)\sum_{i=0}^{n}\binom{n}{i}B(i,n-i,2k+1) \quad \text{for } n \text{ odd,} \end{aligned} \tag{14}$$

where for n even, odd, respectively

$$\begin{aligned} B(i,n-i,2k+2) &= \begin{cases} \sum_{l=0}^{k+1}A(i,2l)A(n-i,2k+2-2l) & \text{if } i \text{ is even,} \\ \sum_{l=0}^{k}A(i,2l+1)A(n-i,2k+1-2l) & \text{if } i \text{ is odd,} \end{cases} \\ B(i,n-i,2k+1) &= \begin{cases} \sum_{l=0}^{k}A(i,2l)A(n-i,2k+1-2l) & \text{if } i \text{ is even,} \\ \sum_{l=0}^{k}A(i,2l+1)A(n-i,2k-2l) & \text{if } i \text{ is odd.} \end{cases} \end{aligned} \tag{15}$$

Indeed, $\varphi_0(x) = \sum_{k=0}^{\infty}(-1)^k cx^{2k}$. Next assuming inductively (13), by (3) we get for n even

$$\begin{aligned} \varphi_{n+1}(x) = \frac{1}{n+1}\Bigg[&\frac{1}{n!}\sum_{k=2}^{\infty}(-1)^{k+n/2}2k(2k-1)(2k-2)A(n,2k)x^{2k-3} \\ &+ \sum_{i=0,i-\text{even}}^{n}\frac{1}{i!(n-i)!}\sum_{k=1}^{\infty}(-1)^{k+n/2}2k\sum_{l=0}^{k}A(i,2l)A(n-i,2k-2l)x^{2k-1} \\ &+ \sum_{i=0,i-\text{odd}}^{n}\frac{1}{i!(n-i)!}\sum_{k=0}^{\infty}(-1)^{k+(n-1)/2}(2k+2) \\ &\qquad\qquad \times\sum_{l=0}^{k}A(i,2l+1)A(n-i,2k+1-2l)x^{2k+1}\Bigg] \\ = \frac{1}{(n+1)!}&\sum_{k=0}^{\infty}(-1)^{k+n/2}A(n+1,2k+1)x^{2k+1}, \end{aligned}$$

where $A(n+1, 2k+1)$ is given by (14). Similarly we get (14) for n odd.

Now if $c < 0$ we easily get (since in (14) we subtract a positive term)

$$A(n, 2k + n \bmod 2) \leq \frac{(2k+3n+n \bmod 2)!}{(2k+n \bmod 2)!}c. \tag{16}$$

So for n even

$$|\varphi_n(0)| = \frac{|A(n,0)|}{n!} \geq \frac{(3n)!}{n!}|c|$$

and taking $K = \{0\}$ in Definition 1 we see that the formal solution (2) does not belong to $G^s(\mathbb{R})$ in time for $0 \leq s < 2$.

For $c > 0$ the estimation (16) does not prove the theorem. Instead, by the Main Lemma we show

Claim. *Let* $0 < c < 5\frac{305}{359}$. *Then*

$$\begin{aligned} &c\frac{(2k+3n+n \bmod 2)!}{(2k+n \bmod 2)!}\left(1 - \sum_{i=1}^{n} \varepsilon(i, 2k+3n+n \bmod 2 - 3i)\right) \\ \leq & A(n, 2k+n \bmod 2) \leq c\frac{(2k+3n+n \bmod 2)!}{(2k+n \bmod 2)!} \end{aligned} \tag{17}$$

with $\varepsilon(i,l)$, $i \in \mathbb{N}, l \in \mathbb{N}_0$, *defined by*

$$(l+3i-1)(l+3i)\varepsilon(i,l) = \begin{cases} 2\frac{9}{70}c & \text{if } l = 0, i \geq 1, \\ \big((l - l \bmod 2)/2 + 2\big)c & \text{if } l \geq 1,\ i \geq 1. \end{cases} \tag{18}$$

Furthermore,

$$\sum_{i=1}^{n} \varepsilon(i, 2k+3n+n \bmod 2 - 3i) \leq \frac{359}{2100}c. \tag{19}$$

Proof. First of all we show (19). To this end we derive for n even, $n \geq 2$ and $k = 0$

$$\sum_{i=1}^{n} \varepsilon(i, 2k+3n-3i) = \frac{\sum_{j=1}^{n/2}(3n-6j+5) + \frac{9}{70}}{3n(3n-1)}c = \frac{3n^2+4n+\frac{18}{35}}{12n(3n-1)}c \leq \frac{359}{2100}c;$$

for n even, $n \geq 2$ and $k \geq 1$

$$\sum_{i=1}^{n} \varepsilon(i, 2k+3n-3i) = \frac{\sum_{j=1}^{n/2}(3n-6j+5)}{(3n+2k)(3n+2k-1)}c = \frac{3n^2+4n}{4(3n+2k)(3n+2k-1)}c \leq \frac{c}{6};$$

finally, for n odd and $k \in \mathbb{N}_0$

$$\begin{aligned}\sum_{i=1}^{n} \varepsilon(i, 2k+3n+1-3i) &= \frac{k+2+\sum_{j=1}^{(n-1)/2}(2k+3n-6j+6)}{(3n+2k)(3n+2k+1)}c \\ &= \frac{3n^2+(4k+6)n-1}{4(3n+2k)(3n+2k+1)}c \le \frac{c}{6}.\end{aligned}$$

To prove (17) observe that it trivially holds for $n=0$ since $A(0,2k)=c$. Next, if $n=1$ then by (14)

$$\begin{aligned}&A(1,2k+1) = (2k+2)(2k+3)(2k+4)c - (2k+2)(k+2)c^2 \\ &\le c\frac{(2k+4)!}{(2k+1)!}\big(1-\varepsilon(1,2k+1)\big) \quad \text{with} \quad \varepsilon(1,2k+1) = c\frac{k+2}{(2k+3)(2k+4)}.\end{aligned}$$

Now fix $m \in \mathbb{N}$ and assume that (17) holds for $n \le m$ and $k \in \mathbb{N}_0$. Since $A(m+1, 2k+1-m \bmod 2)$ is given by (14) (with $n=m$) and by (19) we subtract a positive term (since $0 < c < 5\frac{305}{359}$), we easily get the estimation from above

$$\begin{aligned}A(m+1,2k+1-m_2) &= (2k+2-m_2)(2k+3-m_2)(2k+4-m_2)A(m,2k+4-m_2) \\ &\le c\frac{(2k+3(m+1)+1-m_2)!}{(2k+1-m_2)!}, \quad \text{where} \quad m_2 = m \bmod 2.\end{aligned}$$

To estimate $A(m+1,2k+1-m_2)$ from below we need to estimate the second term of (14) from above. By the inductive assumption, (15) and (4) we derive for m even

$$\begin{aligned}\sum_{i=0}^{m}\binom{m}{i}B(i,m-i,2k+2) &\le \sum_{i=0, i-\text{even}}^{m}\binom{m}{i}\sum_{l=0}^{k+1}c^2\frac{(2l+3i)!}{(2l)!}\frac{(2k+2-2l+3m-3i)!}{(2k+2-2l)!} \\ &+ \sum_{i=0, i-\text{odd}}^{m}\binom{m}{i}\sum_{l=0}^{k}c^2\frac{(2l+3i+1)!}{(2l+1)!}\frac{(2k-2l+3m-3i+1)!}{(2k-2l+1)!} \\ &= c^2\frac{(2k+3m+2)!}{(2k+2)!}\cdot C(k,m),\end{aligned}$$

and for m odd

$$\begin{aligned}\sum_{i=0}^{m}\binom{m}{i}B(i,m-i,2k+1) &\le \sum_{i=0, i-\text{even}}^{m}\binom{m}{i}\sum_{l=0}^{k}c^2\frac{(2l+3i)!}{(2l)!}\frac{(2k-2l+3m-3i+1)!}{(2k-2l+1)!} \\ &+ \sum_{i=0, i-\text{odd}}^{m}\binom{m}{i}\sum_{l=0}^{k}c^2\frac{(2l+3i+1)!}{(2l+1)!}\frac{(2k-2l+3m-3i)!}{(2k-2l)!} \\ &= c^2\frac{(2k+3m+1)!}{(2k+1)!}\cdot C(k,m).\end{aligned}$$

So, by the Main Lemma and (18) we get

$$A(m+1,2k+1-m_2) \geq c\frac{(2k+3m+4-m_2)!}{(2k+1-m_2)!}\Big(1-\sum_{i=1}^{m}\varepsilon(i,2k+4-m_2+3m-3i)\Big)$$
$$-c\frac{(2k+3m+4-m_2)!}{(2k+1-m_2)!}\varepsilon(m+1,2k+1-m_2).$$

Hence (17) holds for $n=m+1$.

Returning to the proof of Theorem 1 take $K=\{0\}$ in Definition 1. Since for n even the Claim implies

$$|\varphi_n(0)| = \frac{A(n,0)}{n!} \geq c\frac{(3n)!}{n!}\Big(1-\sum_{i=1}^{n}\varepsilon(i,3n-3i)\Big) \geq c\big(1-\frac{359}{2100}c\big)\frac{(3n)!}{n!}$$

the formal solution (2) does not belong to $G^s(\mathbb{R})$ in time for $0 \leq s < 2$ if $0 < c < 5\frac{305}{359}$.

4. The representation of $C(k,n)$. The proof of the Main Lemma is based on the inequality $C(k,n) \geq C(k,n+2)$. In order to prove it we represent $C(k,n)$ as a finite sum of sequences ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n)$ given by (6) with some $\alpha,\beta,\gamma,\delta \in \mathbb{N}_0$. Next, we prove that the sequences ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}$ appearing in the sum are decreasing. The actual form of the sum depends on $k \bmod 3$ and $n \bmod 2$.

Lemma 2. *The $C(k,n)$ given by (4) can be represented as follows*
Case A. If $k=3\overline{k}$ and $n=2\overline{n}$ with $\overline{k},\overline{n} \in \mathbb{N}_0$ then

$$\begin{aligned}
C(3\overline{k},2\overline{n}) = {} & 2\sum_{i=0}^{\lfloor \overline{k}/2\rfloor} a(3\overline{k},6i)\ {}^{0}_{0}D^{12i+2}_{6i}(\overline{n}+\overline{k}-2i)\\
& + 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2\rfloor} a(3\overline{k},6i+2)\ {}^{1}_{0}D^{12i+8}_{6i+2}(\overline{n}+\overline{k}-2i-1)\\
& + 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2\rfloor} a(3\overline{k},6i+3)\ {}^{2}_{1}D^{12i+14}_{6i+6}(\overline{n}+\overline{k}-2i-2)\\
& + 2\sum_{i=0}^{\lfloor \overline{k}/2-1\rfloor} a(3\overline{k},6i+5)\ {}^{1}_{0}D^{12i+14}_{6i+6}(\overline{n}+\overline{k}-2i-2)\\
& + \sum_{i=0}^{\lfloor \overline{k}/2\rfloor} b(3\overline{k},6i+1)\ {}^{2}_{1}D^{12i+8}_{6i+4}(\overline{n}+\overline{k}-2i-1)\\
& + \sum_{i=0}^{\lfloor (\overline{k}-1)/2\rfloor} b(3\overline{k},6i+4)\ {}^{0}_{0}D^{12i+8}_{6i+4}(\overline{n}+\overline{k}-2i-1),
\end{aligned} \tag{20A}$$

where

$$
(21A)\qquad
\begin{aligned}
a(3\overline{k},3l) &= 3^{3l}\binom{2\overline{k}+l}{3l} && \text{for}\quad 0\le l\le \overline{k},\\
a(3\overline{k},3l+2) &= 3^{3l+2}\binom{2\overline{k}+l+1}{3l+2} && \text{for}\quad 0\le l\le \overline{k}-1,\\
b(3\overline{k},3l+1) &= \frac{3^{3l}(6\overline{k}+2)}{3l+1}\binom{2\overline{k}+l}{3l} && \text{for}\quad 0\le l\le \overline{k}.
\end{aligned}
$$

Case B. If $k=3\overline{k}+1$ and $n=2\overline{n}$ with $\overline{k},\overline{n}\in\mathbb{N}_0$ then

$$
(20B)\qquad
\begin{aligned}
C(3\overline{k}+1,2\overline{n}) = {} & 2\sum_{i=0}^{\lfloor \overline{k}/2\rfloor} a(3\overline{k}+1,6i)\ {}_0^1D_{6i}^{12i+4}(\overline{n}+\overline{k}-2i)\\
&+2\sum_{i=0}^{\lfloor \overline{k}/2\rfloor} a(3\overline{k}+1,6i+1)\ {}_1^2D_{6i+4}^{12i+10}(\overline{n}+\overline{k}-2i-1)\\
&+2\sum_{i=0}^{\lfloor (\overline{k}-1)/2\rfloor} a(3\overline{k}+1,6i+3)\ {}_0^1D_{6i+4}^{12i+10}(\overline{n}+\overline{k}-2i-1)\\
&+2\sum_{i=0}^{\lfloor (\overline{k}-1)/2\rfloor} a(3\overline{k}+1,6i+4)\ {}_0^0D_{6i+4}^{12i+10}(\overline{n}+\overline{k}-2i-1)\\
&+\sum_{i=0}^{\lfloor \overline{k}/2\rfloor} b(3\overline{k}+1,6i+2)\ {}_0^0D_{6i+2}^{12i+4}(\overline{n}+\overline{k}-2i)\\
&+\sum_{i=0}^{\lfloor (\overline{k}-1)/2\rfloor} b(3\overline{k}+1,6i+5)\ {}_1^2D_{6i+8}^{12i+16}(\overline{n}+\overline{k}-2i-2),
\end{aligned}
$$

where

$$
(21B)\qquad
\begin{aligned}
a(3\overline{k}+1,3l) &= 3^{3l}\binom{2\overline{k}+l+1}{3l} && \text{for}\quad 0\le l\le \overline{k},\\
a(3\overline{k}+1,3l+1) &= 3^{3l+1}\binom{2\overline{k}+l+1}{3l+1} && \text{for}\quad 0\le l\le \overline{k},\\
b(3\overline{k}+1,3l+2) &= \frac{3^{3l+1}(6\overline{k}+4)}{3l+2}\binom{2\overline{k}+l+1}{3l+1} && \text{for}\quad 0\le l\le \overline{k}.
\end{aligned}
$$

Case C. If $k=3\overline{k}+2$ and $n=2\overline{n}$ with $\overline{k},\overline{n}\in\mathbb{N}_0$ then

$$
C(3\overline{k}+2,2\overline{n}) = \sum_{i=0}^{\lfloor (\overline{k}+1)/2\rfloor} b(3\overline{k}+2,6i)\ {}_0^0D_{6i}^{12i}(\overline{n}+\overline{k}-2i+1)
$$

$$
\begin{aligned}
&+ \sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} b(3\overline{k}+2, 6i+3)\ {}^{2}_{1}D^{12i+12}_{6i+6}(\overline{n}+\overline{k}-2i-1)\\
&+ 2 \sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(3\overline{k}+2, 6i+1)\ {}^{1}_{0}D^{12i+6}_{6i+2}(\overline{n}+\overline{k}-2i)\\
&+ 2 \sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(3\overline{k}+2, 6i+2)\ {}^{0}_{0}D^{12i+6}_{6i+2}(\overline{n}+\overline{k}-2i)\\
&+ 2 \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(3\overline{k}+2, 6i+4)\ {}^{1}_{0}D^{12i+12}_{6i+4}(\overline{n}+\overline{k}-2i-1)\\
&+ 2 \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(3\overline{k}+2, 6i+5)\ {}^{2}_{1}D^{12i+18}_{6i+8}(\overline{n}+\overline{k}-2i-2),
\end{aligned} \tag{20C}
$$

where

$$
\begin{aligned}
a(3\overline{k}+2, 3l+1) &= 3^{3l+1}\binom{2\overline{k}+l+2}{3l+1} && \text{for} \quad 0 \le l \le \overline{k},\\
a(3\overline{k}+2, 3l+2) &= 3^{3l+2}\binom{2\overline{k}+l+2}{3l+2} && \text{for} \quad 0 \le l \le \overline{k},\\
b(3\overline{k}+2, 0) &= 1,\\
b(3\overline{k}+2, 3l) &= \frac{3^{3l-1}(6\overline{k}+6)}{3l}\binom{2\overline{k}+l+1}{3l-1} && \text{for} \quad 1 \le l \le \overline{k}+1.
\end{aligned} \tag{21C}
$$

Case D. If $k = 3\overline{k}$ and $n = 2\overline{n}+1$ with $\overline{k}, \overline{n} \in \mathbb{N}_0$ then

$$
\begin{aligned}
C(3\overline{k}, 2\overline{n}+1) = {}& 2 \sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(3\overline{k}, 6i)\ {}^{1}_{0}D^{12i+4}_{6i}(\overline{n}+\overline{k}-2i)\\
&+ 2 \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(3\overline{k}, 6i+1)\ {}^{2}_{1}D^{12i+10}_{6i+4}(\overline{n}+\overline{k}-2i-1)\\
&+ 2 \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(3\overline{k}, 6i+3)\ {}^{1}_{0}D^{12i+10}_{6i+4}(\overline{n}+\overline{k}-2i-1)\\
&+ 2 \sum_{i=0}^{\lfloor \overline{k}/2-1 \rfloor} a(3\overline{k}, 6i+4)\ {}^{0}_{0}D^{12i+10}_{6i+4}(\overline{n}+\overline{k}-2i-1)\\
&+ \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} b(3\overline{k}, 6i+2)\ {}^{0}_{0}D^{12i+4}_{6i+2}(\overline{n}+\overline{k}-2i)
\end{aligned} \tag{20D}
$$

$$+ \sum_{i=0}^{\lfloor \overline{k}/2-1 \rfloor} b(3\overline{k}, 6i+5)\ {}_1^2D_{6i+8}^{12i+16}(\overline{n}+\overline{k}-2i-2),$$

where

$$(21D)\qquad \begin{aligned} a(3\overline{k}, 3l) &= 3^{3l}\binom{2\overline{k}+l}{3l} && \text{for } \ 0 \le l \le \overline{k}, \\ a(3\overline{k}, 3l+1) &= 3^{3l+1}\binom{2\overline{k}+l}{3l+1} && \text{for } \ 0 \le l \le \overline{k}-1, \\ b(3\overline{k}, 3l+2) &= \frac{3^{3l+1}(6\overline{k}+1)}{3l+2}\binom{2\overline{k}+l}{3l+1} && \text{for } \ 0 \le l \le \overline{k}-1. \end{aligned}$$

Case E. If $k = 3\overline{k}+1$ and $n = 2\overline{n}+1$ with $\overline{k}, \overline{n} \in \mathbb{N}_0$ then

$$(20E)\qquad \begin{aligned} C(3\overline{k}+1, 2\overline{n}+1) = & \sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} b(3\overline{k}+1, 6i)\ {}_0^1D_{6i}^{12i}(\overline{n}+\overline{k}-2i+1) \\ & + \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} b(3\overline{k}+1, 6i+3)\ {}_1^2D_{6i+6}^{12i+12}(\overline{n}+\overline{k}-2i-1) \\ & + 2\sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(3\overline{k}+1, 6i+1)\ {}_0^1D_{6i+2}^{12i+6}(\overline{n}+\overline{k}-2i) \\ & + 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(3\overline{k}+1, 6i+2)\ {}_0^0D_{6i+2}^{12i+6}(\overline{n}+\overline{k}-2i) \\ & + 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(3\overline{k}+1, 6i+4)\ {}_0^1D_{6i+4}^{12i+12}(\overline{n}+\overline{k}-2i-1) \\ & + 2\sum_{i=0}^{\lfloor \overline{k}/2-1 \rfloor} a(3\overline{k}+1, 6i+5)\ {}_1^2D_{6i+8}^{12i+18}(\overline{n}+\overline{k}-2i-2), \end{aligned}$$

where

$$(21E)\qquad \begin{aligned} a(3\overline{k}+1, 3l+1) &= 3^{3l+1}\binom{2\overline{k}+l+1}{3l+1} && \text{for } \ 0 \le l \le \overline{k}, \\ a(3\overline{k}+1, 3l+2) &= 3^{3l+2}\binom{2\overline{k}+l+1}{3l+2} && \text{for } \ 0 \le l \le \overline{k}-1, \\ b(3\overline{k}+1, 0) &= 1, \\ b(3\overline{k}+1, 3l) &= \frac{3^{3l-1}(6\overline{k}+3)}{3l}\binom{2\overline{k}+l}{3l-1} && \text{for } \ 1 \le l \le \overline{k}. \end{aligned}$$

Case F. If $k = 3\overline{k} + 2$ and $n = 2\overline{n} + 1$ with $\overline{k}, \overline{n} \in \mathbb{N}_0$ then

$$
\begin{aligned}
C(3\overline{k}+2, 2\overline{n}+1) = {} & 2 \sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(3\overline{k}+2, 6i)\ {}_0^0D_{6i}^{12i+2}(\overline{n}+\overline{k}-2i+1) \\
& + 2 \sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(3\overline{k}+2, 6i+2)\ {}_0^1D_{6i+2}^{12i+8}(\overline{n}+\overline{k}-2i) \\
& + 2 \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(3\overline{k}+3, 6i+3)\ {}_1^2D_{6i+6}^{12i+14}(\overline{n}+\overline{k}-2i-1) \\
& + 2 \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(3\overline{k}+2, 6i+5)\ {}_0^1D_{6i+6}^{12i+14}(\overline{n}+\overline{k}-2i-1) \\
& + \sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} b(3\overline{k}+2, 6i+1)\ {}_1^2D_{6i+4}^{12i+8}(\overline{n}+\overline{k}-2i) \\
& + \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} b(3\overline{k}+2, 6i+4)\ {}_0^0D_{6i+4}^{12i+8}(\overline{n}+\overline{k}-2i),
\end{aligned} \tag{20F}
$$

where

$$
\begin{aligned}
a(3\overline{k}+2, 3l) &= 3^{3l} \binom{2\overline{k}+l+1}{3l} && \text{for } 0 \le l \le \overline{k}, \\
a(3\overline{k}+2, 3l+2) &= 3^{3l+2} \binom{2\overline{k}+l+2}{3l+2} && \text{for } 0 \le l \le \overline{k}, \\
b(3\overline{k}+2, 3l+1) &= \frac{3^{3l}(6\overline{k}+5)}{3l+1} \binom{2\overline{k}+l+1}{3l} && \text{for } 0 \le l \le \overline{k}.
\end{aligned} \tag{21F}
$$

Proof. We shall prove Lemma 1 only in Case A, since the proofs of the other cases are analogous. So let $n = 2\overline{n}$ be even. Assuming that $\binom{m}{i} = 0$ if $|m - 2i| > m$ with $m \in \mathbb{N}_0, i \in \mathbb{Z}$, we get by (4)

$$
\begin{aligned}
C(k, 2\overline{n}) &= \sum_{i=0}^{\overline{n}} \sum_{l=0}^{k+1} \frac{\binom{2\overline{n}}{2i}\binom{2k+2}{2l}}{\binom{2k+6\overline{n}+2}{2l+6i}} + \sum_{i=0}^{\overline{n}-1} \sum_{l=0}^{k} \frac{\binom{2\overline{n}}{2i+1}\binom{2k+2}{2l+1}}{\binom{2k+6\overline{n}+2}{2l+6i+4}} \\
&= \sum_{j=0}^{\overline{n}+\lfloor (k+1)/3 \rfloor} \frac{\sum_{i=0}^{\overline{n}} \binom{2\overline{n}}{2i}\binom{2k+2}{6j-6i} + \sum_{i=0}^{\overline{n}-1} \binom{2\overline{n}}{2i+1}\binom{2k+2}{6j-6i-3}}{\binom{2k+6\overline{n}+2}{6j}} \\
&+ \sum_{j=0}^{\overline{n}+\lfloor k/3 \rfloor} \frac{\sum_{i=0}^{\overline{n}} \binom{2\overline{n}}{2i}\binom{2k+2}{6j-6i+2} + \sum_{i=0}^{\overline{n}-1} \binom{2\overline{n}}{2i+1}\binom{2k+2}{6j-6i-1}}{\binom{2k+6\overline{n}+2}{6j+2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=0}^{\overline{n}+\lfloor (k-1)/3\rfloor} \frac{\sum_{i=0}^{\overline{n}} \binom{2\overline{n}}{2i}\binom{2k+2}{6j-6i+4} + \sum_{i=0}^{\overline{n}-1} \binom{2\overline{n}}{2i+1}\binom{2k+2}{6j-6i+1}}{\binom{2k+6\overline{n}+2}{6j+4}} \\
&= \sum_{j=0}^{\overline{n}+\lfloor (k+1)/3\rfloor} \frac{1}{\binom{2k+6\overline{n}+2}{6j}} \sum_{l=0}^{\lfloor (2k+2)/3\rfloor} \binom{2k+2}{3l}\binom{2\overline{n}}{2j-l} \\
&+ \sum_{j=0}^{\overline{n}+\lfloor k/3\rfloor} \frac{1}{\binom{2k+6\overline{n}+2}{6j+2}} \sum_{l=0}^{\lfloor 2k/3\rfloor} \binom{2k+2}{3l+2}\binom{2\overline{n}}{2j-l} \\
&+ \sum_{j=0}^{\overline{n}+\lfloor (k-1)/3\rfloor} \frac{1}{\binom{2k+6\overline{n}+2}{6j+4}} \sum_{l=0}^{\lfloor (2k+1)/3\rfloor} \binom{2k+2}{3l+1}\binom{2\overline{n}}{2j+1-l} \\
&=: C_1(k,2\overline{n}) + C_2(k,2\overline{n}) + C_3(k,2\overline{n}).
\end{aligned}
$$

In Case A we have $k = 3\overline{k}$. So

$$
C_1(3\overline{k},2\overline{n}) = \sum_{j=0}^{\overline{n}+\overline{k}} \binom{6\overline{k}+6\overline{n}+2}{6j}^{-1} \sum_{l=0}^{2\overline{k}} \binom{6\overline{k}+2}{3l}\binom{2\overline{n}}{2j-l}.
$$

Now we apply

$$
\sum_{l=l_1}^{2\overline{k}-l_2} \binom{2\overline{k}-l_1-l_2}{l-l_1}\binom{2\overline{n}}{2j-l} = \binom{2\overline{n}+2\overline{k}-l_1-l_2}{2j-l_1}
$$

with $l_1 = l_2 = 0$ and (6) with $\alpha = \beta = \delta = 0$, $\gamma = 2$, to get

$$
C_1(3\overline{k},2\overline{n}) = {}^{0}_{0}D^{2}_{0}(\overline{n}+\overline{k}) + \sum_{j=1}^{\overline{n}+\overline{k}} \binom{6\overline{k}+6\overline{n}+2}{6j}^{-1} \sum_{l=1}^{2\overline{k}} \left\{ \binom{6\overline{k}+2}{3l} - \binom{2\overline{k}}{l} \right\} \binom{2\overline{n}}{2j-l}.
$$

Continuing the above procedure and noting that ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n) = {}^{\alpha}_{\alpha-\beta}D^{\gamma}_{\gamma-\delta}(n)$ we prove inductively that for $m \in \{0, 1, ..., \lfloor \overline{k}/2 \rfloor + 1\}$

$$
C_1(3\overline{k},2\overline{n}) = \sum_{i=0}^{m-1} A(i) + \sum_{j=m}^{\overline{n}+\overline{k}-m} B(j), \tag{22}
$$

where

$$
\begin{aligned}
A(i) &= a(3\overline{k},6i)\, {}^{0}_{0}D^{12i+2}_{6i}(\overline{n}+\overline{k}-2i) + a(3\overline{k},6i+2)\, {}^{1}_{0}D^{12i+8}_{6i+2}(\overline{n}+\overline{k}-2i-1) \\
&+ a(3\overline{k},6i+3)\, {}^{2}_{1}D^{12i+14}_{6i+6}(\overline{n}+\overline{k}-2i-2) + a(3\overline{k},6i+5)\, {}^{1}_{0}D^{12i+14}_{6i+6}(\overline{n}+\overline{k}-2i-2),
\end{aligned}
$$

$$B(j)=\binom{6\overline{n}+6\overline{k}+2}{6j}^{-1}\cdot\sum_{l=2m}^{2\overline{k}-2m}\binom{2\overline{n}}{2j-l}\Bigg\{\binom{6\overline{k}+2}{3l}$$
$$-\sum_{i=0}^{m-1}\Bigg[a(3\overline{k},6i)\binom{2\overline{k}-4i}{l-2i}+a(3\overline{k},6i+2)\binom{2\overline{k}-4i-1}{l-2i-1}$$
$$+a(3\overline{k},6i+3)\binom{2\overline{k}-4i-2}{l-2i-1}+a(3\overline{k},6i+5)\binom{2\overline{k}-4i-3}{l-2i-2}\Bigg]\Bigg\}$$

and the coefficients $a(3\overline{k},3l), a(3\overline{k},3l+2), l\in\mathbb{N}_0$, satisfy recurrence relations

$$a(3\overline{k},3l)=\binom{6\overline{k}+2}{3l}-\sum_{j=0}^{l-1}\Bigg[a(3\overline{k},3j)\binom{2\overline{k}-2j}{l-j}+a(3\overline{k},3j+2)\binom{2\overline{k}-2j-1}{l-j-1}\Bigg], \tag{23}$$

$$a(3\overline{k},3l+2)=\binom{6\overline{k}+2}{3l+2}-\sum_{j=0}^{l-1}\Bigg[a(3\overline{k},3j)\binom{2\overline{k}-2j}{l-j}+a(3\overline{k},3j+2)\binom{2\overline{k}-2j-1}{l-j}\Bigg]$$
$$-a(3\overline{k},3l).$$

Next, by the symmetry of binomial coefficients we easily get

$$C_2(3\overline{k},2\overline{n})=C_1(3\overline{k},2\overline{n}).$$

As about $C_3(3\overline{k},2\overline{n})$ one can prove inductively that for $m\in\{0,1,...,\lfloor\overline{k}/2\rfloor+1\}$

$$C_3(3\overline{k},2\overline{n})=\sum_{i=0}^{m-1}A(i)+\sum_{j=m}^{\overline{n}+\overline{k}-m-1}B(j), \tag{24}$$

where

$$A(i)=b(3\overline{k},6i+1)\;{}^{2}_{1}D^{12i+8}_{6i+4}(\overline{n}+\overline{k}-2i-1)+b(3\overline{k},6i+4)\;{}^{0}_{0}D^{12i+8}_{6i+4}(\overline{n}+\overline{k}-2i-1),$$

$$B(j)=\binom{6\overline{n}+6\overline{k}+2}{6j+4}^{-1}\cdot\sum_{l=2m}^{2\overline{k}-2m}\binom{2\overline{n}}{2j+1-l}\Bigg\{\binom{6\overline{k}+2}{3l+1}$$
$$-\sum_{i=0}^{m-1}\Bigg[b(3\overline{k},6i+1)\binom{2\overline{k}-4i}{l-2i}+b(3\overline{k},6i+4)\binom{2\overline{k}-4i-2}{l-2i-1}\Bigg]\Bigg\}$$

and the coefficients $b(3\overline{k},3l+1), l\in\mathbb{N}_0$, satisfy a recurrence relation

$$b(3\overline{k},3l+1)=\binom{6\overline{k}+2}{3l+1}-\sum_{j=0}^{l-1}b(3\overline{k},3j+1)\binom{2\overline{k}-2j}{l-j}. \tag{25}$$

Finally, since for $m=\lfloor\overline{k}/2\rfloor+1$ the second summands in (22) and in (24) vanish, by Lemma 3 (stated below) we get (20A) and (21A), proving Lemma 2 in Case A. The proofs of Cases B – E are done in the same way.

LEMMA 3. *Let $a(3\overline{k},3l)$, $a(3\overline{k},3l+2)$ and $b(3\overline{k},3l+1),\overline{k},l\in\mathbb{N}_0$, satisfy recurrence relations* (23) *and* (25)*. Then* (21A) *holds.*

Proof. For $l=0$ we clearly have

$$a(3\overline{k},0)=1,\quad a(3\overline{k},2)=\binom{6\overline{k}+2}{2}-1=3^2\binom{2\overline{k}+1}{2},\quad b(3\overline{k},1)=6\overline{k}+2.$$

To prove (21A) for $l\geq 1$ put $2\overline{k}=\tilde{k}$ and note that it is sufficient to show that

$$\binom{3\tilde{k}+2}{3l}=\sum_{j=0}^{l}3^{3j}\left[\binom{\tilde{k}+j}{3j}\binom{\tilde{k}-2j}{l-j}+9\binom{\tilde{k}+j+1}{3j+2}\binom{\tilde{k}-2j-1}{l-j-1}\right],\tag{26}$$

$$\binom{3\tilde{k}+2}{3l+2}=\sum_{j=0}^{l}3^{3j}\left[\binom{\tilde{k}+j}{3j}\binom{\tilde{k}-2j}{l-j}+9\binom{\tilde{k}+j+1}{3j+2}\binom{\tilde{k}-2j-1}{l-j}\right],\tag{27}$$

$$\binom{3\tilde{k}+2}{3l+1}=\sum_{j=0}^{l}\frac{3^{3j}(3\tilde{k}+2)}{3j+1}\binom{\tilde{k}+j}{3j}\binom{\tilde{k}-2j}{l-j}.\tag{28}$$

To show (26) for a fixed $l\in\mathbb{N}$ we observe that the left-hand side is a polynomial on $\tilde{k}$ of degree $3l$, with the leading coefficient $3^{3l}/(3l)!$, vanishing at $\tilde{k}=0,...,l-1$ and at $\tilde{k}=-1/3+m,\tilde{k}=-2/3+m$ with $m=0,...,l-1$. Clearly, the first two statements also hold for the right-hand side of (27). So we only need to prove the third one. To this end we compute for $j=0,...,l$

$$\begin{aligned}&3^{3j}\left[\binom{\tilde{k}+j}{3j}\binom{\tilde{k}-2j}{l-j}+9\binom{\tilde{k}+j+1}{3j+2}\binom{\tilde{k}-2j-1}{l-j-1}\right]\\&=(\tilde{k}-l+1)\cdot\ldots\cdot\tilde{k}\times\left(1+\frac{9(l-j)(\tilde{k}+j+1)}{(3j+1)(3j+2)}\right)\\&\times\frac{3^{3j}(\tilde{k}-l-j+1)\cdot\ldots\cdot(\tilde{k}-l)\cdot(\tilde{k}+1)\cdot\ldots\cdot(\tilde{k}+j)}{(l-j)!(3j)!}.\end{aligned}$$

So it is sufficient to show that the polynomial

$$\begin{aligned}W_l(\tilde{k})=1+\frac{9l(\tilde{k}+1)}{2}+\sum_{j=1}^{l}\frac{l!}{(l-j)!}&\left(1+\frac{9(l-j)(\tilde{k}+j+1)}{(3j+1)(3j+2)}\right)\\&\times\frac{3^{3j}(\tilde{k}-l-j+1)\cdot\ldots\cdot(\tilde{k}-l)\cdot(\tilde{k}+1)\cdot\ldots\cdot(\tilde{k}+j)}{(3j)!}\end{aligned}$$

vanishes for $\tilde{k}=-1/3+m$ and $\tilde{k}=-2/3+m$ with $m=0,...,l-1$. But for $m=0,...,l-1$

we derive

$$W_l(-1/3+m) = 1 + \frac{3l(3m+2)}{2} + \sum_{j=1}^{l} (-1)^j \frac{l!}{(l-j)!j!} \left(1 + \frac{3(l-j)(3m+3j+2)}{(3j+1)(3j+2)}\right)$$
$$\times \prod_{i=0}^{j-1} \frac{(3l-3m+3i+1)(3m+3i+2)}{(3i+1)(3i+2)} = \sum_{j=0}^{l} (-1)^j \binom{l}{j} P_l(j) = 0,$$

since for $m = 0, 1, ..., l-1$

$$P_l(j) = \frac{(9l-9m+3)j + 9lm + 6l + 2}{(3j+1)(3j+2)} \prod_{i=0}^{l-m-1} \frac{3j+3i+1}{3i+1} \prod_{i=0}^{m-1} \frac{3j+3i+2}{3i+2}$$

is a polynomial of degree $l-1$ and for such a polynomial P, $\sum_{j=0}^{l}(-1)^j\binom{l}{j}P(j) = 0$. Analogously for $m = 0, ..., l-1$

$$W_l(-2/3+m) = 1 + \frac{3l(3m+1)}{2} + \sum_{j=1}^{l} (-1)^j \frac{l!}{(l-j)!j!} \left(1 + \frac{3(l-j)(3m+3j+2)}{(3j+1)(3j+2)}\right)$$
$$\times \prod_{i=0}^{j-1} \frac{(3m+3i+1)(3l-3m+3i+2)}{(3i+1)(3i+2)} = \sum_{j=0}^{l} (-1)^j \binom{l}{j} P_l(j) = 0,$$

where

$$P_l(j) = \frac{(9l-9m+6)j + 9lm + 3l + 2}{(3j+1)(3j+2)} \prod_{i=0}^{m-1} \frac{3j+3i+1}{3i+1} \prod_{i=0}^{l-m-1} \frac{3j+3i+2}{3i+2}$$

is a polynomial of degree $l-1$. The proofs of (27) and (28) go along the same lines.

5. An auxiliary lemma. In the proof of the Main Lemma we shall also need

LEMMA 4. *Let* ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}$ *be given by* (6). *Assume one of the cases*
Case 1°. $\alpha = 0, \beta = 0$, $\delta \geq 0$, $\gamma = 2\delta + \eta$ *with* $\eta = 0, 2$;
Case 2°. $\alpha = 1, \beta = 0$, $\delta \geq 0$, $\gamma = 2\delta + \eta$ *with* $\eta = 2, 4$;
Case 3°. $\alpha = 2, \beta = 1$, $\delta \geq 2$, $\gamma = 2\delta + \eta$ *with* $\eta = 0, 2$;
Then for $n \geq 2$

$$ {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n) \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1). \tag{29}$$

Furthermore, (29) *holds for* $n = 0$ *if* $\delta \geq 1$, *and for* $n = 1$ *except Case* 1° *with* $\delta = \gamma = 0$.

Proof. For $n, l \in \mathbb{N}_0$ put

$$ {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n,l) = \binom{2n+\alpha}{2l+\beta} \Big/ \binom{6n+\gamma}{6l+\delta}$$

Note that it is sufficient to show that for n even the following inequalities hold

$$ {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n,l) \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,l) \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+2,l) \quad \text{for } l = 0, ..., n/2-1, \quad n \geq 2; \tag{IN1}$$

$$(IN2) \qquad \begin{array}{l} {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n,l) \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,l+1) \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+2,l+2) \\ \text{for} \quad l = n/2+1,...,n, \quad n \geq 2; \end{array}$$

$$(IN3) \qquad \begin{array}{l} {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n,n/2) \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,n/2) + {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,n/2+1) \\ \text{for} \quad n \geq 2 \quad \text{and} \quad \text{for} \quad n=0 \quad \text{if} \quad \delta \geq 1; \end{array}$$

$$(IN4) \qquad \begin{array}{c} {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,n/2) + {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,n/2+1) \\ \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+2,n/2) + {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+2,n/2+1) + {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+2,n/2+2) \\ \text{for} \quad n \geq 2 \quad \text{and for} \quad n=0 \quad \text{except Case 1} \quad \text{with} \quad \gamma=\delta=0. \end{array}$$

Proof of $(IN1)$. Using the definition of ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n,l)$, expanding the binomial coefficients and cancelling similar factors we see that the first inequality in $(IN1)$ is equivalent to

$$(30) \qquad \prod_{i=1}^{2} \frac{2n+\alpha+i}{2n-2l+\alpha-\beta+i} \prod_{j=1}^{6} \frac{6n-6l+\gamma-\delta+j}{6n+\gamma+j} \leq 1.$$

which in tern is implied by (since $\delta \geq 0$)

$$(30a) \qquad \Big(1 - \frac{6l+\delta}{6n+\gamma+5}\Big)^3 \leq 1 - \frac{2l+\beta}{2n+\alpha+1},$$

$$(30b) \qquad \Big(1 - \frac{6l+\delta}{6n+\gamma+6}\Big)^3 \leq 1 - \frac{2l+\beta}{2n+\alpha+2}.$$

To show $(30a)$ we fix $l \in \mathbb{N}_0$ and put $n = 2l+2m+2$ with some $m \in \mathbb{N}_0$,

$$x = \frac{6l+\delta}{12l+12m+\gamma+17}, \quad y = \frac{2l+\beta}{4l+4m+\alpha+5}.$$

Next note that since $x \geq 0$ it is sufficient to show that $3x - 3x^2 \geq y$. But $\gamma = 2\delta + \eta$ with $\eta \geq -17$ implies that $x \leq 1/2$ and so $3x - 3x^2 \geq \frac{3}{2}x$. Now, $\frac{3}{2}x \geq y$ is equivalent to

$$(31a) \qquad \begin{array}{l} 24l^2 + (24m + 12\delta - 4\gamma + 18\alpha - 24\beta + 22)l \\ \qquad + (12\delta - 24\beta)m + 3\delta(\alpha+5) - (2\gamma+34)\beta \geq 0 \end{array}$$

which in Cases 1°, 2°, 3° holds for any $l \in \mathbb{N}, m \in \mathbb{N}_0$ and for $l = 0, m \in \mathbb{N}_0$ except Case 3° with $\delta = \eta = 2$. $(30b)$ we treat in the same way with

$$x = \frac{6l+\delta}{12l+12m+\gamma+18}, \quad y = \frac{2l+\beta}{4l+4m+\alpha+6}$$

and $(31a)$ replaced by

$$(31b)\qquad \begin{aligned}&24l^2+(24m+12\delta-4\gamma+18\alpha-24\beta+36)l\\&\quad+(12\delta-24\beta)m+3\delta(\alpha+6)-(2\gamma+36)\beta\geq 0.\end{aligned}$$

Finally, we directly show (30) for $l=0$ in the exceptional Case 3° with $\delta=2,\gamma=6$.

The second inequality in $(IN1)$ is equivalent to

$$(32)\qquad \prod_{i=3}^{4}\frac{2n+\alpha+i}{2n-2l+\alpha-\beta+i}\ \prod_{j=7}^{12}\frac{6n-6l+\gamma-\delta+j}{6n+\gamma+j}\leq 1.$$

which in tern is implied by (since $\delta\geq 0$)

$$(32a)\qquad \Big(1-\frac{6l+\delta}{6n+\gamma+11}\Big)^3\leq 1-\frac{2l+\beta}{2n+\alpha+3},$$

$$(32b)\qquad \Big(1-\frac{6l+\delta}{6n+\gamma+12}\Big)^3\leq 1-\frac{2l+\beta}{2n+\alpha+4}.$$

To show $(32a)$ we follow the proof of $(30a)$ with

$$x=\frac{6l+\delta}{12l+12m+\gamma+23},\quad y=\frac{2l+\beta}{4l+4m+\alpha+7}.$$

and $(31a)$ replaced by

$$(33a)\qquad \begin{aligned}&24l^2+(24m+12\delta-4\gamma+18\alpha-24\beta+34)l\\&\quad+(12\delta-24\beta)m+3\delta(\alpha+7)-(2\gamma+46)\beta\geq 0\end{aligned}$$

which in Cases 1°, 2°, 3° holds for any $l\in\mathbb{N},m\in\mathbb{N}_0$ and for $l=0,m\in\mathbb{N}_0$ except Case 3° with $\delta=\eta=2$. $(32b)$ we treat in the same way with

$$x=\frac{6l+\delta}{12l+12m+\gamma+24},\quad y=\frac{2l+\beta}{4l+4m+\alpha+8}$$

and $(33a)$ replaced by

$$(33b)\qquad \begin{aligned}&24l^2+(24m+12\delta-4\gamma+18\alpha-24\beta+48)l\\&\quad+(12\delta-24\beta)m+3\delta(\alpha+8)-(2\gamma+48)\beta\geq 0.\end{aligned}$$

Finally, we directly show (32) for $l=0$ in the exceptional Case 3° with $\delta=2,\gamma=6$.

Proof of $(IN2)$. Note that $(IN2)$ is equivalent to

$${}_{\alpha-\beta}^{\ \alpha}D_{\gamma-\delta}^{\gamma}(n,l)\geq\ {}_{\alpha-\beta}^{\ \alpha}D_{\gamma-\delta}^{\gamma}(n+1,l)\geq\ {}_{\alpha-\beta}^{\ \alpha}D_{\gamma-\delta}^{\gamma}(n+2,l)$$

for $l'=0,...,n/2-1$. Hence, we have to show $(IN1)$ in the cases

Case 1′. $\alpha=0,\beta=0,\ \delta\geq 0,\ \gamma=2\delta+\eta$ with $\eta=0,-2$;
Case 2′. $\alpha=1,\beta=1,\ \delta\geq 2,\ \gamma=2\delta+\eta$ with $\eta=-2,-4$;
Case 3′. $\alpha=2,\beta=1,\ \delta\geq 2,\ \gamma=2\delta+\eta$ with $\eta=0,-2$;
In these cases $(31a)$, $(31b)$, $(33a)$ and $(33b)$ hold for any $l,m\in\mathbb{N}_0$ except Case 2′ with $\delta=2,\gamma=2$ and $l=0$. In this exceptional case we directly check (30) and (32).

Proof of $(IN3)$. Expanding the binomial coefficients and cancelling the similar factors we see that $(IN3)$ is equivalent to

$$\prod_{i=1}^{2}\frac{2n+\alpha+i}{n+\alpha-\beta+i}\prod_{j=1}^{6}\frac{3n+\gamma-\delta+j}{6n+\gamma+j}+\prod_{i=1}^{2}\frac{2n+\alpha+i}{n+\beta+i}\prod_{j=1}^{6}\frac{3n+\delta+j}{6n+\gamma+j}\le 1. \tag{34}$$

Case 1°. Then (34) is implied by

$$4\cdot\frac{2n+1}{n+2}\prod_{j=1}^{6}\frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j}\le 1$$

which holds for $n\ge 2$, and for $n=0$ if $\delta\ge 1$.

Case 2°. Then (34) takes the form

$$\frac{2n+2}{n+2}\cdot\frac{2n+3}{n+3}\prod_{j=1}^{6}\frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j}+2\cdot\frac{2n+3}{n+1}\prod_{j=1}^{6}\frac{3n+\delta+j}{6n+2\delta+\eta+j}\le 1$$

which holds for $n\ge 2$, and for $n=0$ if $\delta\ge 1$.

Case 3°. Then (34) takes the form

$$2\cdot\frac{2n+3}{n+3}\left(\prod_{j=1}^{6}\frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j}+\prod_{j=1}^{6}\frac{3n+\delta+j}{6n+2\delta+\eta+j}\right)\le 1$$

which holds for $n\ge 2$, and for $n=0$ if $\delta\ge 1$.

Proof of $(IN4)$. Expanding the binomial coefficients and cancelling the similar factors we see that $(IN4)$ is equivalent to

$$\begin{aligned}&\prod_{i=1}^{4}\frac{2n+\alpha+i}{n+\alpha-\beta+i}\prod_{j=1}^{12}\frac{3n+\gamma-\delta+j}{6n+\gamma+j}+\prod_{i=1}^{4}\frac{2n+\alpha+i}{n+\beta+i}\prod_{j=1}^{12}\frac{3n+\delta+j}{6n+\gamma+j}\\&+\prod_{i=1}^{2}\frac{2n+\alpha+i}{n+\alpha-\beta+i}\prod_{i=1}^{2}\frac{2n+\alpha+i+2}{n+\beta+i}\prod_{j=1}^{6}\frac{3n+\gamma-\delta+j}{6n+\gamma+j}\prod_{j=1}^{6}\frac{3n+\delta+j}{6n+\gamma+j+6}\\&\le\prod_{i=1}^{2}\frac{2n+\alpha+i}{n+\alpha-\beta+i}\prod_{j=1}^{6}\frac{3n+\gamma-\delta+j}{6n+\gamma+j}+\prod_{i=1}^{2}\frac{2n+\alpha+i}{n+\beta+i}\prod_{j=1}^{6}\frac{3n+\delta+j}{6n+\gamma+j}.\end{aligned}\tag{35}$$

Note that for $\gamma=2\delta+\eta$ with $0\le\eta\le 6$

$$2\prod_{j=1}^{6}\frac{3n+\gamma-\delta+j}{6n+\gamma+j}\prod_{j=1}^{6}\frac{3n+\delta+j}{6n+\gamma+j+6}\le\prod_{j=1}^{12}\frac{3n+\gamma-\delta+j}{6n+\gamma+j}+\prod_{j=1}^{12}\frac{3n+\delta+j}{6n+\gamma+j}.$$

Hence, (35) is implied by

$$(35a)\qquad \left(\prod_{i=3}^{4}\frac{2n+\alpha+i}{n+\alpha-\beta+i}+\frac{1}{2}\prod_{i=1}^{2}\frac{2n+\alpha+i+2}{n+\beta+i}\right)\prod_{j=7}^{12}\frac{3n+\gamma-\delta+j}{6n+\gamma+j}\leq 1,$$

$$(35b)\qquad \left(\prod_{i=3}^{4}\frac{2n+\alpha+i}{n+\beta+i}+\frac{1}{2}\prod_{i=1}^{2}\frac{2n+\alpha+i+2}{n+\alpha-\beta+i}\right)\prod_{j=7}^{12}\frac{3n+\delta+j}{6n+\gamma+j}\leq 1.$$

Case 1°. Then $(35b)$ is weaker then $(3a)$ which takes the form

$$\left(\frac{2n+3}{n+3}\cdot\frac{2n+4}{n+4}+\frac{2n+3}{n+1}\right)\prod_{j=7}^{12}\frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j}\leq 1$$

and it clearly holds for $n\geq 2$, and for $n=0$ if $\delta\geq 4$. If $n=0$ we directly check (35) for $\delta=1,2,3$, $\eta=0$ and for $\delta=0,1,2,3$, $\eta=2$.

Case 2°. Then $(35a)$ and $(35b)$ take the form

$$\left(\frac{2n+4}{n+4}\cdot\frac{2n+5}{n+5}+\frac{2n+5}{n+1}\right)\prod_{j=7}^{12}\frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j}\leq 1,$$

$$\left(\frac{2n+4}{n+3}\cdot\frac{2n+5}{n+4}+\frac{2n+5}{n+3}\right)\prod_{j=7}^{12}\frac{3n+\delta+j}{6n+2\delta+\eta+j}\leq 1.$$

Clearly, both inequalities hold for $n\geq 2$, and for $n=0$ if $\delta\geq 8$. If $n=0$ we directly check (35) for $\delta=0,...,7$, $\eta=2,4$.

Case 3°. Then $(35b)$ is weaker then $(3a)$ which takes the form

$$\left(\frac{2n+5}{n+4}\cdot\frac{2n+6}{n+5}+\frac{2n+5}{n+2}\right)\prod_{j=7}^{12}\frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j}\leq 1$$

and it clearly holds for $n\geq 2$, and for $n=0$ if $\delta\geq 4$. If $n=0$ we directly check (35) for $\delta=1,2,3$, $\eta=0,2$.

6. Proof of the Main Lemma. First of all observe that all sequences ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}$ appearing in the representation of $C(k,n)$ given by $(20A)-(20F)$ fall within one of the cases of Lemma 4. Next, the coefficients $a(k,l), b(k,l)$ in $(20A)-(20F)$ are given by $(21A)-(21F)$, and so they are nonnegative. Hence, by Lemma 4 we get $C(k,n)\geq C(k,n+2)$ for $n\geq 2$, for $n\geq 1$ if $k\geq 2$ and for $n\geq 0$ if $k\geq 3$. But $C(1,0)=3>C(1,2)=2\frac{37}{105}$, $C(1,1)=2\frac{2}{5}>C(1,3)=2\frac{103}{770}$ and $C(2,0)=4>C(2,2)=2\frac{67}{110}$. So,

$$C(k,n)\geq C(k,n+2)\quad\text{for}\quad n\geq 2 \text{ if } k=0\quad\text{and for}\quad n\geq 0\quad\text{if}\quad k\geq 1.$$

Now, if $n=0$ we easily get $C(k,0)=k+2$ for $k\in\mathbb{N}_0$. Next, if $n=1$ we derive for $k\in\mathbb{N}_0$

$$C(k,1)=2\sum_{l=0}^{k}\frac{(2l+2)(2l+3)(2l+4)}{(2k+2)(2k+3)(2k+4)}=\frac{(k+2)(k+3)}{2k+3}\leq k+2.$$

Hence, $C(k,n)\leq C(k,0)=k+2$ for $n\in\mathbb{N}_0$ if $k\geq 1$. To finish the proof we compute $C(0,2)=2\frac{9}{70}>C(0,3)=2\frac{2}{70}$.

References

[Ba] W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*, Springer-Verlag, New York, 2000.

[Bo] R. P. Boas, *Entire functions*, Academic Press, New York, 1954.

[Bou] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part II: The KdV-equation*, Geom. and Functional Anal. 3 (1993), 209–262.

[ByH] P. Byers, A. Himonas, *Non-analytic solutions of the KdV equation*, to appear in Abstract and Applied Analysis.

[DBHK] A. De Bouard, N. Hayashi and K. Kato, *Regularizing effect for the (generalized) Korteweg-de Vries equation and nonlinear Schrödinger equation*, Ann. Inst. Henri Poincaré, Anal. non-linéare 9 (1995), 673–725.

[H] N. Hayashi, *Analytic function spaces and their applications to nonlinear evolution equations*, Analytic extension formulas and their applications, Kluwer, 2001, 73–86.

[JK] A. Jeffery and T. Kakutani, *Weak nonlinear dispersive waves: A discussion centered around Korteweg-de Vries equation*, SIAM Review 14 (1972), 582–643.

[KM] T. Kato and M. Masuda, *Nonlinear evolutions equations and analyticity. I*, Ann. Inst. Henri Poincaré, Anal. non-linéare 3 (1986), 455–467.

[KO] K. Kato and T. Ogawa, *Analyticity and smoothing effect for the Korteweg de Vries equation with a single point singularity*, Math. Ann. 316 (2000), 577–608.

[KPV] C. E. Kenig, G. Ponce and L. Vega, *A bilinear estimate with the application to the KdV equation*, Trans. Amer. Math. Soc. 348 (1996), 573–603.

[KdV] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag. 39 (1895), 422–443.

[K] M. D. Kruskal, *Nonlinear wave equations*, Dynamical Systems, Theory and Applications, J. Moser, ed., Springer-Verlag, New York, 1975, 310–354.

[L] P. D. Lax, *Almost periodic solutions of the KdV equation*, SIAM Review 18 (1976), 351–375.

[Ły] G. Łysik, *Nonanalyticity of solutions to* $\partial_t u = \partial_x^2 u + u^2$, to appear in Colloquium Math.

[M] R. M. Miura, *The Korteweg-de Vries equation: a survey of results*, SIAM Review 18 (1976), 412–459.

[PBM] A. P. Prudnikov, Yu. A. Brychkov and O.I. Marichev, *Integrals and series. Elementary functions*, Nauka, Moscow, 1981 (in Russian); English transl.: Gordon and Breach, 1986.

[T] E. Trubowitz, *The inverse problem for periodic potentials*, Comm. Pure Appl. Math. 30 (1977), 321–337.

[Z] P. E. Zhidkov, *Korteweg-de Vries and Nonlinear Schrödinger Equations: Qualitative Theory*, Lecture Notes in Mathematics 1756, Springer-Verlag, Berlin, 2001.