A NOTE ON STOCHASTIC BURGERS' SYSTEM OF EQUATIONS

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ABSTRACT

We consider a stochastic version of a system of two equations formulated by Burgers in [2] with the aim to describe the laminar and turbulent motions of a fluid in a channel. The existence and uniqueness theorem for a global solution is established. The paper generalizes the result from the paper [11] by Da Prato and Gątarek dealing with the equation describing only the turbulent motion.

1 Introduction

The paper is concerned with the stochastic version of two hydrodynamic equations for the turbulent flow in a channel between parallel walls. The original non-stochastic model was first proposed by Burgers in [2]. The system is derived from the theory of turbulent fluid motion and has similar properties as the Navier-Stokes equation, but is simpler to study.

Let $U = U(t)$ denote the *primary* velocity of the fluid, parallel to the walls of the channel, whereas the second one $v = v\,(t, x)$ denote the *secondary* velocity of the turbulent motion. Let P, ρ and μ be constants representing, respectively, an exterior force, analogous to the mean pressure gradient in the hydrodynamic case, the density of the fluid and its viscosity. Set $\nu = \frac{\mu}{\rho} > 0$.

According to [2], the functions $U(t)$, $v(t,\cdot)$, $t \geq 0$, should satisfy the following system of equations

$$\frac{dU(t)}{dt} = P - \nu U(t) - \int_0^1 v^2(t,x)\,dx \qquad \text{for } t > 0, \tag{1}$$

$$\frac{\partial v(t,x)}{\partial t} = \nu \frac{\partial^2 v(t,x)}{\partial x^2} + U(t)\,v(t,x) - \frac{\partial}{\partial x}\left(v^2(t,x)\right) \tag{2}$$

with the initial and boundary conditions

$$U(0) = U_{0,}\ v(0,x) = v_0(x),\ \ v(t,0) = v(t,1) = 0,\ \ x \in (0,1),\ \ t > 0. \tag{3}$$

The simplified version consisting of one equation on v only ($U(t) \equiv 0$), $t \geq 0$,

$$\frac{\partial v(t,x)}{\partial t} = \nu \frac{\partial^2 v(t,x)}{\partial x^2} - \frac{\partial}{\partial x}\left(v^2(t,x)\right) \tag{4}$$

with the initial and boundary conditions

$$v(0,x) = v_0(x), \qquad v(t,0) = v(t,1) = 0 \tag{5}$$

for $x \in (0,1)$ and for $t > 0$, was investigated by many authors, e.g., in [20] and [25]. For the stochastic version of such equation see e.g. to the papers [7], [8], [10], [18], [21] and [23].

The system (1)-(3) was analysed in [2] and [3]. The existence and uniqueness theorem for the global solution of the system was examined by Dłotko in [9], using the Galerkin method. Other properties of such systems were studied by Cholewa and Dłotko in [5].

The Burger's system (1)-(3) as well as the Burger's equation do not display any chaotic phenomena and therefore a stochastic perturbations of (4) was proposed as a better model, see [4], [6], [19].

The stochastic Burgers' equation is of the form

$$\frac{\partial v(t,x)}{\partial t} = \nu \frac{\partial^2 v(t,x)}{\partial x^2} - \frac{\partial}{\partial x}\left(v^2(t,x)\right) + g(v(t,x))\,\frac{\partial^2 B(t,x)}{\partial t \partial x} \tag{6}$$

with the initial and boundary conditions (5), where B is a Brownian sheet on $[0,\infty) \times (0,1)$ and $\frac{\partial^2 B(t,x)}{\partial t \partial x}$ is the time-space white noise.

The existence and uniqueness theorem for (6) with additive noise $g \equiv 1$ was established in [12] and the case of general g, in [11].

Our paper generalizes the existence and uniqueness result from [11] to the system

$$\frac{dU(t)}{dt} = P - \nu U(t) - \int_0^1 v^2(t,x)\,dx \qquad \text{for } t > 0, \tag{7}$$

$$\begin{aligned} \frac{\partial v(t,x)}{\partial t} &= \nu \frac{\partial^2 v(t,x)}{\partial x^2} + U(t)\, v(t,x) \\ &\quad - \frac{\partial}{\partial x}\left(v^2(t,x)\right) + g\left(v(t,x)\right) \frac{\partial^2 B(t,x)}{\partial t \partial x} \end{aligned} \tag{8}$$

with the initial and boundary conditions (3). We adapt the method from [11]. We prove first the existence of a local solution by proper modification of the drift terms and Banach fixed point argument and then we establish a priori estimates to get global existence.

2 Preliminaries and formulation of the main result

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space on which an increasing and right-continuous family $(\mathcal{F}_t)_{t\in[0,T]}$ of sub$-\sigma-$algebras of $\mathcal{F}$ is defined such that $\mathcal{F}_0$ contains all P-null sets in $\mathcal{F}$. We model mathematicaly the space-time white noise B as the distributional derivative of the cylindrical Wiener process W

$$W(t) = \sum_{k=1}^{\infty} W_k(t) e_k. \tag{9}$$

Here (e_k) is an orthonormal basis of $L^2 = L^2(0,1)$,

$$e_k(x) = \sqrt{\frac{2}{\pi}} \sin k\pi x,\ x \in (0,1),\ k = 1, 2, ... \tag{10}$$

The scalar product in L^2 is denoted by $(\cdot,\cdot)$,

$$(h,\psi) = \int_0^1 h(x)\psi(x)dx$$

and the norm by $\| \cdot \|$.

We consider the following stochastic one-dimensional Burgers' problem

$$\frac{dU(t)}{dt} = P - \nu U(t) - \int_0^1 v^2(t,x)\,dx \qquad \text{for } t>0, \tag{11}$$

$$\begin{aligned}\frac{\partial v(t,x)}{\partial t} &= \nu \frac{\partial^2 v(t,x)}{\partial x^2} + U(t)\,v(t,x) \\ &\quad - \frac{\partial}{\partial x}\left(v^2(t,x)\right) + g(v(t,x))\frac{\partial W(t)}{\partial t} \text{ for}\end{aligned} \tag{12}$$

with the initial and boundary conditions

$$\begin{aligned} U(0) &= U_{0,} \\ v(0,x) &= v_0(x) \quad \text{for } x\in(0,1), \\ v(t,0) &= v(t,1) = 0 \quad \text{for } t>0. \end{aligned} \tag{13}$$

We assume that g is a real valued Lipschitz continuous and bounded function.

Notice that if we replace v by $-u$ in (12), then we obtain an equivalent form of equation (12) with the positive sign before $\frac{\partial}{\partial x}\left(v^2(t,x)\right)$.

We have the following definition

Definition 1 *A pair of processes* $\begin{pmatrix} U \\ v \end{pmatrix}$ *is a weak solution to problem (11)-(13) if and only if* $U(t)$, $t \geq 0$, *and* $v(t)$, $t \geq 0$, *are adapted continuous processes with values in* $\mathbb{R}^1$ *and* L^2, *respectively,* $U(0) = U_0$, $v(0) = v_0$ *and :*

(i) for arbitrary $t \geq 0$:

$$U(t) = U_0 + tP - \nu \int_0^t U(s)ds - \int_0^t \| v(s) \|^2 ds, \quad P\text{-a.s.}, \tag{14}$$

(ii) for arbitrary $t \geq 0$ and arbitrary $\varphi \in C_0^\infty(0,1)$:

$$\begin{aligned}(v(t),\varphi) &= (v_0,\varphi) + \int_0^t U(s)\,(v(s),\varphi)ds \\ &\quad + \int_0^t (v(s), \nu\frac{\partial^2}{\partial x^2}\varphi)ds + \int_0^t (v^2(s), \frac{\partial}{\partial x}\varphi)ds \\ &\quad + \int_0^t (\varphi, g(v(s))dW(s)), \qquad \text{P-a.s.}\end{aligned} \tag{15}$$

Notice that from the very definition of the distributional derivative $\frac{\partial}{\partial x}v^2$, for arbitrary $v \in L^2$:

$$(\frac{\partial}{\partial x}v^2, \varphi) = -\int_0^t v^2(x)\frac{\partial}{\partial x}\varphi(x)ds = -(v^2, \frac{\partial}{\partial x}\varphi).$$

We introduce now an equivalent concept of the *integral solution.* Let $S(t)$, $t \geq 0$, be the classical heat semigroup on L^2 . Then, for $v \in L^2$:

$$S(t)v = \sum_{k=1}^{\infty} e^{-\frac{\pi^2}{\nu}k^2 t}(v, e_k)e_k \tag{16}$$

with the convergence of the series in L^2. It is well known that the generator A of the semigroup $S(t)$, $t \geq 0$, is identical with the second derivative operator $\frac{\partial^2}{\partial x^2}$ on the domain $D(A)$ consisting of functions v such that v, $\frac{\partial v}{\partial x}$ are absolutely continuous with $\frac{\partial^2 v}{\partial x^2} \in L^2$, $v(0) = v(1) = 0$. In some places $S(t)$, $t \geq 0$, will be denoted by e^{At}, $t \geq 0$.

We need the following lemma with the proof postponed to Appendix.

Lemma 1 *The operator $S(t)$, $t \geq 0$, can be extended linearly to the space of all distributions of the form $\frac{\partial}{\partial x}v$, $v \in L^1(0,1)$, in such a way that it takes values in L^2 and*

$$\| S(t)\frac{\partial}{\partial x}v \| \leq \| v \|_{L^1(0,1)} (\sum_{k=1}^{\infty} \frac{2\pi}{\sqrt{\nu}}k^2 e^{-\frac{2\pi^2}{\nu}k^2 t})^{1/2}. \tag{17}$$

Definition 2 *A pair of continuous adapted processes $\begin{pmatrix} U \\ v \end{pmatrix}$ with values in $\mathbb{R}^1$ and L^2, respectively, is said to be an integral solution to problem (11)-(13) if*

$$U(t) = e^{-\nu t}U_0 + \int_0^t e^{-\nu(t-s)}(P - \| v(s) \|^2)ds \tag{18}$$

and

$$v(t) = S(t)v_0 + \int_0^t S(t-s)U(s)\, v(s)ds \tag{19}$$

$$+ \int_0^t S(t-s)\frac{\partial}{\partial x}v^2(s)ds + \int_0^t S(t-s)g(v(s))dW(s).$$

In the integral

$$\int_0^t S(t-s)\frac{\partial}{\partial x}v^2(s)ds,\ t>0$$

we use the extension of the operator $S(t-s)$ described in Lemma 1.

We have the following result which proof can be found for instance in [24].

Proposition 2 *A continuous adapted process* $\begin{pmatrix} U \\ v \end{pmatrix}$ *is an integral solution to problem (11)-(13) if and only if it is a weak solution to problem (11)-(13).*

The main result of the paper is contained in the following

Theorem 3 *System (11)-(13) has a unique weak solution.*

The proof is given in the following sections.

3 Existence of a local solution

Let $\pi_{n,1} : \mathbb{R}^1 \to B_1(0,n)$ be the projection onto the interval $B_1(0,n) = \{U \in \mathbb{R}^1 :\mid U \mid\leq n\}$ and let $\pi_{n,2} : L^2 \to B_2(0,n)$ be the projection onto the ball $B_2(0,n) = \{v \in L^2 : \parallel v \parallel\leq n\}$, where

$$\pi_{n,1}(U) = \begin{cases} U \text{ if } \mid U \mid\leq n, \\ \frac{nU}{|U|} \text{ if } \mid U \mid> n. \end{cases} \tag{20}$$

and

$$\pi_{n,2}(v) = \begin{cases} v \text{ if } \parallel v \parallel\leq n, \\ \frac{nv}{\|v\|} \text{ if } \parallel v \parallel> n. \end{cases} \tag{21}$$

Let Z_T^p, $p>1$, denote the space of all continuous adapted processes $X(t) = \begin{pmatrix} U(t) \\ v(t) \end{pmatrix}$ on $[0,T]$ with values on $\mathbb{R}^1 \times L^2$ such that

$$\begin{aligned} &\parallel X \parallel_{Z_T^p} = \parallel \begin{pmatrix} U \\ v \end{pmatrix} \parallel_T && (22) \\ &= (E(\sup_{t\in[0,T]} \mid U(t) \mid^p))^{1/p} + (E(\sup_{t\in[0,T]} \parallel v(t) \parallel^p))^{1/p} < \infty \end{aligned}$$

with fixed initial conditions $U(0) = U_0$, $v(0) = v_0$. We define

$$\| \begin{pmatrix} U \\ v \end{pmatrix} \|_T = \| U \|_{1,T} + \| v \|_{2,T} . \tag{23}$$

Now we prove

Proposition 4 *For arbitrary $p > 4$ and each $n = 1, 2, ...$ the following system of equations*

$$U(t) = e^{-\nu t}U_0 + \int_0^t e^{-\nu(t-s)}(P - \| \pi_{n,2}v(s) \|^2)ds \tag{24}$$

and

$$v(t) = S(t)v_0 + \int_0^t S(t-s)\pi_{n,1}U(s)\,\pi_{n,2}v(s)ds \tag{25}$$
$$+ \int_0^t S(t-s)\frac{\partial}{\partial x}(\pi_{n,2}v(s))^2 ds + \int_0^t S(t-s)g(v(s))dW(s),$$
$$t \in [0, T]$$

has a unique weak solution in the space Z_T^p.

Let us stress that we look for a continuous and adapted process $v(s)$, $s \geq 0$, with values in L^2, and such that $\frac{\partial}{\partial x}(\pi_{n,2}v(s))^2$ is the derivative in the distribution theory sense (on the interval $(0,1)$) of the function belonging to $L^1(0,1)$ (because $(\pi_{n,2}v(s))^2 \in L^1(0,1)$). From Lemma 1 we have that S can be extended to the derivatives of the functions from $L^1(0,1)$. Therefore, equation (25) has a clear meaning.

Proof of Proposition 4. We introduce nonlinear operators F_n, G, H_n and I_n acting on processes $U(t)$, $t \geq 0$, and $v(t)$, $t \geq 0$, according to the following formulae:

$$\begin{aligned} F_n(U,v)(t) &= e^{-\nu t}U_0 + \int_0^t e^{-\nu(t-s)}(P - \| \pi_{n,2}v(s) \|^2)ds \\ &= e^{-\nu t}U_0 + \frac{1-e^{-\nu t}}{\nu}P - \int_0^t e^{-\nu(t-s)} \| \pi_{n,2}v(s) \|^2)ds, \end{aligned} \tag{26}$$

$$G(U,v)(t) = \int_0^t S(t-s)g(v(s))dW(s), \tag{27}$$

$$H_n(U,v)(t) = \int_0^t S(t-s)\frac{\partial}{\partial x}(\pi_{n,2}v(s))^2 ds \tag{28}$$

and

$$I_n(U,v)(t) = S(t)v_0 + \int_0^t S(t-s)\pi_{n,1}U\,(s)\,\pi_{n,2}v(s)ds. \tag{29}$$

Observe that system (24)-(25) is equivalent to fixed point problem:

$$U = F_n(U,v), \tag{30}$$

$$v = G(U,v) + H_n(U,v) + I_n(U,v). \tag{31}$$

We shall show that for arbitrary n the mapping

$$\begin{pmatrix} U \\ v \end{pmatrix} \to \begin{pmatrix} F_n(U,v) \\ G(U,v) + H_n(U,v) + I_n(U,v) \end{pmatrix} \tag{32}$$

is a contraction in the space $Z^p_{T_n}$, for properly chosen T_n. Therefore, system (30)-(31) has a unique solution on the interval $[0,T_n]$. By the standard iteration procedure system (30)-(31) has a unique global solution denoted by $\begin{pmatrix} U_n \\ v_n \end{pmatrix}$.

First we shall show that for each $n = 1, 2, ...$ and $T > 0$ there exists a constant $C_{T,n}$ such that for $X = \begin{pmatrix} U \\ v \end{pmatrix}$, $\overline{X} = \begin{pmatrix} \overline{U} \\ \overline{v} \end{pmatrix} \in Z^p_T$:

$$\begin{aligned} &\left\| \begin{pmatrix} F_n(U,v) \\ G(U,v) + H_n(U,v) + I_n(U,v) \end{pmatrix} - \begin{pmatrix} F_n(\overline{U},\overline{v}) \\ G(\overline{U},\overline{v}) + H(\overline{U},\overline{v}) + I_n(\overline{U},\overline{v}) \end{pmatrix} \right\|_T \\ \leq\ & C_{T,n} \left\| \begin{pmatrix} U \\ v \end{pmatrix} - \begin{pmatrix} \overline{U} \\ \overline{v} \end{pmatrix} \right\|_T . \end{aligned} \tag{33}$$

We have from (23):

$$\begin{aligned} &\left\| \begin{pmatrix} F_n(U,v) \\ G(U,v) + H_n(U,v) + I_n(U,v) \end{pmatrix} - \begin{pmatrix} F_n(\overline{U},\overline{v}) \\ G(\overline{U},\overline{v}) + H(\overline{U},\overline{v}) + I_n(\overline{U},\overline{v}) \end{pmatrix} \right\|_T \\ =\ & \| F_n(U,v) - F_n(\overline{U},\overline{v}) \|_{1,T} \\ +\ & \| (G(U,v) + H_n(U,v) + I_n(U,v) - (G(\overline{U},\overline{v}) + H(\overline{U},\overline{v}) + I_n(\overline{U},\overline{v})) \|_{2,T} . \end{aligned} \tag{34}$$

Step 1^0. First we consider

$$\begin{aligned} & F_n(U,v)(t) - F_n(\overline{U},\overline{v})(t) \\ = & \int_0^t e^{-\nu(t-s)}[\| \pi_{n,2}v(s) \|^2 - \| \pi_{n,2}\overline{v}(s) \|^2]ds. \end{aligned}$$

We shall find a constant $C^1_{T,n}$ such that

$$\| F_n(U,v) - F_n(\overline{U},\overline{v}) \|_{1,T} \leq C^1_{T,n} \| \begin{pmatrix} U \\ v \end{pmatrix} - \begin{pmatrix} \overline{U} \\ \overline{v} \end{pmatrix} \|_T . \tag{35}$$

Since $\nu > 0$ and

$$|\| \pi_n a \| - \| \pi_n b \|| \leq \| a - b \|, \qquad a, b \in L^2,$$

therefore,

$$\begin{aligned} | & \quad F_n(U,v)(t) - F_n(\overline{U},\overline{v})(t) | \\ \leq & \int_0^t e^{-\nu(t-s)} \, |\| \pi_{n,2}v(s) \|^2 - \| \pi_{n,2}\overline{v}(s) \|^2| \, ds \\ = & \int_0^t e^{-\nu(t-s)} \, | \, (\| \pi_{n,2}v(s) \| \\ - \; \| & \quad \pi_{n,2}\overline{v}(s) \|)(\| \pi_{n,2}v(s) \| + \| \pi_{n,2}\overline{v}(s) \|) \, | \, ds \\ \leq & \; 2n \int_0^t e^{-\nu(t-s)} \, |\| \pi_{n,2}v(s) \| - \| \pi_{n,2}\overline{v}(s) \||| \, ds \\ \leq & \; 2n \int_0^t \| v(s) - \overline{v}(s) \| \, ds. \end{aligned}$$

From the Hölder inequality, if $q = \frac{p}{p-1}$, we have

$$\begin{aligned} E(\sup_{t\in[0,T]}(2n \int_0^t & \quad \| \quad v(s) - \overline{v}(s) \| \, ds)^p \\ & \leq \; (2n)^p E[(\int_0^T \| v(s) - \overline{v}(s) \| \, ds)^p] \\ & \leq \; (2n)^p E(\int_0^T \| v(s) - \overline{v}(s) \|^p \, ds)(\int_0^T ds)^{\frac{p}{q}} \\ & \leq \; (2n)^p T^p E(\sup_{s\leq T} \| v(t) - \overline{v}(t) \|^p). \end{aligned}$$

Hence

$$\begin{aligned} & \parallel F_n(U,v) - F_n(\overline{U},\overline{v}) \parallel_{1,T} \\ = & (E(\sup_{t\in[0,T]} \mid F_n(U,v)(t) - F_n(\overline{U},\overline{v})(t) \mid^p))^{\frac{1}{p}} \\ \leq & 2nT \parallel v - \overline{v} \parallel_{2,T} . \end{aligned}$$

So we can set

$$C^1_{T,n} = 2nT. \tag{36}$$

To go further let us recall that, see (25),

$$\begin{aligned} & I_n(U,v)(t) + H_n(U,v)(t) + G(U,v)(t) \\ = & S(t)v_0 + \int_0^t S(t-s)\pi_{n,1}U_n\,(s)\,\pi_{n,2}v_n(s)ds \\ & + \int_0^t S(t-s)\frac{\partial}{\partial x}(\pi_{n,2}v(s))^2 ds + \int_0^t S(t-s)g(v(s))dW(s). \end{aligned}$$

Step 2^0. We estimate now

$$\parallel G(U,v) - G(\overline{U},\overline{v}) \parallel_{2,T} .$$

To treat the stochastic integral

$$\int_0^t S(t-s)[g(v(s)) - g(\overline{v}(s))]dW(s)$$

we use the factorization procedure similarly as in [26], [11] (see also [24]). Let us fix γ such that $\frac{1}{p} < \gamma < \frac{1}{4}$ and define on $L^p([0,T],L^2)$ for $t \in [0,T]$:

$$R_\gamma h(t) = \int_0^t (t-s)^{\gamma-1} e^{A(t-s)} h(s)ds,$$

$h \in L^p([0,T],L^2)$. Then for $t \in [0,T]$:

$$R_\gamma Y(t) = \int_0^t S(t-s)[g(v(s)) - g(\overline{v}(s))]dW(s)$$

where

$$Y(t) = \frac{\sin \pi\gamma}{\gamma} \int_0^t (t-s)^{-\gamma} e^{A(t-s)} [g(v(s)) - g(\overline{v}(s))] dW(s), \ t \in [0,T].$$

By Hölder inequality, for $0 \leq t \leq T$, $h \in L^p([0,T], L^2)$

$$\| R_\gamma h(t) \| \leq (\frac{t^{(\gamma-1)q+1}}{(\gamma-1)q+1})^{\frac{1}{q}} \| h \|_{L^p([0,T],L^2)} .$$

Therefore R_γ is a bounded operator from $L^p([0,T], L^2)$ to $C([0,T], L^2)$ and

$$\begin{aligned} \sup_{0\leq t\leq T} \quad \| \quad R_\gamma h(t) \| &\leq (\frac{T^{(\gamma-1)q+1}}{(\gamma-1)q+1})^{\frac{1}{q}} \| h \|_{L^p([0,T],L^2)} \\ &\leq \quad (\frac{T^{(\gamma-1)\frac{p}{p-1}+1}}{(\gamma-1)\frac{p}{p-1}+1})^{\frac{p-1}{p}} \| h \|_{L^p([0,T],L^2)}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. So

$$\| R_\gamma \| \leq (\frac{T^{(\gamma-1)\frac{p}{p-1}+1}}{(\gamma-1)\frac{p}{p-1}+1})^{\frac{p-1}{p}}. \tag{37}$$

Note that

$$(\gamma-1)\frac{p}{p-1} + 1 > 0.$$

We therefore have

$$\begin{aligned} E(\sup_{0\leq t\leq T} \quad \| \quad & G(U,v)(t) - G(\overline{U},\overline{v})(t) \|^p) \\ \leq \quad & \| R_\gamma \|^p \ E \| Y \|^p_{L^p([0,T],L^2)} . \end{aligned} \tag{38}$$

Denote by $\| K \|_{HS}$ the Hilbert-Schmidt norm of the operator K. Thus

$$\| K \|^2_{HS} = \sum_{j=1}^{\infty} \| K f_j \|^2,$$

where (f_j) is an orthonormal basis of L^2.

By Burkholder's inequality, for arbitrary adapted operator valued process ϕ and $p \geq 2$,

$$\begin{aligned} E(\sup_{0\leq t\leq T} \quad | \quad & \int_0^t \phi(s) dW(s) \ |^p \\ \leq \quad & (\frac{p}{p-1})^p E(\int_0^T \| \phi(s) \|^2_{HS} \ ds)^{\frac{p}{2}}. \end{aligned}$$

Therefore

$$\begin{aligned} E \quad \| \quad Y \|^p_{L^p([0,T],L^2)} &= \int_0^T E \| Y(t) \|^p \, dt \\ &\leq \quad (\frac{p}{p-1})^p \mid \frac{\sin \pi\gamma}{\gamma} \mid^p \\ \int_0^T [E(\int_0^t \quad \| \quad & e^{A(t-s)}(t-s)^{-\gamma}[g(v(s)) - g(\overline{v}(s))] \|^2_{HS} \, ds)^{\frac{p}{2}}]dt. \end{aligned}$$

Note that

$$\begin{aligned} \| \quad & e^{A(t-s)}(t-s)^{-\gamma}[g(v(s)) - g(\overline{v}(s))] \|^2_{HS} \\ = \quad & (t-s)^{-2\gamma} \| e^{A(t-s)}[g(v(s)) - g(\overline{v}(s))] \|^2_{HS}, \end{aligned}$$

and, for an orthonormal basis (f_j) in L^2,

$$\begin{aligned} \| \quad & e^{A(t-s)}[g(v(s)) - g(\overline{v}(s))] \|^2_{HS} \\ = \quad & \sum_{j=1}^{\infty} \| e^{A(t-s)}[g(v(s)) - g(\overline{v}(s))] f_j \|^2 \\ = \quad & \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} e^{-2\frac{\pi^2}{\nu}k^2(t-s)}((g(v(s)) - g(\overline{v}(s)) f_j, e_k)^2 \\ = \quad & \sum_{k=1}^{\infty} e^{-\frac{\pi^2}{\nu}k^2(t-s)} \| (g(v(s)) - g(\overline{v}(s)) e_k \|^2 . \end{aligned}$$

Moreover

$$\begin{aligned} \| \quad & (g(v(s)) - g(\overline{v}(s)) e_k \|^2 \\ = \quad & \int_0^1 \mid (g(v(s,x)) - g(\overline{v}(s,x)) e_k(x) \mid^2 dx \\ \leq \quad & \frac{2}{\pi} \| g \|^2_{Lip} \int_0^1 \mid v(s,x) - \overline{v}(s,x) \mid^2 dx \\ \leq \quad & \frac{2}{\pi} \| g \|^2_{Lip} \| v(s) - \overline{v}(s) \|^2 \end{aligned}$$

and

$$\begin{aligned} \| \quad & e^{A(t-s)}[g(v(s)) - g(\overline{v}(s))] \|^2_{HS} \\ \leq \quad & \frac{2}{\pi} \| g \|^2_{Lip} \| v(s) - \overline{v}(s) \|^2 \, (\sum_{k=1}^{\infty} e^{-2\frac{\pi^2}{\nu}k^2(t-s)}). \end{aligned}$$

Consequently

$$\int_0^T E \parallel Y(t) \parallel^p dt$$
$$\leq (\frac{p}{p-1})^p \mid \frac{\sin \pi\gamma}{\gamma} \mid^p$$
$$\int_0^T [E(\int_0^t (t-s)^{-2\gamma} \frac{2}{\pi} \parallel g \parallel_{Lip}^2 \parallel v(s) - \overline{v}(s) \parallel^2 (\sum_{k=1}^{\infty} e^{-2\frac{\pi^2}{\nu}k^2(t-s)})ds)^{\frac{p}{2}}]dt.$$

But

$$\int_0^t (t-s)^{-2\gamma} (\sum_{k=1}^{\infty} e^{-2\frac{\pi^2}{\nu}k^2(t-s)})ds \leq \int_0^{+\infty} s^{-2\gamma} (\sum_{k=1}^{\infty} e^{-2\frac{\pi^2}{\nu}k^2 s})ds = a_\gamma.$$

Since $\gamma < \frac{1}{4}$, therefore $a_\gamma < +\infty$. Consequently

$$\parallel G(U,v) - G(\overline{U},\overline{v}) \parallel_{2,T}^p \leq T(\frac{p}{p-1})^p \mid \frac{\sin \pi\gamma}{\gamma} \mid^p (\frac{2}{\pi} \parallel g \parallel_{Lip}^2)^{\frac{p}{2}} (a_\gamma)^{\frac{p}{2}}$$
$$E(\sup_{s\leq T} \parallel v(s) - \overline{v}(s) \parallel^p), \tag{39}$$

and we can set

$$C_{T,n}^2 = T^{\frac{1}{p}} (\frac{p}{p-1})(\frac{\sin \pi\gamma}{\gamma})(\frac{2}{\pi})^{\frac{1}{2}} \parallel g \parallel_{Lip} (a_\gamma)^{\frac{1}{2}}. \tag{40}$$

Step 3^0. We shall show that for each $n = 1, 2, ...$ and $T > 0$ there exists a constant $C_{T,n}^3$ such that for $X = \begin{pmatrix} U \\ v \end{pmatrix}$, $\overline{X} = \begin{pmatrix} \overline{U} \\ \overline{v} \end{pmatrix} \in Z_T^p$:

$$\parallel H_n(U,v) - H_n(\overline{U},\overline{v}) \parallel_{2,T} \tag{41}$$
$$\leq C_{T,n}^3 \parallel \begin{pmatrix} U \\ v \end{pmatrix} - \begin{pmatrix} \overline{U} \\ \overline{v} \end{pmatrix} \parallel_T .$$

Let us recall that

$$H_n(U,v)(t) - H_n(\overline{U},\overline{v}) = \int_0^t S(t-s)(\frac{\partial}{\partial x}[(\pi_{n,2}v(s))^2 - \frac{\partial}{\partial x}(\pi_{n,2}\overline{v}(s))^2])ds.$$

By Proposition 11, (see also Lemma 2.1 in [11]), there exists a constant C such that for all $t \in [0,T]$

$$\begin{aligned}\int_0^t &\parallel S(t-s)\frac{\partial}{\partial x}[(\pi_{n,2}v(s))^2 - (\pi_{n,2}\overline{v}(s))^2] \parallel ds \\ \leq &\ Ct^{\frac{1}{4}}\sup_{s\leq T} \parallel (\pi_{n,2}v(s))^2 - (\pi_{n,2}\overline{v}(s))^2 \parallel_{L^1(0,1)} .\end{aligned} \tag{42}$$

Since

$$\parallel \pi_n a - \pi_n b \parallel \leq \parallel a - b \parallel, \qquad a, b \in L^2,$$

for every $s \in [0,T]$,

$$\parallel (\pi_{n,2}v(s))^2 - (\pi_{n,2}\overline{v}(s))^2 \parallel_{L^1(0,1)} \leq 2n \parallel v(s) - \overline{v}(s) \parallel .$$

Consequently

$$\sup_{t\leq T} \parallel H_n(U,v) - H(\overline{U},\overline{v}) \parallel \leq 2CnT^{1/4}\sup_{t\leq T} \parallel v(t) - \overline{v}(t) \parallel ,$$

$$\parallel H_n(U,v) - H(\overline{U},\overline{v}) \parallel_{2,T} \leq 2CnT^{\frac{1}{4}} \parallel v - \overline{v} \parallel_{2,T} . \tag{43}$$

We can set

$$C^3_{T,n} = 2CnT^{1/4}. \tag{44}$$

Step 4^0. We shall find a constant $C^4_{T,n}$ such that:

$$\begin{aligned}\parallel &\ I_n(U,v) - I_n(\overline{U},\overline{v}) \parallel_{2,T} \\ \leq &\ C^4_{T,n} \parallel \begin{pmatrix} U \\ v \end{pmatrix} - \begin{pmatrix} \overline{U} \\ \overline{v} \end{pmatrix} \parallel_T .\end{aligned} \tag{45}$$

Since $\parallel S(t) \parallel \leq 1$ for every $t \geq 0$,

$$\begin{aligned}\parallel &\ I_n(U,v)(t) - I_n(\overline{U},\overline{v})(t) \parallel \\ \leq &\int_0^t \parallel S(t-s) \parallel \parallel \pi_{n,1}U(s)\,\pi_{n,2}v(s) - \pi_{n,1}\overline{U}(s)\pi_{n,2}\overline{v}(s) \parallel ds \\ \leq &\int_0^t \parallel \pi_{n,1}U(s)\,\pi_{n,2}v(s) - \pi_{n,1}\overline{U}(s)\pi_{n,2}\overline{v}(s) \parallel ds.\end{aligned}$$

But notice that for all $s \geq 0$

$$\begin{aligned}
& \| \quad \pi_{n,1}U(s)\,\pi_{n,2}v(s) - \pi_{n,1}\overline{U}(s)\pi_{n,2}\overline{v}(s) \ \| \\
& \leq \ \| \ (\pi_{n,1}U(s) - \pi_{n,1}\overline{U}(s))\pi_{n,2}v(s) \ \| \\
+ & \ \| \quad \pi_{n,1}\overline{U}(s)(\pi_{n,2}v(s) - \pi_{n,2}\overline{v}(s)) \ \| \\
& \leq \ | \ \pi_{n,1}U(s) - \pi_{n,1}\overline{U}(s) \ |\| \ \pi_{n,2}\overline{v}(s) \ \| \\
+ & \ | \quad \pi_{n,1}\overline{U}(s) \ |\| \ \pi_{n,2}v(s) - \pi_{n,2}\overline{v}(s) \ \| \\
& \leq \ n \ | \ U(s) - \overline{U}(s) \ | + n \ \| \ v(s) - \overline{v}(s) \ \| \ .
\end{aligned}$$

By the Hölder inequality

$$\begin{aligned}
& \| \quad I_n(U,v) - I_n(\overline{U},\overline{v}) \ \|_{2,T}^p \\
\leq & \ E(\sup_{t\leq T}[n\int_0^t (| \ U(s) - \overline{U}(s) \ | + \| \ v(s) - \overline{v}(s) \ \|)ds]^p \\
\leq & \ n^p E(\int_0^T (| \ U(s) - \overline{U}(s) \ | + \| \ v(s) - \overline{v}(s) \ \|)^p ds)(\int_0^T ds)^{\frac{p}{q}}.
\end{aligned}$$

Since, for non-negative a, b, $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, we have

$$\begin{aligned}
& \| \quad I_n(U,v)(t) - I_n(\overline{U},\overline{v}) \ \|_{2,T}^p \\
& \leq \ 2^{p-1}n^pT^{p-1}\{E(\int_0^T (| \ U(s) - \overline{U}(s) \ |^p)ds) \\
+E(\int_0^T (& \ \| \quad v(s) - \overline{v}(s) \ \|^p)ds)\} \\
& \leq \ 2^{p-1}n^pT^{p-1}\{TE \ (\sup_{s\leq T} | \ U(s) - \overline{U}(s) \ |^p) \\
+TE \ (\sup_{s\leq T} & \ \| \quad v(s) - \overline{v}(s) \ \|^p)\}
\end{aligned}$$

However $(a+b)^\alpha \leq a^\alpha + b^\alpha$ for $a, b \geq 0$, $0 < \alpha \leq 1$, and therefore

$$\begin{aligned}
& \| \quad I_n(U,v)(t) - I_n(\overline{U},\overline{v}) \ \|_{2,T} \\
\leq & \ Tn2^{\frac{p-1}{p}}(\| \ U - \overline{U} \ \|_{1,T}^p + \| \ v - \overline{v} \ \|_{2,T}^p)^{\frac{1}{p}} \\
\leq & \ Tn2^{\frac{p-1}{p}}(\| \ U - \overline{U} \ \|_{1,T} + \| \ v - \overline{v} \ \|_{2,T}) \\
\leq & \ Tn2^{\frac{p-1}{p}} \ \| \ X - \overline{X} \ \|_T \ .
\end{aligned}$$

And we can set

$$C_{T,n}^4 = Tn2^{\frac{p-1}{p}}. \tag{46}$$

Step 5^0. Finally set

$$C_{T,n} = \max\ (C^i_{T,n},\ i = 1, 2, 3, 4), \tag{47}$$

then (33) holds.

Taking into account explicit expressions for the constants $C^i_{T,n}$, $i = 1, 2, 3, 4$, there exists such T_n that $C_{T_n,n} < 1$.

Step 6^0. By Banach fixed point theorem there exists a unique fixed point of the operator $\begin{pmatrix} U \\ v \end{pmatrix} \to \begin{pmatrix} F_n(U,v) \\ G(U,v) + H_n(U,v) + I_n(U,v) \end{pmatrix}$ in the space $Z^p_{T_n}$. Hence there exists a unique solution $\begin{pmatrix} U_n \\ v_n \end{pmatrix}$ of problem (24)-(25). By a standard iteration procedure there exists a unique solution to problem (24)-(25) on arbitrary time interval $[0, T]$. ■

4 Proof of Theorem 3

Let $X_n(t) = \begin{pmatrix} U_n(t) \\ v_n(t) \end{pmatrix}$, $t \geq 0$, be the solution to problem (11)-(13). Define

$$\tau_n = \min\ [\inf\{t \geq 0 :\mid U_n(t) \mid^2 \geq n^2\},\ \inf\{t \geq 0 :\parallel v_n(t) \parallel^2 \geq n^2\}]. \tag{48}$$

Notice that $X_n(t) = X_m(t)$ for $m \geq n$ and $t \leq \tau_n$. Therefore, we can set $X(t) = X_n(t)$ if $t \leq \tau_n$ and this is a solution to problem (11)-(13) on the time interval $[0, \tau_\infty)$, where

$$\tau_\infty = \lim_{n\to\infty} \tau_n.$$

We shall prove that $\tau_\infty = +\infty$.

Let $X(t) = \begin{pmatrix} U(t) \\ v(t) \end{pmatrix}$ be a possibly exploding solution to problem (11)-(13) defined on $[0, \tau_\infty)$. We set

$$V(t) = v(t) - Z(t), \tag{49}$$

that is,

$$\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = \begin{pmatrix} U(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} 0 \\ Z(t) \end{pmatrix},$$

where

$$Z(t) = \int_0^t e^{A(t-s)} g(v(s)) \chi_{s<\tau_\infty} dW(s), \ Z(0) = 0.$$

Recall that by the Sobolev imbedding theorem (see [1], Theorem 7.57, p. 217) we have for a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary that if

$$s > 0, 1 < p < n, n > sp \text{ and } p \leq r \leq np/(n - sp), \tag{50}$$

then $W^{s,p}(\Omega)$ is continuously imbedded into $L^r(\Omega)$:

$$W^{s,p}(\Omega) \hookrightarrow L^r(\Omega).$$

Therefore, if $n = 1$, $\Omega = (0, 1)$, $p = 2$, $s = \frac{1}{4}$ and $r = 4$ then (50) holds and

$$H^{\frac{1}{4}}(0, 1) \hookrightarrow L^4(\Omega),$$

where we use notation $W^{s,2}(0, 1) = H^s(0, 1)$. Notice that $H^{\frac{1}{4}}(0, 1) \hookrightarrow L^4(\Omega)$ means that there exists $c > 0$ such that for all $u \in H^{\frac{1}{4}}(0, 1)$

$$\| u \|_{L^4} \leq \| u \|_{H^{\frac{1}{4}}(0,1)} .$$

Moreover, there exists $c > 0$ such that

$$c \| (-\frac{d^2}{dx^2})^{\frac{1}{8}} u \| \geq \| u \|_{H^{\frac{1}{4}}(0,1)} .$$

The following Proposition can be obtained by factorization procedure (see [26], [13] and [17]).

Proposition 5 *Let A be a self-adjoint non-positive operator generating the semigroup $S(t), t \geq 0,$ on a Hilbert space H such that*

$$\int_0^T \| S(t) \|_{HS}^2 \, dt < \infty.$$

Let $0 \leq \gamma + \frac{1}{p} < \frac{1}{2}$ and ξ is an adapted stochastic process with values in the space $L(H) = L(H, H)$ of linear operators in H. Then there exists a constant $C > 0$ such that

$$E(sup_{0\leq t\leq T} \| (-A)^\gamma \int_0^t S(t-s)\xi(s)dW(s) \|^p)$$
$$\leq CE(\int_0^T \| \xi(s) \|_{L(H,H)}^p \, ds).$$

Applying Proposition 5 with $\gamma = \frac{1}{8}$, $p = 4$, and $\xi(s)$ the multiplication operator by $g(v(s))\chi_{s<\tau_\infty}$;

$$\begin{aligned} E(\sup_{0\le t\le T} \quad & \| \quad (-\frac{d^2}{dx^2})^{\frac{1}{8}} Z(t) \|^4) \\ & \le \quad CE(\int_0^T \| \xi(s) \|^4_{L(L^2,L^2)} ds) < CT\sup_\sigma | g(\sigma) | < \infty. \end{aligned}$$

Let

$$\mu = \sup_{t\in[0,T]} \| Z(t) \|^4_{L^4} . \tag{51}$$

From Proposition 5 and the above estimates we have

$$\begin{aligned} E\mu &= E(\sup_{t\in[0,T]} \| Z(t) \|^4_{L^4}) \le CE(\sup_{t\in[0,T]} \| Z(t) \|^4_{H^{\frac{1}{4}}}) \\ &\le CE(\sup_{t\in[0,T]} \| (-\frac{d^2}{dx^2})^{\frac{1}{8}} Z(t) \|^4) < \infty. \end{aligned}$$

Thus

$$E\mu < \infty.$$

The following is a standard result on interpolation inequalities ([22], Corollary 1.1.8).

Corollary 6 *Let $(X, Y)_{\theta,p}$ and $(X, Y)_p$ be interpolation spaces for $0 < \theta < 1$, $1 \le p \le \infty$. There is $C(\theta, p)$ such that*

$$\| y \|_{(X,Y)_{\theta,p}} \le C(\theta, p) \| y \|^{1-\theta}_X \| y \|^\theta_Y \quad \textit{for every } y \in Y.$$

Then, see [22] (Example 1.1.3, pp. 13-14) we get that there exists a constant c such that for $u \in H^1(0, 1)$ and $0 < \theta < 1$

$$\| u \|_{H^\theta(0,1)} \le c \| u \|^{1-\theta} \| u \|^\theta_{H^1(0,1)} . \tag{52}$$

We shall prove the following basic estimate.

Lemma 7 *There exist a constant C such that for arbitrary $\alpha > 0$ and $V \in H^1_0$, $Z \in L^4$ we have*

$$| \int VZ\frac{\partial V}{\partial x} dx | \le C \| V \|^{\frac{3}{4}} \| V \|^{\frac{5}{4}}_{H^1_0} \| Z \|_{L^4} \tag{53}$$

and

$$\begin{aligned} & \| V \|^{\frac{3}{4}} \| V \|^{\frac{5}{4}}_{H^1_0} \| Z \|_{L^4} \\ \le & \frac{1}{4} \| V \|^2 \| Z \|^4_{L^4} + \frac{5}{8}\alpha^2 \| V \|^2_{H^1_0} + \frac{1}{8\alpha^2} \| V \|^2 . \end{aligned} \tag{54}$$

Proof. Observe that from the Schwartz inequality we have

$$
\begin{aligned}
& \left| \int VZ\frac{\partial V}{\partial x}dx \right| \leq \left(\int V^4 dx\right)^{\frac{1}{4}} \left(\int Z^4 dx\right)^{\frac{1}{4}} \left(\int \left\| \frac{\partial V}{\partial x} \right\|^2 dx\right)^{\frac{1}{2}} \\
= & \ \| V \|_{L^4} \| Z \|_{L^4} \| V \|_{H_0^1} .
\end{aligned}
$$

From the Sobolev imbedding inequality we get

$$\| V \|_{L^4} \leq c_1 \| V \|_{H^{\frac{1}{4}}}$$

and from (52) we obtain

$$\| V \|_{H^{\frac{1}{4}}} \leq c_2 \| V \|^{\frac{3}{4}} \| V \|^{\frac{1}{4}}_{H_0^1} .$$

Since $V \in H_0^1$

$$\| V \|_{L^4} \leq c_3 \| V \|^{\frac{3}{4}} \| V \|^{\frac{1}{4}}_{H_0^1} .$$

Therefore there exists c_4 such that

$$
\begin{aligned}
& \left| \int VZ\frac{\partial V}{\partial x}dx \right| \leq c_4 \| V \|^{\frac{3}{4}} \| V \|^{\frac{1}{4}}_{H_0^1} \| Z \|_{L^4} \| V \|_{H_0^1} \qquad (55) \\
\leq & \ c_4 \| V \|^{\frac{3}{4}} \| V \|^{\frac{5}{4}}_{H_0^1} \| Z \|_{L^4}
\end{aligned}
$$

and (53) holds.

To prove (54) we observe that using the generalized Young inequality for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $p, q, r > 0$, with $p = 4$, $q = \frac{8}{5}$, $r = 8$, we get

$$
\begin{aligned}
& \| V \|^{\frac{3}{4}} \| V \|^{\frac{5}{4}}_{H_0^1} \| Z \|_{L^4} \\
= & \ \| Z \|_{L^4} \| V \|^{\frac{2}{4}} \| \alpha V \|^{\frac{5}{4}}_{H_0^1} \| \frac{1}{\alpha} V \|^{\frac{1}{4}} \\
\leq & \ \frac{\| Z \|^4_{L^4} \| V \|^2}{4} + \frac{(\| \alpha V \|^{\frac{5}{4}}_{H_0^1})^q}{q} + \frac{(\| \frac{1}{\alpha} V \|^{\frac{1}{4}})^r}{r} \\
\leq & \ \frac{1}{4} \| V \|^2 \| Z \|^4_{L^4} + \frac{5}{8}\alpha^2 \| V \|^2_{H_0^1} + \frac{1}{8\alpha^2} \| V \|^2 .
\end{aligned}
$$

From (53) and (54) we get

$$
\begin{aligned}
\frac{1}{C} & \left| \int VZ\frac{\partial V}{\partial x}dx \right| \leq \| V \|^{\frac{3}{4}} \| V \|^{\frac{5}{4}}_{H_0^1} \| Z \|_{L^4} \\
& \leq \frac{1}{4} \| V \|^2 \| Z \|^4_{L^4} + \frac{5}{8}\alpha^2 \| V \|^2_{H_0^1} + \frac{1}{8\alpha^2} \| V \|^2 . \ \blacksquare
\end{aligned}
$$

Now we prove

Proposition 8 *If for $V \in C([0,T], L^2)$, $Z \in L^\infty([0,T],\ L^4(0,1)\ )$ and continuous function U*

$$\frac{\partial V}{\partial t} = \nu \frac{\partial^2 V}{\partial x^2} + U(V+Z) - \frac{\partial}{\partial x}(V+Z)^2, \tag{56}$$

$$V(0) = v_0, \tag{57}$$

then there exists a constant C such that for all $t \in [0,T]$,

$$\| V \|^2 + U^2 \leq C(\mu + \| v_0 \|^2 + U(0)^2 + 1)e^{(C\mu+1)t}, \tag{58}$$

where μ is given by (51).

Proof. We can assume that V is a strong solution to (56). We have

$$(\frac{\partial V}{\partial t}, V) = \nu(\frac{\partial^2 V}{\partial x^2}, V) + U(V+Z, V) - (\frac{\partial}{\partial x}(V+Z)^2, V)$$

so

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(V,V) &= -\nu(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial x}) + U(V,V) + U(Z,V) \\ &\quad -(\frac{\partial}{\partial x}(V+Z)^2, V) \\ &= -\nu(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial x}) + U(V,V) + U(Z,V) \\ &\quad +(V^2, \frac{\partial}{\partial x}V) + 2(VZ, \frac{\partial}{\partial x}V) + \\ &\quad (Z^2, \frac{\partial}{\partial x}V). \end{aligned}$$

Since

$$\begin{aligned} (V^2, \frac{\partial}{\partial x}V) &= -(\frac{\partial}{\partial x}V^2, V) = -2(\frac{\partial V}{\partial x}V, V) \\ &= -2(\frac{\partial V}{\partial x}, V^2) \end{aligned}$$

so

$$(V^2, \frac{\partial}{\partial x}V) = 0$$

and we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(V,V) &= -\nu(\frac{\partial V}{\partial x},\frac{\partial V}{\partial x}) + U(V,V) + U(V,Z). \\ & +2(VZ,\frac{\partial}{\partial x}V) + (Z^2,\frac{\partial}{\partial x}V)\end{aligned}$$

or

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} &\parallel V \parallel^2 + \nu \parallel V \parallel^2_{H_0^1} = U \parallel V \parallel^2 + U(V,Z) \\ & +2\int VZ\frac{\partial V}{\partial x}dx + \int Z^2\frac{\partial V}{\partial x}dx.\end{aligned}$$

Further we have

$$\begin{aligned}| & (Z^2,\frac{\partial}{\partial x}V) | = | \int_0^1 Z^2(x)\frac{\partial V}{\partial x}(x)dx | \\ \leq & (\int_0^1 Z^4(x)dx)^{\frac{1}{2}} \parallel \frac{\partial}{\partial x}V \parallel \\ \leq & \parallel Z \parallel^2_{L^4} \parallel \frac{\partial}{\partial x}V \parallel = \parallel Z \parallel^2_{L^4} \parallel V \parallel_{H_0^1} \\ \leq & \frac{\varepsilon}{2} \parallel V \parallel^2_{H_0^1} + \frac{1}{2\varepsilon} \parallel Z \parallel^4_{L^4} .\end{aligned}$$

But from (53) and (54) we have

$$\begin{aligned}| & \int VZ\frac{\partial}{\partial x}V dx | \\ \leq & C[\frac{1}{4} \parallel V \parallel^2 \parallel Z \parallel^4_{L^4} + \frac{5}{8}\alpha^2 \parallel V \parallel^2_{H_0^1} + \frac{1}{8\alpha^2} \parallel V \parallel^2].\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} &\parallel V \parallel^2 + \nu \parallel V \parallel^2_{H_0^1} \leq U \parallel V \parallel^2 + U(V,Z) \\ +2\{C[\frac{1}{4} &\parallel V \parallel^2 \parallel Z \parallel^4_{L^4} + \frac{5}{8}\alpha^2 \parallel V \parallel^2_{H_0^1} + \frac{1}{8\alpha^2} \parallel V \parallel^2] \\ +\frac{\varepsilon}{2} &\parallel V \parallel^2_{H_0^1} + \frac{1}{2\varepsilon} \parallel Z \parallel^4_{L^4} .\end{aligned} \tag{59}$$

Now we consider equation

$$\frac{1}{2}\frac{d}{dt}U^2 + \nu U^2 = U(P - \parallel V + Z \parallel^2). \tag{60}$$

Adding (59) and (60) we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}[&\parallel V \parallel^2 + U^2] + \nu \parallel V \parallel^2_{H_0^1} + \nu U^2 \\
&\leq U \parallel V \parallel^2 + U(V, Z) + 2C[\frac{1}{4} \parallel V \parallel^2 \parallel Z \parallel^4_{L^4} + \frac{5}{8}\alpha^2 \parallel V \parallel^2_{H_0^1} + \frac{1}{8\alpha^2} \parallel V \parallel^2] \\
+\frac{\varepsilon}{2} &\parallel V \parallel^2_{H_0^1} + \frac{1}{2\varepsilon} \parallel Z \parallel^4_{L^4} + U(P - \parallel V + Z \parallel^2) \\
&\leq -U(V, Z) + 2C[\frac{1}{4} \parallel V \parallel^2 \parallel Z \parallel^4_{L^4} + \frac{5}{8}\alpha^2 \parallel V \parallel^2_{H_0^1} + \frac{1}{8\alpha^2} \parallel V \parallel^2] \\
+\frac{\varepsilon}{2} &\parallel V \parallel^2_{H_0^1} + \frac{1}{2\varepsilon} \parallel Z \parallel^4_{L^4} + UP - U \parallel Z \parallel^2
\end{aligned}
$$

because

$$
\begin{aligned}
U(P- &\parallel V + Z \parallel^2) \\
&= U(P - \parallel V \parallel^2 - 2(V, Z) - \parallel Z \parallel^2) \\
&= UP - U \parallel V \parallel^2 - 2U(V, Z) - U \parallel Z \parallel^2 .
\end{aligned}
$$

Observe that from the Young inequality

$$
\begin{aligned}
-U(V, Z) + UP - U &\parallel Z \parallel^2 \\
&\leq |\, (V, ZU) \,| + \frac{U^2}{2} + \frac{1}{2}P^2 + \frac{U^2}{2} + \frac{1}{2} \parallel Z \parallel^2_{L^4} \\
&\leq \parallel V \parallel \parallel U \parallel \parallel Z \parallel + \frac{U^2}{2} + \frac{1}{2}P^2 + \frac{U^2}{2} + \frac{1}{2} \parallel Z \parallel^2_{L^4} \\
&\leq \frac{1}{2} \parallel V \parallel^2 + \frac{1}{2}U^2 \parallel Z \parallel^2 + \frac{U^2}{2} + \frac{1}{2}P^2 + \frac{U^2}{2} + \frac{1}{2} \parallel Z \parallel^2_{L^4} .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}[&\parallel V \parallel^2 + U^2] + \nu \parallel V \parallel^2_{H_0^1} + \nu U^2 \\
&\leq \frac{1}{2} \parallel V \parallel^2 + \frac{1}{2}U^2 \parallel Z \parallel^2 \\
+\frac{U^2}{2} + \frac{1}{2}P^2 + \frac{U^2}{2} + \frac{1}{2} &\parallel Z \parallel^2_{L^4} \\
+2C\{\frac{1}{4} &\parallel V \parallel^2 \parallel Z \parallel^4_{L^4} + \frac{5}{8}\alpha^2 \parallel V \parallel^2_{H_0^1} \\
+\frac{1}{8\alpha^2} &\parallel V \parallel^2\} + \frac{\varepsilon}{2} \parallel V \parallel^2_{H_0^1} + \frac{1}{2\varepsilon} \parallel Z \parallel^4 .
\end{aligned}
$$

Now we choose α and ε to get

$$\nu = \frac{\varepsilon}{2} + \frac{5}{8} \cdot 2C\alpha^2.$$

Therefore

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}[\quad \| \quad & V \|^2 + U^2] + \nu U^2 \\ \leq \quad & \frac{1}{2} \| V \|^2 + \frac{1}{2} U^2 \| Z \|^2 \\ + \frac{U^2}{2} + \frac{1}{2} P^2 + \frac{U^2}{2} + \frac{1}{2} \quad \| \quad & Z \|_{L^4}^2 \\ +2C\{\frac{1}{4} \quad \| \quad & V \|^2 \| Z \|_{L^4}^4 + \frac{1}{8\alpha^2} \| V \|^2 \} + \frac{1}{2\varepsilon} \| Z \|^4 . \end{aligned}$$

For arbitrary $Z \in L^4$,

$$\| Z \| \leq \| Z \|_{L^4}, \qquad \| Z \|_{L^4}^2 \leq \| Z \|_{L^4}^4 + 1,$$

therefore, neglecting the term νU^2 in the left hand side of the inequality, we arrive at

$$\begin{aligned} \frac{d}{dt}[\quad \| \quad & V \|^2 + U^2] \\ \leq \quad & C(\| V \|^2 + U^2)(\| Z \|_{L^4}^4 + 1) + C(\| Z \|_{L^4}^4 + 1), \end{aligned}$$

where C is the maximal number among:

$$\frac{C}{2}, \ \frac{1}{2} + \frac{1}{8\alpha^2}, \ \frac{3}{2}, \ \frac{1}{2}P^2 + \frac{1}{2}, \ \frac{1}{2\varepsilon} + \frac{1}{2}.$$

Consequently

$$\begin{aligned} \| \quad & V(t) \|^2 + U^2(t) \\ \leq \quad & e^{\int_0^t C(\|Z\|_{L^4}^4 + 1) ds} (\| v_0 \|^2 + U^2(0)) + C \int_0^t e^{2 \int_s^t (\|Z(\sigma)\|_{L^4}^4 + 1) d\sigma} (\| Z(s) \|_{L^4}^4 + 1) ds \end{aligned}$$

So the required estimate holds. ■

Continuation of the proof of Theorem 3

Let $X_n(t) = \begin{pmatrix} U_n(t) \\ v_n(t) \end{pmatrix}$ be a, possibly exploding, solution to problem (11)-(13), where $U_n(t)$ is the solution to (18) and $v_n(t)$ is the solution to (19).

By (58) (compare Lemma 3.1 of [11]) there exists a constant $C_1 \geq 1$ such that

$$\begin{aligned} | & U_n(t) |^2 + \| v_n(t) \|^2 + 1 \\ \leq & C_1(\mu + | U_n(0) |^2 + \| v_0 \|^2 + 1) e^{(C\mu+1)t} + 1 \\ \leq & C_1(\mu + | U_n(0) |^2 + \| v_0 \|^2 + 2) e^{(C\mu+1)t} \end{aligned}$$

so

$$\begin{aligned} \log (\quad | & U_n(t) |^2 + \| v_n(t) \|^2 + 1) \\ \leq & \log C_1 + \log(\mu + | U_n(0) |^2 + \| v_0 \|^2 + 2) \\ & + C(\mu + 1)T \end{aligned}$$

so

$$\begin{aligned} E[\log \sup_{t \leq T}(\quad | & U_n(t) |^2 + \| v_n(t) \|^2 + 1) \\ \leq & \log C_1 + \log(E\mu + | U_n(0) |^2 + \| v_0 \|^2 + 2) \\ & + C(E\mu + 1)T. \end{aligned}$$

By Jensen inequality it follows that

$$\begin{aligned} E(\sup_{t \in [0,T]} \log(\quad | & \quad U_n(t) |^2 + \| v_n(t) \|^2 + 1) \\ & \leq \log C_1 + \log(E\mu + | U_n(0) |^2 + \| v_0 \|^2 + 2) \\ +C(E\mu + 1)T & = K_T. \end{aligned}$$

Since by the Chebyshev inequality

$$\begin{aligned} P(\tau_n & \leq T) \\ & = P(\sup_{t \in [0,T]} \log (| U_n(t) |^2 + \| v_n(t) \|^2 + 1) \geq \log (n+1)) \\ & \leq \frac{E(\sup_{t \in [0,T]} \log (| U_n(t) |^2 + \| v_n(t) \|^2 + 1)}{\log(n+1)} \end{aligned}$$

we get, for a new constant K'_T

$$P(\tau_n \leq T) \leq \frac{K'_T}{\log\,(n+1)} \to 0$$

as $n \to \infty$. Hence $\tau_\infty = \infty$. ■

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Appendix

An estimate for extended heat semigroup

Let us recall that $S(t)$ is the heat semigroup introduced in (16). We prove now

Lemma 1. *The operators $S(t)$, $t>0$, can be extended linearly in the space of all distributions of the form $\frac{\partial}{\partial x}v$, $v\in L^1(0,1)$, in a way such that*

$$\| S(t)\frac{\partial}{\partial \xi}v \| \leq \| v \|_{L^1(0,1)} (\sum_{k=1}^{\infty}\frac{2\pi}{\sqrt{\nu}}k^2 e^{-2\frac{\pi^2}{\nu}k^2 t})^{1/2}.$$

Proof. Set $\nu=1$. By Parseval's identity

$$\| S(t)u \|^2 = \sum_{k=1}^{\infty} e^{-2\pi^2k^2t}(u,e_k)^2,\ u\in L^2.$$

Let $v\in L^2$ be an absolutely continuous function such that $\frac{\partial}{\partial \xi}v\in L^2$. Then

$$\| S(t)\frac{\partial}{\partial \xi}v \|^2 = \sum_{k=1}^{\infty} e^{-2\pi^2k^2t}(\int_0^1 \frac{\partial}{\partial \xi}v(\xi)e_k(\xi)d\xi)^2.$$

Integrating by parts

$$\sqrt{\frac{2}{\pi}}\int_0^1 \frac{\partial}{\partial \xi}v(\xi)\ \sin\, k\pi\xi\ d\xi = -\sqrt{\frac{2}{\pi}}k\pi\int_0^1 v(\xi)\ \cos\, k\pi\xi\ d\xi.$$

Therefore

$$| \int_0^1 \frac{\partial}{\partial \xi} v(\xi) e_k(\xi) d\xi | \leq \sqrt{2}\pi k \int_0^1 | v(\xi) | d\xi$$

and consequently

$$\begin{aligned} \| \quad S(t) \frac{\partial}{\partial x} v \| \leq \| \, v \, \|_{L^1(0,1)} \\ \times (\sum_{k=1}^{\infty} 2\pi k^2 e^{-2\pi^2 k^2 t})^{1/2}. \end{aligned}$$

Since absolutely continuous function with square integrable derivatives are dense in $L^1(0,1)$ the required extension of S exists. It will be denoted with the same symbol $S(t)$, $t \geq 0$. From this lemma follows. ■

Our aim is to prove in an elementary way the following result from [12].

Proposition 9 *For arbitrary $T > 0$ there exists C such that for $t \leq T$ and for measurable, bounded, $L^1(0,1)$-valued function $v(s)$, $s \in (0,t)$:*

$$\int_0^t \| \, S(\sigma) \frac{\partial}{\partial \xi} v(\sigma) \, \| \, d\sigma \leq C t^{1/4} \; sup_{s \leq t} \| \, v(s) \, \|_{L^1(0,1)} \; .$$

Proof. Set $\nu = 1$. We have to show that for a constant $C > 0$ and $T > 0$

$$\int_0^T (\sum_{k=1}^{\infty} e^{-2\pi^2 k^2 t} k^2)^{1/2} dt \leq C T^{1/4}.$$

The function

$$h(t) = \sum_{k=1}^{\infty} e^{-2\pi^2 k^2 t} k^2, \; t \geq 0$$

is the Laplace transform of purely atomic measure μ which associates with points $2\pi^2 k^2$ masses k^2, $k = 1, 2, ...$

Let

$$U(\sigma) = \mu((0,\sigma]) = \sum_{2\pi^2 k^2 \leq \sigma} k^2 = \sum_{k \leq \frac{1}{\pi}\sqrt{\frac{\sigma}{2}}} k^2.$$

One easily finds that U is slowly varying and

$$\lim_{\sigma \to \infty} \frac{U(\sigma y)}{U(\sigma)} = y^{3/2}, \; y > 0.$$

Consequently, by tauberian theorems (see [15], p. 422-423)

$$\lim_{\,t\to 0}\frac{h(t)}{U(\frac{1}{t})} = \Gamma(\frac{5}{2}).$$

But $U(\frac{1}{t}) \sim \frac{1}{3}\frac{1}{t^{3/2}}$ as $t \to +\infty$ and therefore

$$h(t) \sim \frac{1}{3\Gamma(\frac{5}{2})}\frac{1}{t^{3/2}}$$

and for a constant C

$$h(t) \leq C\frac{1}{t^{3/2}},\ t \leq T_0.$$

Finally

$$\int_0^T h^{1/2}(t)dt \leq C\int_0^T \frac{1}{t^{3/4}}dt = 4CT^{1/4},\ T \leq T_0.$$

and therefore, the required inequality follows. ■