ON THE ŁOJASIEWICZ EXPONENT AND EFFECTIVE NULLSTELLENSATZ

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Dedicated to Professor Arkadiusz Płoski.

ABSTRACT. Let $\mathbb{K}$ be an algebraically closed field and let $X \subset \mathbb{K}^m$ be an affine variety. Assume that $f_1, ..., f_m$ are polynomials which have no common zeros on X. We estimate degrees of polynomials $A_i \in \mathbb{K}[X]$ such that $1 = \sum_{i=1}^m A_i f_i$ on X. In the more general case, when $f_1, ..., f_m$ have finite number of zeros we give a sharp estimation of Łojasiewicz exponent at infinity (with respect to arbitrary non-trivial absolute value on $\mathbb{K}$) of the mapping $f = (f_1, ..., f_m) : X \to \mathbb{K}^m$. At the end of this paper we estimate the degree of the set of non-properness for every generically-finite separable mapping $f = (f_1, ..., f_m) : X \to \mathbb{K}^m$ and we show that such a mapping must be finite, provided it has a sufficiently large geometric degree.

1. INTRODUCTION.

In recent years there has been a great deal of interest in the problem of finding effective versions of Hilbert Nullstellensatz (see e.g. [1], [3], [6], [7]). The classical theorem states if polynomials $f_1, ..., f_m \in \mathbb{C}[x_1, ..., x_n]$, have no common zeroes in $\mathbb{C}^n$, then they generate the unit ideal, i.e., there exists $g_j \in \mathbb{C}[x_1, ..., x_n]$ such that

$$\sum g_j f_j = 1.$$

The problem is to bound the degrees of the g_j in terms of the f_j. This question was considered by D. Brownawell [1], who proved that if deg $f_j \le d$ for all j, then one can find g_j such that

$$\deg\, g_j \le n^2 d^n + nd.$$

Moreover, he proved that if f_i are as above then there exists a constant $C > 0$, such that

$$\max|f_i(x)| \ge C||x||^{1-(n-1)D^{\min\{n,k\}}}.$$

Shortly thereafter J. Kollar (see [6]) improved results of Brownawell and obtained the following theorem:

Theorem 1.1. *([6], Corollary 1.9) Let $\mathbb{K}$ be a field and let $f_1, ..., f_k \in \mathbb{K}[x_1, ..., x_n]$ be non-zero polynomials without common zeros. Assume that* deg $f_i = d_i$, *where $d_1 \ge d_2 \ge ... \ge d_k > 2$. There exists polynomials g_i, such that*

a) deg $f_i g_i \le B(d_1, ..., d_k; n)$,

b) $1 = \sum_{i=1}^k f_i g_i$.

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Recall that for a sequence $d_1 \geq d_2 \geq ... \geq d_k$, we have $B(d_1, ..., d_k; n) = \prod_{i=1}^{k} d_i$ for $k \leq n$, $B(d_1, ..., d_k; n) = (\prod_{i=1}^{n-1} d_i)d_k$ for $k > n > 1$ and $B(d_1, ..., d_k; n) = d_1 + d_k - 1$ for $k > n = 1$.

If l numbers of d_i equal to 2, then the Kollar method gives a weaker estimation - deg $f_i g_i \leq (3/2)^l B(d_1, ..., d_k, n)$. Kollar also improved the second result of Brownawell:

Theorem 1.2. *([6], Proposition 1.10) Let $f_1, ..., f_k \in \mathbb{C}[x_1, ..., x_n]$ such that the f_i have only finitely many common zeros. Let $d_i = \deg f_i$ (where $d_1 \geq d_2 \geq ... \geq d_k$) and assume that $n \geq 2$. Then there is a constant $C > 0$, such that*

$$\max |f_i(x)| \geq C||x||^{d_k - B(d_1, ..., d_k; n)}$$

for $||x||$ sufficiently large.

In this paper we consider the analogous problems for polynomials $f_1, ..., f_k$ on affine algebraic variety X over an algebraically closed field $\mathbb{K}$. Moreover, we generalize Theorem 1.2 to all algebraically closed fields with arbitrary non-trivial absolute values.

We always assume that X is of positive dimension and $f_i|_X \not\equiv 0$. For a sequence $d_1 \geq d_2 \geq ... \geq d_k$ let $N(d_1, ..., d_k; n) = B(d_1, ..., d_k; n)$ if $n > 1$ and $N(d_1, ..., d_k; 1) = d_1$. We prove the following theorem:

Theorem 1.3. *Let $\mathbb{K}$ be an algebraically closed field, $X \subset \mathbb{K}^m$ be an affine irreducible $n-$dimensional variety of degree D. Let $f_1, ..., f_k \in \mathbb{K}[x_1, ..., x_n]$ be non-zero polynomials without common zeros on X. Assume that* deg $f_i = d_i$, *where $d_1 \geq d_2 \geq ... \geq d_k$. There exists polynomials g_i, with deg $f_i g_i \leq DN(d_1, ..., d_k; n)$ if $k \leq n$ and deg $f_i g_i \leq 2DN(d_1, ..., d_k; n) - 1$ if $k > n$, such that*

$$1 = \sum_{i=1}^{k} f_i g_i \text{ on } X.$$

Observe that in the particular case if $X = \mathbb{K}^n$, our result gives in some cases the estimation which is better that the Kollar estimation (in fact for $k \leq n$ we obtain the best possible result). However in the general case ($k > n$) our estimation is slightly weaker. We also prove the following generalization of Theorem 1.2:

Theorem 1.4. *Let $\mathbb{K}$ be an algebraically closed field with a non-trivial absolute value $| - |_v : \mathbb{K} \to \mathbb{R}$. Let $X \subset \mathbb{K}^m$ be an affine $n-$dimensional variety of degree D. Assume that $f_1, ..., f_k \in \mathbb{K}[x_1, ..., x_m]$ have only finite number ν of zeros on X. Let $d_i =$ deg f_i (where $d_1 \geq d_2 \geq ... \geq d_k$). Then there is a constant $C > 0$, such that for $x \in X$*

$$\max |f_i(x)|_v \geq C||x||_v^{d_k - DN(d_1, ..., d_k; n) + \nu},$$

for $||x||_v$ sufficiently large.

Here for $x = (x_1, ..., x_n) \in \mathbb{K}^n$ by $||x||_v$, we mean max $|x_i|_v$. If $\mathbb{K} = \mathbb{C}$ and the norm is the usual Euclidean norm we have a stronger result:

Corollary 1.1. *Let $X \subset \mathbb{C}^m$ be an affine $n-$dimensional variety of degree D. Assume that $f_1, ..., f_k \in \mathbb{C}[x_1, ..., x_m]$ have only finitely many zeros on X. Let $d_i = \deg f_i$ (where $d_1 \geq d_2 \geq ... \geq d_k$). Then there is a constant $C > 0$, such that for $x \in X$*

$$\max |f_i(x)| \geq C||x||^{d_k - DN(d_1, ..., d_k; n) + \sum_{f(a)=0} \mu_a(f)},$$

for $||x||$ sufficiently large.

Here $\mu_a(f)$ stands for the local multiplicity of the mapping $f|_X$ at a point a (see Definition 3.1). In particular Corollary 1.1 generalizes Proposition 1.10 from [6] and Theorem 7.3 from [2].

In the last section we give a sharp estimation of the degree of the set of non-properness of generically finite separable polynomial mapping $f : X \to \mathbb{K}^n$. Moreover, we prove that such a mapping must be finite provided it has a sufficiently large geometric degree.

Most of our results hold for arbitrary infinite field $\mathbb{K}$. All our methods are elementary and the paper is self-contained.

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2. Terminology

We assume that $\mathbb{K}$ is an algebraically closed field. If $X \subset \mathbb{K}^m$ is an affine variety of codimension k, then by deg X we mean the number of common points of X and sufficiently general linear subspace M of dimension k. In particular if $X = \mathbb{K}^m$, then deg $X = 1$.

If $X \subset \mathbb{K}^m$ is an affine variety and $g \in \mathbb{K}[X]$ is a polynomial, then we put

$$\deg\, g = \min\{\deg\, G : G \in \mathbb{K}[x_1, ..., x_m] \text{ and } G|_X = g\}.$$

If $f : X \to Y$ is a polynomial generically-finite mapping of affine varieties, then we define the geometric degree of f, denoted $\mu(f)$, to be the number $[\mathbb{K}(X) : f^*\mathbb{K}(Y)]$. If the mapping f is separable, then it is well-known that the $\mu(f)$ is equal to the number of points in a generic fiber of f.

3. Main results

We start with the simple lemma:

Lemma 3.1. *Let $f \in k[x_1, ..., x_n]$ be a polynomial and let $\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$. The polynomial f is reduced if and only if ∇f does not vanish identically on any irreducible component of the set $Z(f) = \{x : f(x) = 0\}$.*

Proof. If f is reduced then ∇f does not vanish identically on irreducible components of $Z(f)$, because the set of smooth points is dense in $Z(f)$. On the other hand if f is not reduced, then $f = g^2 h$ and $\nabla f = g(2\nabla gh + g\nabla h) = 0$ on $Z(g)$. □

Corollary 3.1. *Let $f \in k[x_1, ..., x_n]$ be a reduced polynomial. Assume that $Z(f)$ does not contain hyperplanes $\{x_i = a_i\}$ (where $a_i \in k$) as irreducible components. Then for any numbers $m_1, ..., m_n$ the polynomial $F = f(T_1^{m_1} + T_1, ..., T_n^{m_n} + T_n)$ is also reduced.*

Proof. Indeed,

$$\nabla F = \nabla f \cdot < m_1 T_1^{m_1-1} + 1, ..., m_n T_n^{m_n-1} + 1 >,$$

(where $\cdot$ denotes the scalar product). By the assumption and Lemma 3.1 this product does not vanish on irreducible components of F. □

Now we are ready to prove the following generalization of the Perron Theorem (see [8] for the classical version):

Theorem 3.1. *(Generalized Perron Theorem) Let* $\mathbb{L}$ *be a field and let* $Q_1, ..., Q_{n+1} \in \mathbb{L}[x_1, ..., x_m]$ *be non-constant polynomials with* $\deg Q_i = d_i$. *Assume that* $X \subset \mathbb{L}^m$ *is an affine variety of dimension* n *and of degree* D. *If the mapping* $Q = (Q_1, ..., Q_{n+1}) : X \to \mathbb{L}^{n+1}$ *is generically finite, then there exists a non-zero polynomial* $W(T_1, ..., T_{n+1}) \in \mathbb{L}[T_1, ..., T_{n+1}]$ *such that*

a) $W(Q_1, ..., Q_{n+1}) = 0$ *on* X,

b) $\deg W(T_1^{d_1}, T_2^{d_2}, ..., T_{n+1}{}^{d_{n+1}}) \leq D \prod_{j=1}^{n+1} d_j$.

Proof. Without loss of generality we can assume that the field $\mathbb{L}$ is algebraically closed. Let $\tilde{X} = \{(x, w) \in X \times \mathbb{L}^{n+1} : Q_i(x) = w_i^{d_i} + w_i\}$. Let W be an irreducible polynomial such that $W(Q_1, ..., Q_{n+1}) = 0$ and take $P(T_1, ..., T_{n+1}) = W(T_1^{d_1} + T_1, ..., T_{n+1}^{d_{n+1}} + T_n)$. Let $Y = \{w \in \mathbb{L}^{n+1} : P(w) = 0\}$. By Corollary 3.1 we have $\deg Y = \deg P$. The sets $\tilde{X}, Y$ are affine sets of pure dimension n. Now consider the mapping

$$\pi : \tilde{X} \ni (x, w) \to w \in Y.$$

It is easy to see that π is a dominant generically finite mapping. Consequently

$$\deg \pi \deg Y \leq \deg \tilde{X}.$$

By the Bezout Theorem we have $\deg \tilde{X} \leq D \prod_{j=1}^{n+1} d_j$. This finishes the proof. □

The following Lemma is obvious:

Lemma 3.2. *Let* $\mathbb{K}$ *be an infinite field. Let* $X \subset \mathbb{K}^m$ *be an affine algebraic variety of dimension* n. *For sufficiently general numbers* $a_{ij} \in \mathbb{K}$ *the mapping*

$$\pi : X \ni (x_1, ..., x_m) \to (\sum_{j=1}^{m} a_{1j} x_j, \sum_{j=2}^{m} a_{2j} x_j, ..., \sum_{j=n}^{m} a_{1j} x_j) \in \mathbb{K}^n$$

is finite. □

First we prove:

Theorem 3.2. *Let* $\mathbb{K}$ *be an algebraically closed field and let* $f_1, ..., f_k \in \mathbb{K}[x_1, ..., x_m]$ *be non-zero polynomials. Let* $X \subset \mathbb{K}^m$ *be an affine algebraic variety of dimension* n. *Assume that* $k \leq n$. *Let* $\deg f_i = d_i$, *where* $d_1 \geq d_2 \geq ... \geq d_k$. *If* $V(f_1, ..., f_k) \cap X = \emptyset$, *then there exists polynomials* g_i, *such that*

a) $\deg f_i g_i \leq D N(d_1, ..., d_k; n)$,

b) $1 = \sum_{i=1}^{k} f_i g_i$ *on* X.

Proof. By assumption there are polynomials A_j such that $\sum_{j=1}^{k} f_j A_j = 1$ on X. Let us consider the mapping

$$\Phi : X \times \mathbb{K} \ni (x, z) \to (x, f_1(x)z, ..., f_k(x)z) \in \mathbb{K}^m \times \mathbb{K}^k.$$

The mapping Φ is an embedding. Indeed, denote variables in the space $\mathbb{K}^m \times \mathbb{K}^k$ in this way that $X_i = x_i, W_j = f_j(x)z$. Now we have $z = \sum_{i=1}^{k} A_i(X) W_i$ and $x_i = X_i$ (on X). This implies that the mapping Φ is an embedding. In particular the mapping Φ is a finite mapping and the set $\Gamma = \Phi(X \times \mathbb{K})$ is closed. Let $\pi : \Gamma \to \mathbb{K}^{n+1}$ be a generic projection. Hence π is a finite mapping. Define $\Psi := \pi \circ \Phi(x, z)$. By Lemma 3.2, we can assume that

$$\Psi = (\sum_{j=1}^{n} \gamma_{1j} f_j z + l_1(x), ..., \sum_{j=n}^{n} \gamma_{nj} f_j z + l_n(x), l_{n+1}(x)),$$

where l_j are linear forms (and $f_j := 0$ for $j > k$). Denote $\Psi = (\Psi_1, ..., \Psi_{n+1})$.

Apply Theorem 3.1 to $\mathbb{L} = \mathbb{K}(z)$, polynomials $\Psi_1, ..., \Psi_{n+1} \in \mathbb{L}[x]$ and to the variety X considered over $\mathbb{L}$. Thus there exists a non-zero polynomial $W(T_1, ..., T_{n+1}) \in \mathbb{L}[T_1, ..., T_{n+1}]$ such that

$$W(\Psi_1, ..., \Psi_{n+1}) = 0 \text{ and } \deg W(T_1^{d_1}, T_2^{d_2}, ..., T_{n+1}{}^{d_{n+1}}) \leq D \prod_{j=1}^{k} d_j.$$

Since coefficients of a polynomial W are in $\mathbb{K}(z)$, we obtain that there is a non-zero polynomial $\tilde{W} \in \mathbb{K}[T_1, ..., T_{n+1}, Y]$ such that

a) $\tilde{W}(\Psi_1(x, z), ..., \Psi_{n+1}(x, z), z) = 0$,

b) $\deg_T \tilde{W}(T_1^{d_1}, T_2^{d_2}, ..., T_{n+1}{}^{d_{n+1}}, Y) \leq D \prod_{j=1}^{k} d_j$, where $\deg_T$ denotes the degree with respect to variables $T = (T_1, ..., T_{n+1})$.

Since the mapping $\Psi = (\Psi_1, ..., \Psi_{n+1}) : \mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ is finite, for every polynomial $G \in \mathbb{K}[X][z]$ there is a minimal (monic with respect to Y) polynomial $P_G(T, Y) \in \mathbb{K}[T_1, ..., T_{n+1}][Y]$, such that $P_G(\Psi_1, ..., \Psi_{n+1}, G) = 0$. Let us take here $G = z$. By the minimality of P_z, we have

$$P_z(T, Y) | \tilde{W}(T, Y),$$

in particular

$$\deg_T P_z(T_1^{d_1}, T_2^{d_2}, ..., T_{n+1}{}^{d_{n+1}}, z) \leq D \prod_{j=1}^{k} d_j.$$

Let N be the degree of the polynomial P_z with respect to Y. Add all terms of the form $z^N Q(x)$ which occur in the expression $P_G(\Psi_1, ..., \Psi_{n+1}, z)$. It is easy to see that Q must be either equal to 1 or must be of a form $f_1^{s_1} ... f_k^{s_k} P(x)$, where $\sum s_i > 0$ and $\deg f_1^{s_1} ... f_k^{s_k} P(x) \leq D \prod_{j=1}^{k} d_j$. Thus we have the equality $1 + \sum f_i g_i = 0$, where $\deg f_i g_i \leq D \prod_{j=1}^{k} d_j$. □

Remark 3.1. In the same way we can show that in the general case ($k > n$) we have the estimation $f_i g_i \leq DN(d_1, ..., d_k; n+1)$. However we will find in this case a better estimation (see the proof of Theorem 1.3).

Theorem 3.3. *(Elimination Theorem) Let $\mathbb{K}$ be an algebraically closed field and let $f_1, ..., f_n \in \mathbb{K}[x_1, ..., x_m]$ be polynomials with $\deg f_i = d_i$. Let $X \subset \mathbb{K}^m$ be an affine algebraic $n-$dimensional variety of degree D. Assume that the set $V(f_1, ..., f_n) \cap X$ is finite. If we take a sufficiently general system of coordinates $(x_1, ..., x_n)$, then there exists polynomials $g_{ij} \in \mathbb{K}[x_1, ..., x_m]$ and polynomials $\phi_i(x_i) \in \mathbb{K}[x_i]$, such that*

a) $\deg g_{ij} f_j \leq DN(d_1, ..., d_n; n)$,

b) $\phi_i(x_i) = \sum_{j=1}^{k} g_{ij} f_j$ for every $i = 1, ..., m$ (on X),

c) if $\phi_j(a) = 0$ for $a \in \mathbb{K}$, then there exist $a_1, a_2, ..., a_{j-1}, a_{j+1}, ..., a_n \in \mathbb{K}$ such that $(a_1, a_2, ..., a_{j-1}, a, a_{j+1}, ..., a_n) \in X$ and $f_l(a_1, a_2, ..., a_{j-1}, a, a_{j+1}, ..., a_n) = 0$ for $l = 1, ..., k$.

Proof. By Theorem 3.2 we can assume that $V(f_1, ..., f_n) \cap X = \{a_1, ..., a_s\}$ is a finite and non-empty set of points. The mapping

$$\Phi : X \times \mathbb{K} \ni (x, z) \to (x, f_1(x)z, ..., f_n(x)z) \in \mathbb{K}^m \times \mathbb{K}^n$$

is a (non-closed) embedding outside the set $\{a_1, ..., a_r\}$. Take $\Gamma = cl(\Phi(X \times \mathbb{K}))$. Let $\pi : \Gamma \to \mathbb{K}^{n+1}$ be a generic projection. Hence π is a finite mapping. Define $\Psi := \pi \circ \Phi(x, z)$.

By Lemma 3.2 we can assume that

$$\Psi = (\sum_{j=1}^{n} \gamma_{1j} f_j z + l_1(x), ..., \sum_{j=n}^{n} \gamma_{nj} f_j z + l_n(x), l_{n+1}(x)),$$

where l_{n+1} is a generic linear form. In particular we can assume that l_{n+1} is a variable x_1 in a generic system of coordinates.

Apply the Generalized Perron Theorem to $\mathbb{L} = \mathbb{K}(z)$, polynomials $\Psi_1, ..., \Psi_{n+1} \in \mathbb{L}[x]$ and to the variety X considered over $\mathbb{L}$. Thus there exists a non-zero polynomial $W(T_1, ..., T_{n+1}) \in \mathbb{L}[T_1, ..., T_{n+1}]$ such that $W(\Psi_1, ..., \Psi_{n+1}) = 0$ on X, and

$$\deg\, W(T_1^{d_1}, T_2^{d_2}, ..., T_{n+1}{}^{d_{n+1}}) \le D \prod_{j=1}^{n} d_j.$$

Since coefficients of a polynomial W are in $\mathbb{K}(z)$, we obtain that there is a non-zero polynomial $\tilde{W} \in \mathbb{K}[T_1, ..., T_{n+1}, Y]$, such that

a) $\tilde{W}(\Psi_1(x,z), ..., \Psi_{n+1}(x,z), z) = 0$,

b) $\deg_T \tilde{W}(T_1^{d_1}, T_2^{d_2}, ..., T_{n+1}{}^{d_{n+1}}, Y) \le D \prod_{j=1}^{n} d_j$, where $\deg_T$ denotes the degree with respect to variables $T = (T_1, ..., T_{n+1})$.

Note that the mapping $\Psi = (\Psi_1, ..., \Psi_{n+1}) : X \times \mathbb{K} \to \mathbb{K}^{n+1}$ is finite outside the set $\bigcup_{j=1}^{s} \{T \in \mathbb{K}^n : T_{n+1} = a_{1j}\}$, where a_{1j} is the first coordinate of a_j (recall that we consider a generic system of coordinates in which $x_1 = l_{n+1}$!). In particular the set of non-properness of the mapping Ψ is contained in the hypersurface $S = \{T \in \mathbb{K}^n : \prod_{j=1}^{s} (T_{n+1} - a_{1j}) = 0\}$ (see section 4 for more information about the set of non-properness).

Since the mapping $\Psi = (\Psi_1, ..., \Psi_{n+1}) : \mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ is finite outside S, for every polynomial $G \in \mathbb{K}[x_1, ..., x_n, z]$ there is a minimal polynomial $P_G(T, Y) \in \mathbb{K}[T_1, ..., T_{n+1}][Y]$, such that $P_G(\Psi_1, ..., \Psi_{n+1}, G) = \sum_{i=0}^{r} b_i(\Psi_1, ..., \Psi_{n+1}) G^{r-i} = 0$ and the coefficient b_0 satisfy $\{T : b_0(T) = 0\} \subset S$ (see Corollary 4.1).

Take here $G = z$. By the minimality of P_z, we have

$$P_z(T, Y) | \tilde{W}(T, Y),$$

in particular $\deg_T P_z(T_1^{d_1}, T_2^{d_2}, ..., T_{n+1}{}^{d_{n+1}}, z) \le D \prod_{j=1}^{n} d_j$. Let N be the degree of P_z with respect to Y. Add all terms of the form $z^N Q(x)$ which occur in the expression $P_z(\Psi_1, ..., \Psi_{n+1}, z)$. It is easy to see that Q must be either equal to $b_0(x_1)$ or must be of a form $f_1^{s_1} ... f_n^{s_n} P(x)$, where $\sum s_i > 0$ and $\deg\, f_1^{s_1} ... f_n^{s_n} P(x) \le D \prod_{j=1}^{n} d_j$. Thus we have the equality $b_0(x_1) + \sum f_i g_i = 0$, where $\deg\, f_i g_i \le D \prod_{j=1}^{n} d_j$. Take $\phi_1 = b_0$. By the construction the polynomial ϕ_1 has zeroes only in $a_{11}, ..., a_{1s}$.

Further, since the form l_{n+1} was generic, we can find n forms of this type which are linearly independent. Hence in a similar way as above we can construct polynomials $\phi_i(x_i),\ i = 2, ..., m$ as in b). Moreover, it is easy to check that ϕ_i satisfy also c). □

Now we are in a position to prove our main results:

Proof of Theorem 1.3. If $f_1, f_2 \in \mathbb{K}[x]$ are two polynomials of one variable of degree at most d without common zeroes, then there exists polynomials $g_1, g_2 \in \mathbb{K}[x]$ such that

*) $\deg\, g_i \le d - 1$ and $f_1 g_1 + f_2 g_2 = 1$.

This can be easily derived from the properties of the resultant of f, g. Now take polynomials $f_1, ..., f_k \in \mathbb{K}[x_1, ..., x_m]$, which have no common zeroes on X. We can assume that $d_2 \ge d_3 \ge ... \ge d_k \ge d_1$. Using a general linear combinations

**) $F_1 = f_1,\ F_i = \sum_{j=i}^k \gamma_{ij} f_j\ ,\ i = 2, ..., k,$

we can assume that $k \leq n+1$. If $V(F_1, ..., F_n) = \emptyset$, then we are done by Theorem 3.2.

Let $k = n+1$. Thus $V(F_1, ..., F_n)$ is a finite non-empty set. If $F_1', ..., F_{n+1}'$ is another sufficiently general linear combination **), then $V(F_1, ..., F_n) \cap V(F_1', ..., F_n') = \emptyset$. By Theorem 3.3 there exist polynomials $g_i, g_i' \in \mathbb{K}[x_1, ..., x_n]$ and polynomials $\phi(x_1), \phi'(x_1) \in \mathbb{K}[x_1]$, such that

a) deg $F_i g_i \leq DN(d_1, ..., d_k; n)$ and deg $F_i' g_i' \leq DN(d_1, ..., d_k; n)$,

b) $\phi(x_1) = \sum_{i=1}^n F_i g_i$ and $\phi'(x_1) = \sum_{i=1}^n F_i' g_i'$ on X,

c) polynomials ϕ and ϕ' have not common zeroes (we assume that the system of coordinates is sufficiently generic).

Now Theorem 1.3 follows directly from *). □

Proof of Theorem 1.4. Take a general linear combination:

$F_1 = f_1,\ F_i = \sum_{j=i}^k \gamma_{ij} f_j\ ,\ i = 2, ..., k,$ (or $F_1 = \sum_{j=1}^k \gamma_j f_j$ for $n = 1$).

Since the number of zeroes of $F_1, ..., F_n$ is finite and greater or equal to ν, we can assume that $k \leq n$.

Now Theorem 1.4 is a direct consequence of the Elimination Theorem (Theorem 3.3). Indeed, we can assume that the system of coordinates is sufficiently generic and there exist polynomials $g_{ij} \in \mathbb{K}[x_1, ..., x_m]$ and polynomials $\phi_i(x_i) \in \mathbb{K}[x_i]$, such that

a) deg $g_{ij} f_j \leq DN(d_1, ..., d_k; n)$,

b) $\phi_i(x_i) = \sum_{j=1}^k g_{ij} f_j$,

c) if $\phi_j(a) = 0$ for $a \in \mathbb{K}$ then there exists $a_1, a_2, ..., a_{j-1}, a_{j+1}, ..., a_n \in \mathbb{K}$ such that $(a_1, a_2, ..., a_{j-1}, a, a_{j+1}, ..., a_n) \in X$ and $f_l(a_1, a_2, ..., a_{j-1}, a, a_{j+1}, ..., a_n) = 0$ for $l = 1, ..., k$.

Observe that we have deg $\phi_i \geq \nu$ (we can assume that all zeroes of $f_1, ..., f_n$ have all different coordinates!). Further, if $G(x_1, ..., x_m) \in \mathbb{K}[x_1, ..., x_m]$ is a polynomial of degree at most d, then

$$||G(x)||_v < C||x||_v^d$$

for large $||x||_v$. On the other hand if $\phi(t)$ is a polynomial of one variable t of degree at least ν, then $|\phi(t)|_v > C'|t|_v^\nu$ for large $|t|_v$. In particular from b) we get

$$A||x||_v^{DN(d_1,...,d_k;n)-d_s}||f(x)|| > B||x||_v^\nu$$

and consequently

$$\max|f_i(x)|_v \geq C||x||_v^{d_k - DN(d_1,...,d_k;n)+\nu},$$

for $||x||_v$ sufficiently large. □

Our next aim is the proof of Corollary 1.1. First we introduce the notion of a local multiplicity:

Definition 3.1. *Let $X \subset \mathbb{C}^m$ be an affine variety and let $f : X \to \mathbb{C}^r$ be a polynomial mapping. Assume that the fiber $f^{-1}(0)$ is finite and let $a \in f^{-1}(0)$. Let $Y = cl(f(X))$ and let $\mathbf{Y_0} = \bigcup_{j=1}^l \mathbf{Y_j}$ be a decomposition of the analytic germ of Y at 0 into irreducible components. We define the local multiplicity of the mapping f at the point a, denoted $\mu_a(f)$, to be the number of points in $U \cap (\bigcup_{j=1}^l f^{-1}(y_j))$, where U is a sufficiently small neighborhood of a (in the classical topology) and $y_j \in \mathbf{Y_j}$ are sufficiently general points.*

Proof of Corollary 1.1. Taking a general linear combination as before, it is not difficult to check that we can assume $k \le n$. If $k < n$, then $V(f_1, ..., f_k) = \emptyset$ and Corollary 1.1 follows directly from Theorem 3.2.

Hence we can assume that $k = n$ and $V(f_1, ..., f_n) \neq \emptyset$. Now it is enough to prove that deg $\phi_i \ge \sum_{f(a)=0} \mu_a(f)$ (the notation is as in the proof of Theorem 1.4). Let $V(f_1, ..., f_n) = \{a_1, ..., a_r\}$. Take a sequence of points $y_l \in \mathbb{C}^n$ which tends to 0 and small neighborhoods U_i of points a_i, such that:

$$\# U_i \cap f^{-1}(y_l) = \mu_{a_i}(f).$$

We can assume that we have a generic system of coordinates, in particular all points a_i and all points from the set $\bigcup_{s=1}^{\infty} f^{-1}(y_k)$ have coordinates, which are pairwise different (i.e., projections $x \to x_i$, $i = 1, ...m$ are injective on this set).

Consider polynomials ϕ_i as in the proof of Theorem 3.3. Take $\phi_1 = \phi$. From the proof of Theorem 3.3 we have the equality:

$$\phi(x_1)z^s + \sum_{j=1}^{s} a_j(\Sigma_{j=1}^n \gamma_{1j} f_j z + l_1(x), ..., \Sigma_{j=n}^n \gamma_{nj} f_j z + l_n(x), x_1) z^{s-j} = 0.$$

In particular

$$\phi(x_1) = -(1/z)(\sum_{j=1}^{s} a_j(\Sigma_{j=1}^n \gamma_{1j} f_1 z + l_1(x), ..., \Sigma_{j=n}^n \gamma_{nj} f_n z + l_n(x), x_1) z^{-j+1}).$$

We can choose a sequence $z_l \to \infty$ ($z_l \in \mathbb{C}$), such that the sequence $y_l z_l$ is bounded. In particular if $f(b_l) = y_l$ and $b_l = (b_{1l}, ..., b_{ml})$, then $\phi(b_{1l}) \to 0$. This implies that ϕ has multiplicity $\mu \ge \mu_{a_i} f$ at the point a_{1i}. Thus deg $\phi \ge \sum_{f(a)=0} \mu_a(f)$. Similarly deg $\phi_i \ge \sum_{f(a)=0} \mu_a(f)$ for every i. $\square$

4. The set S_f for polynomial mappings.

Let $\mathbb{K}$ be an algebraically closed field and let X, Y be affine varieties over $\mathbb{K}$. Recall the following (see [4], [5]):

Definition 4.1. Let $f : X \to Y$ be a generically-finite dominant polynomial mapping of affine varieties. We say that f *is finite at a point* $y \in Y$, if there exists a Zariski open neighborhood U of y such that the mapping $\mathrm{res}_{f^{-1}(U)} f : f^{-1}(U) \to U$ is finite.

It is well-known that the set S_f of points at which the mapping f is not finite, is either empty or it is a hypersurface (see [4], [5] and Theorem 4.1). We say that the set S_f is *the set of non-properness* of the mapping f.

Let $X \subset \mathbb{K}^m$ be an affine variety of dimension n. In this section we give a sharp estimation of the degree of the hypersurface S_f for a generically-finite dominant polynomial mapping $f = (f_1, ..., f_n) : X \to \mathbb{K}^n$. As a corollary, we show that if the geometric degree $\mu(f)$ of the mapping f is sufficiently large (relatively to the degree of X and the degrees of polynomials f_j), then the mapping f must be finite.

First we recall our result about the set S_f (see [4], [5]). Let X be an affine $n-$dimensional variety and let $f : X \to \mathbb{K}^n$ be a dominant, generically-finite polynomial mapping. We have:

Theorem 4.1. *Let $f : X \to \mathbb{K}^n$ be a dominant generically finite polynomial map and let $k(f_1, \dots, f_n) \subset \mathbb{K}(X)$ be the induced field extension. Let $\mathbb{K}[X] = \mathbb{K}[g_1, ..., g_r]$ and*

$$t^{n_i} + \sum_{k=1}^{n_i} a^i_k(f) t^{n_i - k} = 0,$$

where the $a^i_k \in \mathbb{K}(f_1, ..., f_n)$ are rational functions, be the minimal equation of g_i over $\mathbb{K}(f_1, \dots, f_n)$. Let S denote the union of poles of all functions a^i_k. Then f is finite at a point y if and only if $y \in \mathbb{K}^n \setminus S$.

Proof. $\Leftarrow$ It is enough to prove that the mapping $f : X \setminus f^{-1}(S) \to \mathbb{K}^n \setminus S$ is finite. If S is the empty set, then $\mathbb{K}[f_1, \dots, f_n] \subset \mathbb{K}[g_1, \dots, g_r]$ is an integral extension and the mapping f is finite. Otherwise S, and hence $f^{-1}(S)$ is a hypersurface. Let $S = \{x : A(x) = 0\}$ for some polynomial A. Let $V = X \setminus f^{-1}(S)$ and let $W = \mathbb{K}^n \setminus S$. Then V, W are affine varieties and $\mathbb{K}[V] = \mathbb{K}[g_1, \dots, g_r][(A(f)^{-1}]$, $\mathbb{K}[W] = \mathbb{K}[x_1, \dots, x_n][A^{-1}]$. Hence $f_*\mathbb{K}[W] = \mathbb{K}[f_1, \dots, f_n][(A(f)^{-1}]$. Since all functions a^i_k are regular in W we conclude that elements g_i are integral over $f_*\mathbb{K}[W]$. Of course a polynomial $A(f)^{-1}$ is also integral, and we get the integral extension $f_*\mathbb{K}[W] \subset \mathbb{K}[V]$.

$\Rightarrow$ The following lemma is well-known:

Lemma 4.1. *Let A, B be integral domains, $B = A[g_1, \dots, g_n]$ such that the field B_0 is finite field extension of the field A_0. Assume that A is a normal ring.*

The ring B is a finite $A-$module if and only if the following condition holds: if $P_i \in A_0[t]$ is the minimal monic polynomial of g_i over A_0, then $P_i \in A[t]$, $i = 1, \dots, n$.

Now let f be finite over $x \in \mathbb{K}^n$. It means that there is a affine neighborhood U of y such that the mapping $\mathrm{res}_{f^{-1}(U)} f : f^{-1}(U) \to U$ is finite. Of course, we can assume that $U = \mathbb{K}^n \setminus \{x : A(x) = 0\}$, where A is a polynomial. By the assumption, the ring $\mathbb{K}[f^{-1}(U)] = \mathbb{K}[g_1, \dots, g_r][(A(f)^{-1}]$ is integral over the ring $f_*\mathbb{K}[U] = \mathbb{K}[f_1, \dots, f_n][(A(f)^{-1}]$. By Lemma 4.1 we have , that the coefficients $a^i_k(f)$ of polynomials

$$t^{n_i} + \sum_{k=1}^{n_i} a^i_k(f) t^{n_i - k}, \ i = 1, ..., m$$

belong to the ring $\mathbb{K}[f_1, \dots, f_n][(A(f))^{-1}]$. Hence $a^i_k \in \mathbb{K}[x_1, \dots, x_n][(A(x))^{-1}]$ and consequently they are regular in U. Thus $U \subset \mathbb{K}^n \setminus S$. □

Corollary 4.1. *Let X be an affine $n-$dimensional variety and let $f = (f_1, ..., f_n) : X \to \mathbb{K}^n$ be a generically finite mapping. Let S be a set of non-properness of the mapping f. Then for every polynomial $G \in \mathbb{K}[X]$, if $W_G(T_1, ..., T_n, t) = \sum_{i=0}^s a_i(T) t^{s-i} \in \mathbb{K}[T, t]$ is irreducible and*

$$W_G(f_1, ..., f_n, G) = 0,$$

then

$$\{T : a_0(T) = 0\} \subset S.$$

Proof. Observe that the mapping $f : X \setminus f^{-1}(S) \to \mathbb{K}^n \setminus S$, is finite. Moreover, $\mathbb{K}[\mathbb{K}^n \setminus S] = \mathbb{K}[x_1, ..., x_n]_h$, where h is the reduced equation of S. Now Corollary 4.1 follows directly from Lemma 4.1. □

Corollary 4.2. *Let $X \subset \mathbb{K}^m$ be an affine $n-$dimensional variety and let $f : X \to \mathbb{K}^n$ be a dominant generically finite and separable polynomial mapping. Assume that $l \in$*

$\mathbb{K}[x_1, ..., x_n]$ and that

$$P_l(f, l) := \sum_{k=1}^{\mu} a_k(f) l^{\mu-k} = 0,$$

(where $a_k(T) \in \mathbb{K}[T_1, ..., T_n]$) is the minimal irreducible equation of l over $\mathbb{K}[f_1, \ldots, f_n]$. Then there is a linear form l, such that

a) $\mu = \mu(f)$,

b) $S_f = \{T \in \mathbb{K}^n : a_0(T) = 0\}$.

Proof. For $t \in \mathbb{K}$ let $\alpha_t = \sum_{i=0}^{m} t^i x_i$. Let S_t denote the set of poles of coefficients of the minimal equation of α_t over $\mathbb{K}(f_1, \ldots, f_n)$. By Theorem 4.1 we have $S_t \subset S_f$. Since the hypersurface S_f has only finite number of irreducible component, we have (by the Dirichlet box principle) that there is an infinite subset $T \subset \mathbb{K}$, such that if $t, t' \in T$ then $S_t = S_{t'}$.

Since the extension $\mathbb{K}(f) \subset \mathbb{K}(X)$ is separable, we have that there is only finite family of fields between $\mathbb{K}(f)$ and $\mathbb{K}(X)$. In particular, we can assume that for $t, t' \in T$ we have $\mathbb{K}(f)(\alpha_t) = \mathbb{K}(f)(\alpha_{t'})$.

Take $t_1, ..., t_m \in T$, where $t_i \neq t_j$ for $i \neq j$. It is easy to check that linear forms α_{t_i}, $i = 1, .., n$ generates the algebra $\mathbb{K}[X]$. By Theorem 4.1, we have that in fact $S_f = \bigcup_{j=1}^{n} S_{t_j}$. Since $S_{t_i} = S_{t_j}$ we have in fact that $S_f = S_{t_i}$. Moreover, since $\mathbb{K}(f)(\alpha_t) = \mathbb{K}(f)(\alpha_{t'})$ for $t \in T$, we obtain that $\mathbb{K}(f)(\alpha_t) = \mathbb{K}(X)$. In particular if we take $l = \alpha_t$ (where $t \in T$), we have $\mu = \mu(f)$ and $S_f = \{T \in \mathbb{K}^n : a_0(T) = 0\}$. □

The following result gives the (sharp) estimation of the degree of the set S_f:

Theorem 4.2. *Let $X \subset \mathbb{K}^m$ be an affine $n-$dimensional variety of degree D and let $f = (f_1, ..., f_n) : X \to \mathbb{K}^n$ be a generically finite dominant and separable mapping. Then the set S_f of non-properness of the mapping f is a hypersurface (or the empty set) and*

$$\deg\ S_f \leq \frac{D(\prod_{i=1}^{n} \deg\ f_i) - \mu(f)}{\min_{1 \leq i \leq n} \deg\ f_i}.$$

Proof. Let l be a linear form as in Corollary 4.2. Consider the mapping

$$\Phi : X \ni x \to (f_1(x), ..., f_n(x), l(x)) \in \mathbb{K}^{n+1}.$$

Let $\deg f_i = d_i$. By the Generalized Perron Theorem (Theorem 3.1) there exists a non-zero polynomial $W(T_1, ..., T_n, Y) \in \mathbb{K}[T_1, ..., T_n, Y]$ such that

a) $W(f_1, ..., f_n, l) = 0$ on X,

b) $\deg\ W(T_1^{d_1}, T_2^{d_2}, ..., T_n^{d_n}, Y) \leq D \prod_{j=1}^{n} d_j$.

Let

$$P_l = \sum_{k=1}^{\mu} a_k(f) l^{\mu-k} = 0,$$

where the $a_k \in \mathbb{K}[f_1, ..., f_n]$, be the minimal irreducible equation of l over $\mathbb{K}[f_1, \ldots, f_n]$. By the minimality of P_l we have

$$P_l(T, Y) | W(T, Y),$$

in particular $\deg\ P_l(T_1^{d_1}, T_2^{d_2}, ..., {T_n}^{d_n}, Y) \leq D \prod_{j=1}^{n} d_j$. Thus

$$\deg\ a_0(T_1^{d_1}, T_2^{d_2}, ..., {T_n}^{d_n}) Y^{\mu(f)} \leq D \prod_{j=1}^{n} d_j.$$

This means that

$$\deg\ a_0(T_1^{d_1}, T_2^{d_2}, ..., T_n{}^{d_n}) \leq D\prod_{j=1}^{n} d_j - \mu(f).$$

Clearly $\deg\ a_0(T_1^{d_1}, T_2^{d_2}, ..., T_n{}^{d_n}) \geq (\min_{1\leq i\leq n}\ d_i)(\deg\ a_0)$. Finally by Corollary 4.2 we have

$$\deg\ S_f \leq \frac{D(\prod_{i=1}^{n} \deg\ f_i) - \mu(f)}{\min_{1\leq i\leq n} \deg\ f_i}.$$

□

Corollary 4.3. *Let $X \subset \mathbb{K}^m$ be an affine $n-$dimensional variety of degree D and let $f = (f_1, ..., f_n) : X \to \mathbb{K}^n$ be a generically finite, separable and dominant mapping. If*

$$\mu(f) > D(\prod_{i=1}^{n} \deg\ f_i) - \min_{1\leq i\leq n} \deg\ f_i,$$

then the mapping f is finite.

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