Portfolio diversification with Markovian prices

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Abstract

The problem of constructing impulsive rebalancing of portfolios, introduced by Pliska and Suzuki, is solved for models with general Markovian prices. Existence of optimal strategy is established and its structure described. Quasi-variational inequalities determining the value function are deduced for multiplicative prices with general Levy noise and the case of Poissonian noise is considered in some detail.

1. Introduction

An important problem for portfolio managers is to respect the diversification requirement to maintain proportions of the capital, that should be invested in different asset groups, constant. It is impossible to rebalance a portfolio continuously, so it usually does not keep exactly to the required proportions. Therefore each manager has to come up with some algorithm to decide the moments of rebalancing.

Pliska and Suzuki [5] (we have recently learned that this article had been published [6]), improving the ideas of Leland [4], considered a model consisting of d assets whose prices satisfy

$$dS_t^i = S_t^i\big(\mu_i dt + \sigma_i dW_t\big), \quad i = 1, \dots, d,$$

where W_t is a m-dimensional Brownian motion, σ_i is a vector and μ_i is any real number. Trading strategy is described by a d-dimensional adapted process $(N_t)_{t\geq 0}$ denoting the number of units of assets held at each moment. They imposed both proportional and constant transaction costs specified in details later.

In the view of constant transaction costs, any trading strategy Π can be described by a sequence of transaction times (stopping times) $\tau_1, \tau_2, \dots$ and resulting portfolio contents $N_{\tau_1}, N_{\tau_2}, \dots$. The process N_t is constant between transaction times i.e. $N_t = N_0 1_{t\in[0,\tau_1[} + \sum_{i=1}^{\infty} N_{\tau_i} 1_{t\in[\tau_i,\tau_{i+1}[}$. We

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introduce a proportion process linked to the strategy Π

$$w_t = W\binom{N_t}{S_t},$$

where, denoting by $N_t S_t$ the scalar product in $\mathbb{R}^d$,

$$W\binom{N_t}{S_t} = \left(\frac{N_t^1 S_t^1}{N_t S_t}, \ldots, \frac{N_t^d S_t^d}{N_t S_t}\right).$$

The transaction costs are expressed in terms of proportions:

$$c(w, v) = K + k \sum_{i=1}^{d} |w^i - v^i|$$

for $K > 0$ and $k \geq 0$. This is a reasonable simplification that enables us to incorporate transactions costs into a cost functional. Pliska and Suzuki introduced the cost functional

$$J(\Pi) = \mathbb{E}\left(\int_0^\infty e^{-\beta t} f(w_t)dt + \sum_{i=1}^{\infty} e^{-\beta\tau_i} c\left(W\binom{N_{\tau_i -}}{S_{\tau_i}}, W\binom{N_{\tau_i}}{S_{\tau_i}}\right) 1_{\tau_i < \infty}\right),$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is a function measuring quality of the portfolio. They specified further that $f(w) = \lambda(w - w^*)'\sigma\sigma'(w - w^*) - (w - w^*)'\mu$, where σ is a matrix consisting of rows σ_i, $i = 1, \ldots, d$, $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{R}^d$, w^* is a target asset mix and $\lambda \in \mathbb{R}$. For $d = m = 2$ they established the existence of optimal strategy and showed that it is characterized by a continuation region of the proportions where trading is not performed.

In the present paper we cover general Markovian price processes. We consider a market modeled by a general d-dimensional positive Markovian process $(S_t)_{t\geq 0}$, representing price movements of different groups of assets or assets themselves. By positivity we mean $\mathbb{P}(S_t^i > 0 \text{ for } t \geq 0, i = 1, \ldots, d) = 1$. Our evaluation procedure, measuring quality of the portfolio at each moment, is determined by a continuous function f defined on proportions. We show that if S_t is a positive Feller process then there exists an optimal trading strategy Π minimizing the cost functional $J(\Pi)$. Moreover, for multiplicative Poissonian prices we show how the optimal strategy can be found explicitly.

The content of the paper is as follows. In §2 we establish a general existence theorem containing as a special case result by Pliska and Suzuki [5], without any reference to quasi-variational inequalities. In §3 multiplicative price processes with general Levy noise are considered and the form of the corresponding quasi-variational inequality for the value function is established. The special case of Poissonian noise is studied in §4. More details on quasi-variational inequalities for discontinuous process are given in the Appendix.

2. Existence of optimal strategy

We approach the problem of finding optimal solution with the impulse control method. First we prove existence of solution to the functional equation connected to our problem. Then we show that obtained solution defines the optimal strategy.

We assume throughout this section that S_t is a Feller process and the function f is continuous.

To formulate the problem in a formal way we take

$$Y_t = \begin{pmatrix} N_t \\ S_t \end{pmatrix} \in \mathbb{R}^{2d}_+$$

as a controlled process. Certainly impulses change only first coordinate, second representing asset prices is present only for technical reasons. Our goal is to construct a process Y_t satisfying following conditions

- there exists an increasing sequence of stopping times $\tau_1, \tau_2, \ldots$ with $\tau_i \uparrow \infty$ such that first coordinate of Y_t changes only in moments defined by $(\tau_i)_{i=1,2,\ldots}$,
- second coordinate of Y_t is equal to an external price process S_t,
- the trading strategy encoded in Y_t is self-financing i.e. $(N_{\tau_i} - N_{\tau_i -})S_{\tau_i} = 0$,
- the portfolio wealth is always positive, i.e. $N_t S_t > 0$,
- the number of shares of each stock is non-negative (no borrowing of shares allowed), i.e. $N_t \geq 0$

and minimizing the functional

$$J(Y) = \mathbb{E}^{Y_0} \left(\int_0^\infty e^{-\beta t} F \begin{pmatrix} N_t \\ S_t \end{pmatrix} dt + \sum_{i=1}^{\infty} e^{-\beta \tau_i} \tilde{C}(N_{\tau_i -}, N_{\tau_i}, S_{\tau_i}) 1_{\tau_i < \infty} \right), \tag{1}$$

where

$$F \begin{pmatrix} N_t \\ S_t \end{pmatrix} = f \left(W \begin{pmatrix} N_t \\ S_t \end{pmatrix} \right)$$

and

$$W \begin{pmatrix} N_t \\ S_t \end{pmatrix} = \left(\frac{N_t^1 S_t^1}{N_t S_t}, \ldots, \frac{N_t^d S_t^d}{N_t S_t} \right)$$

is the proportion function, $Y_0 = (N_0, S_0)$ is the initial point ($S_0 > 0$, $N_0 \geq 0$, $N_0 S_0 > 0$). The cost of impulses is defined as

$$\tilde{C}(N_1, N_2, S) = c \left(W \begin{pmatrix} N_1 \\ S \end{pmatrix}, W \begin{pmatrix} N_2 \\ S \end{pmatrix} \right).$$

Note that between impulses the dynamics of Y_t is governed by the semigroup

$$P_t^* v \begin{pmatrix} n \\ s \end{pmatrix} = P_t \left(v \begin{pmatrix} n \\ \cdot \end{pmatrix} \right) (s),$$

for $v \in C(\mathbb{R}^{2d}, \mathbb{R})$ and P_t – the semigroup for S_t.

To derive a functional equation connected to the problem (1) we recall the assumption that $\mathbb{P}(S_t > 0 \, \forall t) = 1$ and denote

$$E = \left\{ \begin{pmatrix} n \\ s \end{pmatrix} \in \mathbb{R}^{2d} : \quad n \geq 0 , \quad n \neq 0 , \quad s > 0 \right\} .$$

It is obvious that the process $\begin{pmatrix} N_t \\ S_t \end{pmatrix}$ starting from any point in E does not exit E.

For functions $v : E \to \mathbb{R}$ we write the equation

$$v\binom{n}{s} = \mathcal{K}v\binom{n}{s} = \inf_{\tau} \mathbb{E}^{(n,s)} \left[\int_0^{\tau} e^{-\beta t} F\binom{N_t}{S_t} dt + e^{-\beta\tau} M v\binom{N_\tau}{S_\tau} \right], \tag{2}$$

where the switching functional is given as

$$M\phi\binom{n}{s} = \inf\left\{ \tilde{C}(n, n+\xi, s) + \phi\binom{n+\xi}{s} : \quad \binom{n+\xi}{s} \in E, \quad \xi s = 0 \right\}. \tag{3}$$

Notice that $\tilde{C}(n, n+\xi, s) = \tilde{C}(0, \xi, s)$.

The following theorem contains as a special case a result by Pliska, Suzuki [5] concerned with the case of S_t being a two dimensional Black-Scholes process. Their method was based on the theory of quasi-variational inequalities (QVI) [1]. We deal directly with the equation (2).

THEOREM 2.1. Assume that S_t is a Feller process and f is a bounded continuous function of E. There exists exactly one bounded continuous solution $v(n, s)$ to the equation (2) and the optimal strategy for the problem (1) is given by

$$\begin{aligned}
\tau_1 &= \inf\{t \geq 0 : \quad Mv(N_t, S_t) = v(N_t, S_t)\}, \\
\tau_i &= \inf\{t > \tau_{i-1} : \quad Mv(N_t, S_t) = v(N_t, S_t)\}, \\
N_{\tau_i} &\in \{n \in \mathbb{R}^d : \quad (n, S_{\tau_i}) \in E, \quad Mv(N_{\tau_{i-1}}, S_{\tau_i}) = v(n, S_{\tau_i}) + C(N_{\tau_{i-1}}, n, S_{\tau_i})\}
\end{aligned}$$

Proof. In order to prove existence of a unique solution to the equation (2) we recall a result from Zabczyk [7]. Define

$$h\binom{n}{s} = \mathbb{E}^{(n,s)} \left[\int_0^{\infty} e^{-\beta t} F\binom{N_t}{S_t} dt \right]$$

and let $C^b(E)$ be the space of bounded continuous functions.

PROPOSITION 2.2. Assume that $\binom{N_t}{S_t}$ is a Feller process, $F \geq 0$, $h \in C^b(E)$, $\gamma h \leq M(0)$ for a positive constant γ and M transforms $C^b(E)$ into $C^b(E)$. Then equation (2) has exactly one solution $v \in C^b(E)$. Moreover, $\mathcal{K}^n h$ tends to v uniformly as $n \to \infty$.

In our setting we have to weaken conditions of the above theorem. We define operators

$$\mathcal{K}^L v\binom{n}{s} = \inf_{\tau} \mathbb{E}^{(n,s)} \left[\int_0^{\tau} e^{-\beta t} \left(F\binom{N_t}{S_t} + L \right) dt + e^{-\beta\tau} M v\binom{N_\tau}{S_\tau} \right]$$

for $L \in \mathbb{R}$. Thus $\mathcal{K}^0 = \mathcal{K}$.

LEMMA 2.3. There exists a unique solution to the equation $v = \mathcal{K}v$ iff there exists a unique solution to the equation $v = \mathcal{K}^L v$. Moreover, if $\tilde{v}$ is the solution of $v = \mathcal{K}^L v$ then $\tilde{v} - \frac{L}{\beta}$ is the solution of $v = \mathcal{K}v$.

Proof. Let $\tilde{v}$ be the solution of $v = \mathcal{K}^L v$. Then

$$\begin{aligned}
\tilde{v}\binom{n}{s} &= \mathcal{K}^L\tilde{v}\binom{n}{s} \\
&= \inf_\tau \mathbb{E}^{(n,s)}\left[\int_0^\tau e^{-\beta t}\left(F\binom{N_t}{S_t} + L\right)dt + e^{-\beta\tau} M v\binom{N_\tau}{S_\tau}\right] \\
&= \inf_\tau \mathbb{E}^{(n,s)}\left[\int_0^\tau e^{-\beta t} F\binom{N_t}{S_t}dt + \frac{L}{\beta} - \frac{L}{\beta}e^{-\beta\tau} + e^{-\beta\tau} M v\binom{N_\tau}{S_\tau}\right] \\
&= \frac{L}{\beta}\inf_\tau \mathbb{E}^{(n,s)}\left[\int_0^\tau e^{-\beta t} F\binom{N_t}{S_t}dt + e^{-\beta\tau} M\left(v - \frac{L}{\beta}\right)\binom{N_\tau}{S_\tau}\right].
\end{aligned}$$

Thus

$$\tilde{v} - \frac{L}{\beta} = \mathcal{K}\left(\tilde{v} - \frac{L}{\beta}\right).$$

A similar reasoning proves second implication. ∎

As a corollary to above results we obtain the following lemma.

LEMMA 2.4. Assume that $\binom{N_t}{S_t}$ is a Feller process, $h \in C^b(E)$ and M transforms $C^b(E)$ into $C^b(E)$. Let F be bounded from below by $(-L)$. If there exists positive constant γ such that

$$\gamma\left(h + L\int_0^\infty e^{-\beta t}dt\right) = \gamma(h + \frac{L}{\beta}) \le M(0)$$

then there exists a unique solution $\tilde{v} \in C^b(E)$ of the equation $v = \mathcal{K}^L v$. Moreover, the function $v = \tilde{v} - \frac{L}{\beta}$ is a unique solution of (2).

Now we shall prove existence of solution to the equation (2) for our specific function F, functional M and process $\binom{N_t}{S_t}$. First notice that $\binom{N_t}{S_t}$ is a Feller process since S_t is a Feller process from previous assumptions and N_t is constant. Observe that F is a continuous function that is defined on proportions. So it must be bounded, since proportions form a compact set D in $\mathbb{R}^d$:

$$D = \Big\{(w_1, \ldots, w_d) \in \mathbb{R}^d : \quad w_i \in [0,1], \quad \sum_{i=1}^d w_i = 1\Big\}.$$

Thus it is straightforward that $h \in C^b(E)$. Let $L = \min(0, -\inf_{x\in E} F(x))$. Since $M(0) \ge K > 0$ one can easily find a positive constant γ such that $\gamma(h + \frac{L}{\beta}) \le M(0)$. Continuity of the cost function $\tilde{C}$ and the multifunction mapping $\binom{n}{s}$ into the set of possible impulse destinations implies that M transforms the set continuous functions into itself. To show that M transforms $C^b(E)$ into $C^b(E)$ take any function $g \in C^b(E)$ with $\alpha = \sup|g|$. Then $Mg\binom{n}{s} \le K + dk + \alpha$ and $Mg\binom{n}{s} \ge K - \alpha$, so $Mg \in C^b(E)$. Therefore by lemma 2.4 there exists a unique continuous and bounded function v that is the solution to $v = \mathcal{K}v$. Thus, we have proved the first assertion of the theorem 2.1.

Now, we derive an impulse control for the main problem. Since we know that there exists solution v to the functional equation (2) we have to prove that the infimum in (2) is attained by some stopping time (this would be the moment of the impulse) and that we can find a transaction (change of N_t) that should be made in this moment. It is well known (see Bensoussan, Lions [1], Zabczyk [7]) that the optimal stopping time is given by $\tau = \inf\{t \geq 0 \,:\, Y_t \in Z\}$, where $Z = \{y \in \mathbb{R}^{2d} \,:\, v(y) = Mv(y)\}$. We only have to prove that for each $(n, s) \in E$ there exists $\xi \in \mathbb{R}^d$ such that $(n + \xi, s) \in E$ and

$$Mv\begin{pmatrix} n \\ s \end{pmatrix} = \tilde{C}(0, \xi, s) + v\begin{pmatrix} n + \xi \\ s \end{pmatrix}.$$

Fix $(n, s) \in E$. Both functions $\tilde{C}$ and v are continuous (v is also bounded). We first prove that the infimum is taken over a closed set. In fact this set can be written as

$$A = \{\xi \,:\, (n + \xi, s) \in E, \xi s = 0\} = \{\xi \,:\, n + \xi \geq 0, \xi s = 0\} \setminus \{-n\}.$$

The self-financing condition $\xi s = 0$ assures that $(-n) \notin A$, so A is closed. Now take a sequence $\xi_n \in A$ such that

$$\tilde{C}(0, \xi_n, s) + v\begin{pmatrix} n + \xi_n \\ s \end{pmatrix} \to Mv\begin{pmatrix} n \\ s \end{pmatrix}.$$

If $||\xi_n|| \nrightarrow \infty$ then ξ_n admits a subsequence converging to some $\xi \in E$. Otherwise, $||\xi_n|| \to \infty$. From self-financing condition and equivalence of all norms on $\mathbb{R}^d$ we obtain that $C(0, \xi_n, s) \geq K + \beta||\xi_n||$ for some $\beta \in \mathbb{R}_+$. Hence, the boundedness of v implies that $C(0, \xi_n, s) + v(\xi_n) \to \infty$, which leads to contradiction. For completeness of the proof we shall show that $\tau_n \to \infty$ a.s. Notice that each impulse adds a cost of a size at least K. Since the value function v is bounded, an infinite number of transactions in finite time is impossible – its discounted transaction costs would sum up to infinity. This completes the proof of theorem 2.1. ■

3. Markov property for proportion process

In this section we assume that the price process is multiplicative i.e.

$$S^i(\gamma s, t) = \gamma S^i(s, t), \quad \gamma \in \mathbb{R}, \quad s \in \mathbb{R}, \quad s \geq 0, \quad t \geq 0, \quad i = 1, \ldots, d. \tag{4}$$

Here $\big(S^i(s, t)\big)_{t \geq 0}$ denotes an i-th coordinate of a price process starting from the point s

$$S(s, 0) = s.$$

An important example of a multiplicative price process is a solution to the Ito equation

$$\begin{aligned} dS_i(s, t) &= S_i(s, t) dZ_i(t), \quad i = 1, \ldots, d, \\ S(s, 0) &= s, \quad s \in \mathbb{R}^d, \quad s \geq 0. \end{aligned} \tag{5}$$

for a Levy process $(Z_1, \ldots, Z_d)$ with jumps greater than -1 granting that the solution is a positive process.

We will show that the proportion process linked to S_t is Markovian and argue that the control problem (1) formulated in terms of proportions has an optimal solution. Let D be a simplex of proportions as defined earlier

$$D = \{(w_1, \ldots, w_d) \in [0, 1]^d : \quad \sum_{i=1}^{d} w_i = 1\}.$$

The process $N(t)$ is constant, so, intuitively, we can incorporate it into $S(s,t)$ using (4). We define $T : \mathbb{R}^d_+ \to D$

$$T(S) = \left(\frac{S_1}{S_1 + \cdots + S_d}, \ldots, \frac{S_d}{S_1 + \cdots + S_d} \right).$$

Then $w(t) = T(S(\tilde{s}, t))$, where $\tilde{s} = (N_1(0)S_1(0), \ldots N_d(0)S_d(0))$ and obviously $w(t)$ is indifferent to scaling of initial condition $\tilde{s}$

$$T(S(\tilde{s}, t)) = T(S(\gamma \tilde{s}, t)), \quad \text{for scalar } \gamma \neq 0. \tag{6}$$

We introduce an operator T^* acting on functions $f : D \to \mathbb{R}$ in the following way:

$$(T^* f)(s) = f\left(T(s)\right), \quad s \in \mathbb{R}^d_+.$$

THEOREM 3.1. Let $\mathcal{A}$ be a generator for the positive price process S i.e. almost all trajectories of $\big(S(s,t)\big)_{t \geq 0}$ are positive for a positive initial condition s. Then the proportion process is Markov with the generator $\tilde{\mathcal{A}}$ given by

$$(\tilde{\mathcal{A}} f)(w) = (\mathcal{A}(T^* f))(w), \quad w \in D.$$

Proof. The proof uses theorem 10.13 in Dynkin [3]. We have to show a few properties of the map T with respect to the transition function of $S(s,t)$. We denote by $\mathcal{B}$ the Borel σ-field in $\mathbb{R}^d_+$ and by $\tilde{\mathcal{B}}$ the Borel σ-field in D. Let $P(t, s, \Gamma)$ be a transition function for the process $S(t)$, $\Gamma \in \mathcal{B}$. We have to check the following conditions:

i) $T(\mathbb{R}^d_+) = D$,

ii) $T(\mathcal{B}) \subseteq \tilde{\mathcal{B}}$,

iii) for all $s, s' \in \mathbb{R}^d$ such that $Ts = Ts'$ and $\Gamma \in \tilde{\mathcal{B}}$ we have

$$P(t, s, T^{-1}\Gamma) = P(t, s', T^{-1}\Gamma).$$

The properties i) and ii) are straightforward. Only the third one requires some consideration. If $Ts = Ts'$ then there exists a scalar $\gamma \neq 0$ such that $s = \gamma s'$. Therefore, $T(S(s,t)) = T(S(s',t))$ from (6). Hence, theorem 10.13 in Dynkin [3] implies that

$$T^* \tilde{\mathcal{A}} = \mathcal{A} T^*.$$

Take $f : D \to \mathbb{R}$, $s \in \mathbb{R}^d$, $s \geq 0$ and notice that

$$\begin{aligned} (T^*(\tilde{\mathcal{A}} f))(s) &= (\mathcal{A}(T^* f))(s), \\ (\tilde{\mathcal{A}} f)(T(s)) &= (\mathcal{A}(T^* f))(s), \\ (\tilde{\mathcal{A}} f)(w) &= (\mathcal{A}(T^* f))(T^{-1} w), \end{aligned}$$

where $w = T(s)$ and $T^{-1}w$ is any element of the counterimage of w, for example s. We can simplify the formula further by noting that $w \in T^{-1}w$. Hence, $(\tilde{\mathcal{A}} f)(w) = (\mathcal{A}(T^* f))(w)$. ∎

We can reformulate our problem solely in the language of the proportion process. Our trading strategy Π consists of a sequence of stopping times $\tau_1, \tau_2, \ldots$ and changes of the proportion process at these times $\tilde{w}_1, \tilde{w}_2, \ldots$. Since the proportion process must be defined on $\mathbb{R}_+$ almost everywhere, we take on the following interpretation of the trading strategy which would allow us to write clearly the cost functional: $w(t) = w(\tilde{w}_i, t - \tau_i)$ for $t \in]\tau_i, \tau_{i+1}]$.

We do not have to limit possible impulses (as in the previous case) to satisfy self-financing condition. It is possible to reach any proportion starting from arbitrary one and satisfying self-financing condition. Hence, the functional looks as follows

$$J(\Pi) = \mathbb{E}\left(\int_0^\infty e^{-\beta t} f(w(t))dt + \sum_{i=1}^\infty e^{-\beta \tau_i} c\big(w(\tau_i), \tilde{w}_i\big) 1_{\tau_i < \infty} \right). \tag{7}$$

We can use a similar approach as in section 2 to prove a counterpart of theorem 2.1

THEOREM 3.2. Assume that w is a Feller process and f is a continuous function on D. There exists exactly one bounded continuous solution $v(w)$ to the equation

$$\begin{aligned} v(x) &= \inf_\tau \mathbb{E}\left(\int_0^\tau e^{-\beta t} f\big(w(x,t)\big)dt + e^{-\beta\tau} \tilde{M}v\big(w(x,\tau)\big) \right), \\ Mv(x) &= \inf_{y \in D} \big(v(y) + c(x,y) \big). \end{aligned} \tag{8}$$

and the optimal strategy for the problem (1) is given by

$$\begin{aligned} \tau_1 &= \inf\{t \geq 0 : \quad Mv(w_t) = v(w_t)\}, \\ \tau_i &= \inf\{t > \tau_{i-1} : \quad Mv(w_t) = v(w_t)\}. \end{aligned}$$

The size of the impulse at the moment τ_i is any number from the set

$$\{w \in D : \quad Mv(w_{\tau_i}) = v(w) + c(w_{\tau_i}, w)\}.$$

Notice that if S is a Feller process then so is w. Take $g \in C^b(D)$, $t \geq 0$ and and consider $\tilde{g}(x) = \mathbb{E}\, g\big(w(x,t)\big)$, $x \in D$. The function $\tilde{g}$ can be written in terms of the price process

$$\tilde{g}(x) = \mathbb{E}\, g\Big(T\big(S(x,t)\big)\Big).$$

Moreover, $g \circ T \in C^b(\tilde{E})$ so $\tilde{g} \in C^b(\tilde{E})$, where $\tilde{E} = [0,\infty[^d \setminus \{0\}$ and $D \subseteq \tilde{E}$.

To find explicit solutions to the equation (8) it is convenient to rewrite it in a differential form as a suitable quasi-variational inequality (QVI). We change the state space in order to have a non empty interior. We remove the last coordinate and take

$$\begin{gathered} D = \{w_1, \ldots, w_{d-1} \in [0,1]^{d-1} : \quad \sum_{i=1}^{d-1} w_i \leq 1\}, \\ c(u,w) = K + k\sum_{i=1}^{d-1} |u^i - w^i| + k|\sum_{i=1}^{d-1} (u^i - w^i)| \quad \text{for } K > 0, k \geq 0. \end{gathered}$$

We denote by $\tilde{\mathcal{A}}$ the generator for the proportion process in the new state space and make obvious modifications to the function f. We introduce a switching functional

$$Mv(w) = \inf_{u \in D} \big(v(u) + c(w,u) \big) \tag{9}$$

for any function $v : D \to \mathbb{R}$. The QVI related to the cost functional (7) takes the form

$$\min\big(\tilde{\mathcal{A}}v(w) - \beta v(w) + f(w),\quad Mv(w) - v(w)\big) = 0,\quad w \in D. \tag{10}$$

Let the price process be two dimensional with the second coordinate being always 1 and the first satisfying the Ito equation

$$\begin{aligned} dS(s,t) &= H\big(S(s,t-)\big)d\zeta(t),\\ S(s,0) &= s,\quad s\in\mathbb{R},\quad s>0. \end{aligned} \tag{11}$$

Here $\zeta(t)$ is a Levy process with the Fourier transform

$$\begin{aligned} &\mathbb{E}\,\exp\big(-is\zeta(t)\big) = \exp\big(-t\psi(s)\big),\\ &\psi(s) = \frac{1}{2}\sigma^2 s^2 - i\mu s - \int_{\mathbb{R}}\Big(e^{isy} - 1 - 1_{|y|\le 1} isy\Big)\nu(dy), \end{aligned}$$

where $\sigma \in \mathbb{R}_+,\ \mu\in\mathbb{R}$ and ν is a σ-finite measure satisfying

$$\int_{\mathbb{R}} (1\wedge y^2)\nu(dy) < \infty.$$

We assume that H is chosen in such a way that (11) has a unique weak solution for any initial condition $s \ge 0$. It can be easily verified that $S(s,t)$ is a multiplicative process (cf. (4)) only if H is a linear function. Therefore, without any loss of generality we assume that $H(x) = x$.

PROPOSITION 3.3. The generator $\tilde{\mathcal{A}}$ for the proportion process for the price process (11) has the form:

$$\begin{aligned} \tilde{\mathcal{A}}u(w) = {}& \frac{1}{2}\sigma^2 w\Big(u''(w)(1-w)^3 - 2u'(w)(1-w)^2\Big)\\ &+ \mu w u'(w)(1-w)\\ &+ \int_{\mathbb{R}}\bigg(u\Big(\frac{w+wy}{1+wy}\Big) - u(w) - 1_{|y|\le 1} wyu'(w)(1-w)\bigg)\nu(dy), \qquad u \in]0,1[,\\ \tilde{\mathcal{A}}u(0) = {}& \tilde{\mathcal{A}}u(1) = 0. \end{aligned}$$

for $u \in C^2(\mathbb{R})$.

Proof. Following Bichteler [2], we write the generator $\mathcal{A}$ for $S(s,t)$. Let $u \in C^2(0,1)$.

$$\mathcal{A}u(s) = \frac{1}{2}\sigma^2 s u''(s) + \mu s u'(s) + \int_{\mathbb{R}}\Big(u\big(s+sy\big) - u(s) - 1_{|y|\le 1} syu'(s)\Big)\nu(dy).$$

Let

$$T(s) = \frac{s}{s+1}.$$

The proportion process is obtained as $T(S(s,t))$. (We recall that the price of the second instrument is equal 1.) We apply theorem 3.1 and observe that

$$\begin{aligned} \frac{d}{ds}u\left(\frac{s}{s+1}\right) &= u'\left(\frac{s}{s+1}\right)\frac{1}{(s+1)^2},\\ \frac{d^2}{ds^2}u\left(\frac{s}{s+1}\right) &= u''\left(\frac{s}{s+1}\right)\frac{1}{(s+1)^4} - 2u'\left(\frac{s}{s+1}\right)\frac{1}{(s+1)^3}. \end{aligned}$$

Moreover, $T^{-1}(w) = \frac{w}{1-w}$, $s+1 = \frac{1}{1-w}$, which implies our result for $w \in (0,1)$.

We extend the generator to the points $0, 1$ in an obvious way. These points are stable for the process i.e. the process cannot move away from them, so $\tilde{\mathcal{A}}u(0) = \tilde{\mathcal{A}}u(1) = 0$. ■

On this stage we can write a QVI for the problem of optimal asset allocation:

$$\begin{aligned} &\min\left(\tilde{\mathcal{A}}v(w) - \beta v(w) + f(w), Mv(w) - v(w)\right) = 0, w \in]0,1[, \\ &\min\big(f(w) - \beta v(w), Mv(w) - v(w)\big) = 0, \quad w = 0, 1, \end{aligned} \tag{12}$$

where

$$Mv(w) = K + \inf_{u\in[0,1]} \big(k|u-w| + v(u)\big), \quad K > 0, \quad k \geq 0.$$

Hence, we conclude that the optimal strategy is described by an impulse region $\{Mv - v = 0\}$.

4. Multiplicative Poissonian prices

For further considerations we restrict ourselves to the case where prices are driven by a Poisson process. We specify

$$\zeta(t) = N(t) - \gamma t,$$

where $N(t)$ is a Poisson process with intensity λ and $\gamma \in \mathbb{R}$. The characteristics (μ, σ, ν) of this Levy process is the following: $\mu = \lambda - \gamma$, $\sigma = 0$, $\nu(\{1\}) = \nu(\mathbb{R}) = \lambda$. By proposition 3.3 the generator for the proportion process is given by

$$\tilde{\mathcal{A}}u(w) = \lambda\left(u\left(\frac{2w}{w+1}\right) - u(w)\right) - \gamma u'(w)w(1-w).$$

We write a QVI for the problem of optimal asset allocation:

$$\begin{aligned} &\min\left(\lambda\left(v\left(\frac{2w}{w+1}\right) - v(w)\right) - \gamma v'(w)w(1-w) - \beta v(w) + f(w), Mv(w) - v(w)\right) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad w \in]0,1[, \\ &\min\big(f(w) - \beta v(w), Mv(w) - v(w)\big) = 0, \quad w = 0, 1, \end{aligned} \tag{13}$$

where

$$Mv(w) = K + \inf_{u\in[0,1]} \big(k|u-w| + v(u)\big), \quad K > 0, \quad k \geq 0.$$

Moreover, the optimal strategy is described by an impulse region $\{Mv - v = 0\}$. However, we have to prove that the QVI (13) has an appropriately smooth bounded solution and that this solution satisfies the functional equation

$$v(w) = \inf_{\tau} \mathbb{E}\left(\int_0^\tau e^{-\beta s} f\big(w(w,s)\big)ds + e^{-\beta\tau} Mv\big(w(w,\tau)\big)\right). \tag{14}$$

The function $v(w)$ defines an optimal strategy as stated in theorem 3.2.

The following theorem could be deduced from general results of Bensoussan and Lions [1] chapter 3, although non-degenerate diffusion term is required there. We therefore present here a direct proof.

THEOREM 4.1. Assume that $\gamma \neq 0$. Let $v(w)$ be a bounded continuous function defined on $[0,1]$, piecewise C^1 and with finite left- and right-hand derivatives. If $v(w)$ satisfies (13) in $0,1$ and almost everywhere in $]0,1[$, then $v(w)$ solves (14).

Proof. We shall use theorem 5.2 in appendix. First observe that $\mathcal{D}(\tilde{\mathcal{A}}) \supset C^1$. Moreover, there exists a sequence v_n of C^1 functions converging to v in sup-norm such that $v_n = v$, $v_n' = v'$ everywhere but intervals of measure converging to 0. We introduce

$$h(w) = \lambda\left(v\left(\frac{2w}{w+1}\right) - v(w)\right) - \gamma v'(w)w(1-w),$$

for $w \in [0,1]$ such that $v'(w)$ is defined and continuous. The process $w(t,w)$ has a nonzero drift, so $P(t,w,\{v_n \neq v, v_n' \neq v'\}) \to 0$ as $n \to \infty$ for any $t > 0$ which proves condition iii) of theorem 5.2. Moreover, similar argument shows that every set of Lebesgue measure zero in $]0,1[$ is of $\tilde{\mathcal{A}}$-measure zero, so iv) is satisfied. Therefore

$$Y(w,t) = \int_0^t e^{-\beta s} f\big(w(w,s)\big)ds + e^{-\beta\tau} v\big(w(w,t)\big)$$

is a submartingale, so $\mathbb{E}\, Y(w,\sigma) \geq \mathbb{E}\, Y(w,0) = v(w)$ for any stopping time σ (f is bounded). The process

$$Z(w,t) = \int_0^t e^{-\beta s} f\big(w(w,s)\big)ds + e^{-\beta\tau} Mv\big(w(w,t)\big)$$

satisfies $Z(w,t) \geq Y(w,t)$ since $Mv(w) - v(w) \geq 0$. This leads to the conclusion that $\mathbb{E}\, Z(w,\sigma) \geq v(w)$ for any stopping time σ. To finish the proof it suffices to show that there exists an optimal stopping time $\tau^*(w)$ such that $\mathbb{E}\, Z(w,\tau^*(w)) = v(w)$. It is true for $\tau^*(w) = \inf\{t \geq 0 : Mv\big(w(w,t)\big) = v\big(w(w,t)\big)\}$. ∎

COROLLARY 4.2. Previous considerations imply that $v(w)$ is a unique bounded continuous solution to (14) and it is the value function for the problem of minimizing (7). Theorem 3.2 shows how to construct the optimal strategy.

¿From now we assume, without any loss of generality (see the lemma 2.3), that $f(w)$ is a positive function. We derive two results.

LEMMA 4.3. If f in non-decreasing on $[0,1]$ then v is non-decreasing. If f in non-increasing on $[0,1]$ then v is non-increasing.

Proof. We sketch the proof of the first fact. The proof of the second one is analogous. Observe that if $x \geq x'$ the proportion process satisfies $w(x,t) \geq w(x',t)$. Denote by v_0 the potential

$$v_0(x) = \mathbb{E}\int_0^\infty e^{-\beta s} f\big(w(x,s)\big)ds, \quad x \in [0,1].$$

Hence v_0 is a non-decreasing function. Let

$$v_n(x) = \inf_\tau \mathbb{E}\Big(\int_0^\tau e^{-\beta s} f\big(w(x,s)\big)ds + e^{-\beta\tau} Mv_{n-1}\big(w(x,\tau)\big)\Big), \quad x \in [0,1].$$

We can show by induction that v_n is a non-decreasing function. By theorem 2.2 v_n converges uniformly to the value function v, so v is non-decreasing. ∎

LEMMA 4.4. If the difference between minimum and maximum of f is smaller then βK the continuation region spans all the interval $[0, 1]$.

Proof. Let $\underline{f} = \min_{u\in[0,1]} f(u)$, $\overline{f} = \max_{u\in[0,1]} f(u)$. The value function v has trivial bounds $\underline{f} \leq \beta v(x) \leq \overline{f}$, $x \in [0, 1]$. Hence $Mv(x) > v(x)$ for all $x \in [0, 1]$, which implies that the optimal strategy prohibits any impulses. ■

Assume now that there are no proportional costs, i.e. $k = 0$. In this case all impulses aim at the same target point $u^* \in [0, 1]$ in which the function v attains its minimum. Hence, if f is non-decreasing impulses can only occur on some interval $[b_0, 1]$ and they aim at $w = 0$ (minimum of v). We know, by direct calculation, that $v(0) = f(0)/\beta$. The potential of f in 1 equals to $f(1)/\beta$. Hence, the impulse interval is nonempty if and only if $v(1) > v(0) + \beta K$. The same reasoning applies to the case of non-increasing f.

4.1. Recursive formulae

We derive a solution to (13) for a specific case of nonempty impulse region around 1 and absence of proportional transaction costs $k = 0$. We do not require monotonicity of f. However, we assume that $f \geq 0$, which is no restriction (see lemma 2.3).

We construct an iterative procedure to find the solution to the QVI (13). We set $v_0(w) = H$, $H \in \mathbb{R}$ for $w \in [b_0, 1]$. The function v_0 is undefined outside of the interval $[b_0, 1]$. A pair $H, b_0 \in \mathbb{R} \times [0, 1]$ is used as an index for the set of solutions.

To formulate the lemma we need to define a sequence

$$b_{n+1} = \frac{b_n}{2 - b_n}, \quad n = 0, 1, \ldots$$

and introduce the equation being a differential part of (13)

$$\lambda \left(v\left(\frac{2w}{w+1}\right) - v(w) \right) - \gamma v'(w) w(1-w) - \beta v(w) + f(w) = 0. \tag{15}$$

We note that the sequence b_n is strictly decreasing with the limit equal to 0.

LEMMA 4.5. Assume that v_n is defined on $[b_n, 1]$ and satisfies (15) for $w \in [b_n, b_0]$. We define v_{n+1} on $[b_{n+1}, 1]$ by the formula:

$$v_{n+1}(w) = \nu(w) \left(\frac{v_n(b_n)}{\nu(b_n)} - \int_w^{b_n} \frac{\lambda v_n\left(\frac{2u}{u+1}\right) + f(u)}{\gamma \nu(u) u(1-u)} du \right), \quad w \in [b_{n+1}, b_n[,$$
$$v_{n+1}(w) = v_n(w), \quad w \in [b_n, 1].$$

where $\xi = \frac{\lambda+\beta}{\gamma}$ and

$$\nu(w) = \left(\frac{1-w}{w}\right)^{\xi}.$$

Then v_{n+1} satisfies (15) for $w \in [b_{n+1}, b_0]$.

We can easily see that

$$v_{n+1}(w) = \Big(\frac{1-w}{w}\Big)^{\xi} \left(\frac{v_n(b_0)}{\Big(\frac{1-b_0}{b_0}\Big)^{\xi}} - \int_w^{b_0} \frac{\lambda v_n\left(\frac{2u}{u+1}\right) + f(u)}{\gamma\left(\frac{1-u}{u}\right)^{\xi} u(1-u)} du \right), \quad w \in [b_{n+1}, b_0[.$$

Proof of the lemma. Here we show the derivation of the above formulas. We solve

$$\lambda v\left(\frac{2w}{w+1}\right) + f(w) - \lambda v(w) - \gamma v'(w) w(1-w) - \beta v(w) = 0, \quad w \in [b_{n+1}, b_n]$$
$$v(w) = v_n(w), \quad w \in [b_n, 1].$$

First we sort out the homogeneous case

$$-(\lambda+\beta)v(w) - \gamma v'(w)w(1-w) = 0,$$

which we simplify to

$$-\xi v(w) = v'(w)w(1-w)$$

for $\xi = \frac{\lambda+\beta}{\gamma}$. We obtain the solution

$$v(w) = c\left(\frac{1-w}{w}\right)^{-\xi}.$$

By setting $c = c(w)$ and plugging into the generic equation we obtain

$$c'(w) = \frac{\lambda v\left(\frac{2w}{w+1}\right) + f(w)}{\gamma\left(\frac{1-w}{w}\right)^{\xi} w(1-w)}.$$

■

Remark. Function $\nu(w)$ is unbounded on $[0,1]$. It converges to ∞ as $w \to 0$ and to 0 as $w \to 1$.

We expect that in the majority of cases the optimal control is determined by the numbers $0 \le a < c < b \le 1$ and consists of making impulses to c when the proportion process exits from the interval $[a, b]$. Such strategies will be denoted by $\Pi_{a,b,c}$. Now we formulate the conditions under which this is really the case.

Let $v_{b_0,H}(w)$ be limit of v_n with initial condition $v_0(w) = H, \quad w \in [b_0, 1]$ in the sense that $v_{b_0,H}(w) = v_n(w), \quad w \in [b_n, 1]$. Define $b^*_{b_0,H} = \sup\{w < b_0 : v_{b_0,H}(w) = H\}$.

THEOREM 4.6. Assume that the following conditions hold:

1) $\inf_{w\in[b^*_{b_0,H},1]} v_{b_0,H}(w) = H - K$,
2) $\sup_{w\in[b^*_{b_0,H},1]} v_{b_0,H}(w) = H$,
3) $f(w) \ge \beta H, \quad w \in [b_0, 1]$,
4) $f(w) + \lambda v_{b_0,H}\big(\frac{2w}{1+w}\big) \ge (\lambda+\beta)H, \quad w \in \big[\frac{b^*_{b_0,H}}{2-b^*_{b_0,H}}, b^*_{b_0,H}\big]$,

5) $f(w) \geq \beta H, \quad w \in \left[0, \frac{b^*_{b_0,H}}{2-b^*_{b_0,H}}\right]$

Then

$$v(w) = 1_{\{w \geq b^*_{b_0,H}\}} v_{b_0,H}(w) + 1_{\{w < b^*_{b_0,H}\}} H$$

is a solution to (13). Moreover, $\Pi_{b^*_{b_0,H},b_0,c}$, where

$$c = \text{arginf}_{w \in [b^*_{b_0,H},1]} v_{b_0,H}(w),$$

is the optimal strategy.

Proof. The conditions grant that v is piecewise C^1 with finite left- and right-hand derivatives and satisfies (13) in all points but $(b_n)_{n\in\mathbb{N}}$. By theorem 4.1 v is a value function for the control problem of minimizing (7). ■

Remark. If $f(b_0) > \beta H$ then $v'(b_0-) < 0$ and the condition 2) is satisfied.

If we know a priori that the function v attains its minimum in the first interval $[b_1, b_0]$ then the target point u^* can be characterized by the following transcendental equation

$$\frac{1}{\nu(u^*)} \frac{\lambda H + f(u^*)}{\lambda + \beta} = \frac{H}{\nu(b_0)} - \int_{u^*}^{b_0} \frac{\lambda H + f(u)}{\gamma \nu(u) u(1-u)} du. \tag{16}$$

To obtain above equation we observe that $v'(u^*) = 0$. From (15) we have

$$\lambda H - (\beta + \lambda) v(u^*) + f(u^*) = 0.$$

Hence

$$v(u^*) = \frac{\lambda H + f(u^*)}{\lambda + \beta}$$

and we take the formula for $v(u^*)$ from lemma 4.5.

If we assume that

$$\frac{1}{\nu(u)} \frac{\lambda H + f(u)}{\lambda + \beta}$$

is decreasing in u then the equation (16) has at most one solution. We recall that $f \geq 0$ which makes the expression under integral non-negative.

Similar, but more complicated equations can be obtained if $[b_1, b_0]$ is replaced by $[b_{k+1}, b_k]$.

4.2. Impulse regions

In this section we will show that for any reasonable region J in $[0, 1]$ there exist an evaluation function f such that the optimal control impulse region is exactly J.

We introduce a family of functions:

- $\left(g^{(1)}_b\right)_{b\in[0,1)} \subset C^1([0,1]), \quad g^{(1)}_b\Big|_{(b,1]} \in [a,1), \quad g^{(1)}_b(b) = 1, \quad \frac{d}{dw} g^{(1)}_b(b) = 0,$
- $\left(g^{(2)}_{l,r,\alpha}\right)_{0\leq l<r\leq 1,\alpha\in\mathbb{R}_+} \subset C^1([0,1]), \quad g^{(2)}_{l,r,\alpha}\Big|_{(l,r)} \in [a,1), \quad g^{(2)}_{l,r,\alpha}(l) = g^{(2)}_{l,r,\alpha}(r) = 1,$
  $\frac{d}{dw} g^{(2)}_{l,r,\alpha}(l) = 0, \quad \frac{d}{dw} g^{(2)}_{l,r,\alpha}(r) = \alpha,$

- $\left(g^{(3)}_{r,\alpha}\right)_{0\leq r\leq 1,\alpha\in\mathbb{R}_+} \subset C^1([0,1]),\quad g^{(3)}_{r,\alpha}\Big|_{[0,r)} \in [a,1),\quad g^{(3)}_{r,\alpha}(r) = 1,\quad \frac{d}{dw} g^{(3)}_{r,\alpha}(r) = \alpha,$

for some $a \in (0,1)$.

Now, we proceed with the construction of the function f starting from the right end. We assume that on the impulse region the value function v is equal to 1. We will grant that the value function is bounded by 1. Hence, by setting the impulse cost $K = 1 - \min v$, we obtain the solution to the QVI. The following lemma shows how to extend the function f so as to keep to the required impulse region. Before, we introduce a notation: $[a,b] \prec c$ if $a < c$ and $b < c$. Analogously, $[a,b] \prec [c,d]$ if $[a,b] \prec c$ and $[a,b] \prec d$.

LEMMA 4.7. Assume that the value function v and f are defined on $[b,1]$ and $v\big|_{[b,\tilde{b}]} = 1$, for some $\tilde{b} > b$. For any interval $0 \preceq [l,r] \prec b$ there exists an extension of f to $[l,1]$ such that v is the solution to the QVI on $[l,1]$ with $[l,r]$ being a part of the impulse region and $]r,b[$ – a part of a continuation region.

$$v\big|_{(r,b)} < 1,\quad v\big|_{[l,r]} = 1,\quad f(w) - (\beta+\lambda) + \lambda v\big(\frac{2w}{1+w}\big) \geq 0,\quad w \in [l,r].$$

Proof. We set

$$\alpha = \frac{\lambda + \beta - \lambda v\left(\frac{2b}{1+b}\right) - f(b)}{\gamma b(1-b)}.$$

We extend f on $[r,b)$ in such a way that $v\big|_{[r,b)} = g^{(2)}_{r,b,\alpha}\big|_{[r,b)}$ i.e.

$$f(w) = \gamma v'(w) w(1-w) + \beta v(w) + \lambda\Big(v(w) - v\Big(\frac{2w}{1+w}\Big)\Big),\quad w \in [r,b).$$

The function f is continuous on $[r,b)$. Moreover, the condition on the derivative $v'(b) = \alpha$ implies continuity in b.

To define f on $[l,r)$ we have to assure that

$$f(w) - (\beta+\lambda) + \lambda v\big(\frac{2w}{1+w}\big) \geq 0,\quad w \in [l,r]. \tag{17}$$

We check that

$$f(r) - (\beta+\lambda) + \lambda v\big(\frac{2r}{1+r}\big) = 0,$$

since $v'(r) = 0$. We extend f to $[l,r)$ in any way that grants the inequality (17) and continuity of f.

■

THEOREM 4.8. Let $(I_n)_{n=1,\ldots,N}$ be a family of closed intervals in $[0,1]$ with non-void interior satisfying $I_{n+1} \prec I_n$. There exists a function f and the impulse cost $K > 0$ such that $\bigcup_{n=1,\ldots,N} I_n$ is the impulse region of the optimal strategy.

Proof. If $I_1 = [b,1]$ we set $v\big|_{[b,1]=1},\quad f\big|_{[b,1]} = \beta$. Otherwise, $I_1 = [l,r] \prec 1$. We put

$$v\big|_{[r,1]} = g^{(1)}_r,\quad v\big|_{[l,r]} = 1,$$

taking appropriate f as in the above lemma. For next intervals, excluding the last, we apply the lemma. Let $I_N = [l, r]$ be the last interval. If $l = 0$ then we apply the lemma. Otherwise, we proceed as follows. We take

$$v\big|_{[0,l)} = g^{(3)}_{l,\alpha}, \quad \alpha = \frac{\lambda + \beta - \lambda v\Big(\frac{2l}{1+l}\Big) - f(l)}{\gamma l(1-l)}.$$

We define f appropriately, as in the lemma. For v to be the solution to the QVI, we have to specify the impulse cost K. We put $K = 1 - \min v$. Now, we observe that f is a continuous function on $[0, 1]$, $v \in [a, 1]$ and v is a solution to the QVI. ■

The above theorem can be generalized to the case of infinite number of disjoint intervals with nonempty interior converging to 0.

5. Appendix A

We state and prove here an auxiliary result needed in the proof of the theorem 4.1.

Let $X(t, x)$ be a Markov process on the space $(E, \mathcal{E})$ with respect to the filtration $(\mathcal{F}_t)$ and a semigroup (P_t). By $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ we denote its generator. We formulate and prove a general result giving the probabilistic interpretation of the solution (in some sense, specified later) to the equation

$$\mathcal{A}v(w) - \beta v(w) + f(w) = 0(\geq 0, \leq 0).$$

DEFINITION 5.1. A set $B \in \mathcal{E}$ is of **null $\mathcal{A}$-measure** if

$$\forall x \in E \; \forall t > 0 \; P_t 1_B(x) = 0.$$

THEOREM 5.2. Let $v : E \to \mathbb{R}$ be a continuous function such that there exists a sequence of functions $v_n \in \mathcal{D}(\mathcal{A})$ and a function h satisfying

i) $v_n \to v$ in sup-norm,

ii) h is defined $\mathcal{A}$-a.s.,

iii)
$$\mathbb{E}\int_0^t e^{-\beta s}\mathcal{A}v_n\big(X(s,x)\big)ds \to \mathbb{E}\int_0^t e^{-\beta s}h\big(X(s,x)\big)ds,$$

iv) $h(x) - \beta v(x) + f(x) \geq 0$ $\mathcal{A}$-a.s., for a continuous function $f : E \to \mathbb{R}$.

Then
$$Y(t,x) := e^{-\beta t}v\big(X(t,x)\big) - v(x) + \int_0^t e^{-\beta s}f\big(X(x,s)\big)ds$$
is a submartingale (if it is well-defined and integrable).

For the proof of the theorem we will need the well-known lemma

LEMMA 5.3. Let $Z(t), t \geq 0$ be an adapted and measurable process in $\mathbb{R}^d$. For any Borel function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$

$$\mathbb{E}\left(\int_a^b f(u, Z(u))du \Big| \mathcal{F}_a\right) = \int_a^b \mathbb{E}\left(f(u, Z(u))\big|\mathcal{F}_a\right)du$$

if left- or right-hand side exists.

Proof of the theorem. The proof consists of two parts. First we show that $\mathbb{E}\, Y(t,x) > 0$. Then we exploit Markov property of $X(t,x)$ to show that it is a submartingale. We have

$$\frac{d}{dt} P_t v_n = P_t \mathcal{A} v_n,$$

since v_n is in the domain of $\mathcal{A}$. Hence

$$\frac{d}{dt} e^{-\beta t} P_t v_n = e^{-\beta t} P_t \mathcal{A} v_n - \beta e^{-\beta t} P_t v_n.$$

We integrate above equation and we obtain

$$e^{-\beta t} P_t v_n - v_n = \int_0^t e^{-\beta s} P_s \mathcal{A} v_n ds - \beta \int_0^t e^{-\beta s} P_s v_n ds.$$

We change the order of integration and get

$$\mathbb{E}\left(e^{-\beta t} v_n\big(X(t,x)\big) - v_n(x) - \int_0^t e^{-\beta s} \Big(\mathcal{A} v_n\big(X(s,x)\big) - \beta v_n\big(X(s,x)\big)\Big) ds \right) = 0.$$

We let $n \to \infty$ and by i), iii) we get

$$\mathbb{E}\left(e^{-\beta t} v\big(X(t,x)\big) - v(x) - \int_0^t e^{-\beta s} \Big(h\big(X(s,x)\big) - \beta v\big(X(s,x)\big)\Big) ds \right) = 0.$$

¿From condition iv) we have $-(h - \beta v) \leq f$, so

$$\mathbb{E}\, Y(t,x) = \mathbb{E}\left(e^{-\beta t} v\big(X(t,x)\big) - v(x) + \int_0^t e^{-\beta s} f\big(X(x,s)\big) ds \right) \geq 0,$$

that can be written equivalently

$$e^{-\beta t} P_t v(x) - v(x) + \int_0^t e^{-\beta u} P_u f(x) du \geq 0. \tag{18}$$

We shall show that $Y(t,x)$ is a submartingale. We take $0 \leq s < t$ and write

$$\begin{aligned}
&\mathbb{E}\left(Y(x,t) - Y(x,s) \,\middle|\, \mathcal{F}_s \right) = \\
&= \mathbb{E}\left(e^{-\beta t} v\big(X(t,x)\big) - e^{-\beta s} v\big(X(s,x)\big) + \int_s^t e^{-\beta u} f\big(X(u,x)\big) du | \mathcal{F}_s \right) \\
&= \mathbb{E}\left(e^{-\beta t} v\big(X(t,x)\big) \,\middle|\, \mathcal{F}_s \right) - \mathbb{E}\left(e^{-\beta s} v\big(X(s,x)\big) \,\middle|\, \mathcal{F}_s \right) + \mathbb{E}\left(\int_s^t e^{-\beta u} f\big(X(u,x)\big) du \,\middle|\, \mathcal{F}_s \right).
\end{aligned}$$

¿From Markov property of $X(t,x)$ we get

$$\mathbb{E}\left(e^{-\beta t} v\big(X(t,x)\big) \,\middle|\, \mathcal{F}_s \right) = e^{-\beta t} P_{t-s} v\big(X(s,x)\big).$$

Lemma 5.3 implies

$$\mathbb{E}\left(\int_s^t e^{-\beta u} f\big(X(u,x)\big) du \,\middle|\, \mathcal{F}_s \right) = \int_s^t e^{-\beta u} P_{u-s} f\big(X(s,x)\big) du.$$

Combining the above results leads to

$$\mathbb{E}\left(Y(x,t) - Y(x,s)\,\middle|\,\mathcal{F}_s\right)$$
$$= e^{-\beta s}\left(e^{-\beta(t-s)}P_{t-s}v\big(X(s,x)\big) - v\big(X(s,x)\big) + \int_s^t e^{-\beta u}P_{u-s}f\big(X(s,x)\big)du\right) \geq 0$$

from (18). ■

COROLLARY 5.4. Under the assumptions of the above theorem, if $h(x) - \beta v(x) + f(x) \leq 0$ $(= 0)$ $\mathcal{A}$-a.s. then $Y(t,x)$ is a supermartingale (martingale).

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