Wavelet moment method for Cauchy problem for the Helmholtz equation

Preliminary report

Teresa Regińska, Andrzej Wakulicz

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- *Institute of Mathematics, Polish Academy of Sciences,*
e-mail: reginska@impan.gov.pl
e-mail: wakulicz@impan.gov.pl

Abstract

The paper is concerned with the problem of reconstruction of acoustic or electromagnetic field from unexact data given on an open part of the boundary of a given domain. A regularization concept is presented for the ill-posed moment problem equivalent to a Cauchy problem for the Helmholtz equation. A method of regularization by projection with application of Meyer wavelet subspaces is introduced and analyzed. The derived formula, describing the projection level in terms of the error bound of unexact Cauchy data, allows to prove convergence and stability of the method.

1 Introduction

Let Ω be a simply connected domain in R^d, $d = 2, 3$ with a sufficiently regular boundary $\partial\Omega$ and, moreover, let $\Gamma \subset \partial\Omega$ be an open and connected part of the boundary. We consider the problem of reconstruction of acoustic or electromagnetic field from unexact data given on Γ. Let u denote a certain component of the considered field. Let us assume further that the field is harmonic with the constant wave number k. In this case the scalar function u satisfies in Ω the Helmholtz equation

$$Lu := \Delta u + k^2 u = 0, \text{ on } \Omega. \tag{1.1}$$

With respect to applications we have some freedom in choice of domain Ω: namely, only the part Γ of the boundary $\partial\Omega$ is given a priori (indicated by measurement possibilities), and in particular we may assume the same regularity of $\partial\Omega$ as of Γ.

Direct problems connected with this equation are typically defined by Dirichlet or Neumann conditions on $\partial\Omega$, or by Dirichlet conditions on a one part of boundary (i.e. Γ) and Neumann conditions on the second one. The inverse problem considered in this paper consists in solving the equation (1.1) under the both Dirichlet and Neumann conditions posed on the same part Γ of the boundary $\partial\Omega$. That means that we deal with the Cauchy problem for the Helmholts equation

$$\begin{cases} Lu := \Delta u + k^2 u = 0, & \text{on } \Omega \\ u = f & \text{on } \Gamma, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \Gamma, \end{cases} \tag{1.2}$$

where ν is the outer unit normal to $\partial\Omega$. In all parts of the paper we assume, that k^2 is not an eigenvalue of the Neumann problem for $-\Delta$, i.e. that $v \equiv 0$ is the unique solution to the following homogeneous boundary-value problem $\Delta v + k^2 v = 0$ in Ω, $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$. We assume that $f \in H^1(\Gamma)$ and $g \in L^2(\Gamma)$ are such that there exists the unique solution $u \in H^{3/2}(\Omega)$. It is known that the Cauchy problem for elliptic equations is ill-posed, which means that the solutions do not depend continuously on Cauchy data, see e.g. [10], [7], [9]. This implies serious numerical difficulties in solving of these problems, especially in the case of perturbed data. However, just this case is important from the point of view of real applications for acoustic and electromagnetic fields (cf. [12], [9], [6], [1], [14]) where the exact Cauchy data are approximated by their measurements.

For a stable solving of ill-posed problems, regularization techniques are required (cf. [8], [15]). Numerical analysis of the Cauchy problem for the Laplace equation is a topic of several papers where different regularization methods were proposed ([2], [13], [3]). Unfortunately, their application to Helmholtz equation requires some modifications and additional analysis because of essential differences between these two problems.

In this paper is developed the idea of a numerical method based on a transformation of the Cauchy problem to a generalized moment problem: find $\varphi \in L^2(\partial\Omega \setminus \Gamma)$ such that

$$\int_{\partial\Omega\setminus\Gamma} \varphi v d\sigma = \mu(v) \qquad \forall v \in V(\Omega), \tag{1.3}$$

where $V(\Omega)$ is a certain subspace of $L^2(\Omega)$ and μ a linear functional on $V(\Omega)$

which will be defined later. This idea was proposed by J. Cheng et al. in [3] for the Cauchy problem for Laplace equation.

The paper is organized as follows. In section 2 the equivalence between the Cauchy problem (1.2) and a moment problem on the boundary $\partial\Omega \setminus \Gamma$ is proved according to idea in [3]. The rest of the paper is devoted to a regularization method for solving the obtained moment problem in the particular case of two-dimensional domain Ω. In Section 3 a characterization of a dense subspace of the space is given. In conclusion, in Section 4, Meyer wavelet projections are choosen for convergent approximation of the solution in the case of the exact data. In Section 5 stability of the method is considered with respect to perturbations of the boundary value functions. Finally, in Section 6 the regularization property of defined wavelet-projection method for the moment problem is established and, as consequence, a stable approximation of the Cauchy problem (1.2) is obtained.

2 Moment problem, general case

Let v be an arbitrary $H^1(\Omega)$ weak solution to the equations

$$\begin{cases} Lv = 0, & \text{on } \Omega \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases} \tag{2.1}$$

Applying Green's formula to the solution u of (1.2) and to v

$$\int_\Omega [vLu - uLv]dx = \int_{\partial\Omega} \left[v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right] d\sigma, \tag{2.2}$$

where σ is the Lebesgue measure on $\partial\Omega$, we get

$$\int_{\partial\Omega\setminus\Gamma} v\frac{\partial u}{\partial \nu}d\sigma + \int_\Gamma vgd\sigma - \int_\Gamma f\frac{\partial v}{\partial \nu}d\sigma = 0. \tag{2.3}$$

Let us define a test space $V(\Omega)$ as follows

$$V(\Omega) := \left\{ v \in H^{3/2}(\Omega) : \quad Lv = 0 \text{ on } \Omega \text{ and } \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \setminus \Gamma \right\}. \tag{2.4}$$

Corollary 2.1 *If there exists the solution of (1.2) such that*

$$\varphi := \left\{ \frac{\partial u}{\partial \nu} \right\}_{\partial\Omega\setminus\Gamma} \in L^2(\partial\Omega \setminus \Gamma) \tag{2.5}$$

then φ is the solution to the following moment problem

$$\int_{\partial\Omega\setminus\Gamma} \varphi v d\sigma = \int_{\Gamma} \left[f \frac{\partial v}{\partial \nu} - g v \right] d\sigma \qquad \forall v \in V(\Omega). \tag{2.6}$$

Going into lines of Cheng et al. reasoning [3] we prove

Theorem 2.2 *Let $g \in L^2(\Gamma)$ and $f \in H^{1/2}(\Gamma)$ and let k^2 does not be an eigenvalue of the Neumann problem for $-\Delta$. If $\varphi \in L^2(\partial\Omega \setminus \Gamma)$ is a solution to the moment problem (2.6) and $\partial\Omega \in C^{1+\epsilon}$ [1], then there exists a solution u to the Cauchy problem (1.2) such that $\frac{\partial u}{\partial \nu} \in L^2(\partial\Omega \setminus \Gamma)$ and $\frac{\partial u}{\partial \nu} = \varphi$.*

PROOF: Let us consider the following Neumann problem

$$\begin{cases} \Delta\alpha + k^2\alpha = 0, & \text{on } \Omega \\ \frac{\partial \alpha}{\partial \nu} = \varphi & \text{on } \partial\Omega \setminus \Gamma, \\ \frac{\partial \alpha}{\partial \nu} = g & \text{on } \Gamma. \end{cases} \tag{2.7}$$

It is known (cf. [5], § 3 of Chapter XI) that for $\partial\Omega \in C^{1+\epsilon}$ the Neumann problem (2.7) for $\varphi \in H^{-\frac{1}{2}}(\partial\Omega \setminus \Gamma)$, $g \in H^{-\frac{1}{2}}(\Gamma)$ admits a unique solution in $H^1(\Omega)$ under Theorem 2.2 assumptions. Classical regularity arguments imply that there exists a unique α in $H^{\frac{3}{2}}(\Omega)$ for $\varphi \in L^2(\partial\Omega \setminus \Gamma)$, $g \in L^2(\Gamma)$ [2].

We are going to prove that

$$\int_{\Gamma} (\alpha - f)^2 d\sigma = 0. \tag{2.8}$$

From (2.6) it follows that for any $v \in V(\Omega)$

$$\int_{\partial\Omega\setminus\Gamma} \frac{\partial \alpha}{\partial \nu} v d\sigma = \int_{\Gamma} \left[f \frac{\partial v}{\partial \nu} - v \frac{\partial \alpha}{\partial \nu} \right] d\sigma,$$

i.e.

$$\int_{\partial\Omega} v \frac{\partial \alpha}{\partial \nu} d\sigma = \int_{\Gamma} f \frac{\partial v}{\partial \nu} d\sigma. \tag{2.9}$$

[1] We say that a regular open set Ω has a boundary $\partial\Omega$ of class $C^{1+\epsilon}$ if in the neighborhood of every point $x \in \partial\Omega$ there exists a normal parametric representation σ and an increasing function $\epsilon \in C(R_+, R_+)$ with $\int_{R_+} \epsilon(r) \frac{dr}{r} < \infty$ such that for any pair $(\widetilde{x}, \widehat{x})$ in the neighborhood of x, where the parametric representation is given,

$$|grad\sigma(\widetilde{x}) - grad\sigma(\widehat{x})| \leq \epsilon(|\widetilde{x} - \widehat{x}|).$$

One can see that Lyapunov surfaces (curves) are in $C^{1+\epsilon}$.

[2] A similar result can be obtained for $\partial\Omega$ beeing Lipschitz continuous under additional assumptions on the geometry of Ω, (cf. [11])

On the other hand, Green's formula gives

$$0 = \int_\Omega [vL\alpha - \alpha L v]dx = \int_{\partial\Omega} \left[v\frac{\partial \alpha}{\partial \nu} - \alpha \frac{\partial v}{\partial \nu} \right] d\sigma,$$

which implies

$$\int_{\partial\Omega} v\frac{\partial \alpha}{\partial \nu} d\sigma = \int_{\partial\Omega} \alpha \frac{\partial v}{\partial \nu} d\sigma.$$

the formula above and (2.9) yields

$$\int_\Gamma (\alpha - f)\frac{\partial v}{\partial \nu} d\sigma = 0. \tag{2.10}$$

Now, let us take a special element $\widetilde{v}$ of $V(\Omega)$ defined as follows

$$\begin{cases} \Delta\widetilde{v} + k^2\widetilde{v} = 0, & \text{in } \Omega \\ \frac{\partial \widetilde{v}}{\partial \nu} = 0 & \text{in } \partial\Omega \setminus \Gamma, \\ \frac{\partial \widetilde{v}}{\partial \nu} = \alpha - f & \text{in } \Gamma, \end{cases} \tag{2.11}$$

Since $(\alpha - f) \in L^2(\Gamma)$, we have by the same arguments as used for (2.7), that there exists a unique solution $\widetilde{v}$ to (2.11) belonging to $H^{3/2}(\Omega)$. Thus, by (2.10)

$$\int_\Gamma (\alpha - f)^2 d\sigma = 0,$$

i.e $\alpha = f$ almost everywhere on Γ. It means that α is a solution to the Cauchy problem (1.2). ∎

Due to Corollary 2.1 and Theorem 2.2 the problem (1.2) can be equivalently formulated as the following moment problem

$$\int_{\partial\Omega\setminus\Gamma} \varphi v d\sigma = \mu(v) \qquad \forall v \in V(\Omega). \tag{2.12}$$

where

$$\mu(v) = \int_\Gamma \left[f\frac{\partial v}{\partial \nu} - gv \right] d\sigma, \tag{2.13}$$

which has at most one solution. In the equation (2.12) we can replace the space V by any dense subset of V, for instance by a dense sequence.

3 Moment problem, for $\Omega \subset R^2$

Now, our consideration is restricted to the case $\Omega \subset R^2$. Let us introduce the subdomain $\Omega_1 \subset \Omega$ with the boundary given by Γ and the closed interval connecting the ends of Γ. The problem of reconstruction u in Ω_1 is now considered. Without lost of generality we may assume that

$$\partial\Omega_1 \setminus \Gamma = \{(x_1, x_2) : x_1 \in [0,1], x_2 = 0\} \tag{3.1}$$

and $\Gamma \subset R \times R^+$ is a sufficiently regular curve which connects the two points $(0,0)$ and $(1,0)$. Let

$$\gamma = max\{x_2 : (x_1, x_2) \in \Gamma\}. \tag{3.2}$$

Let us denote $\Omega_1' := R \times (0, \gamma)$ and introduce the space W

$$W := \left\{ w \in H^2(\Omega_1') : \quad Lw = 0 \text{ on } \Omega_1' \text{ and } \frac{\partial w}{\partial x_2}(x, 0) = 0 \text{ for } x \in R \right\}. \tag{3.3}$$

If $w \in W$, then $w|_{\Omega_1} \in V(\Omega_1)$.

Remark 3.1 *The set $W(\Omega_1) = \{w|_{\Omega_1} : w \in W\}$ is dense in $V(\Omega_1)$.*

Remark 3.2 *In the moment problem (2.12) the space $V(\Omega_1)$ can be replaced by $W(\Omega_1)$.*

We are going to define such a basis in the space W which will allow us to formulate a stable approximation method of solving the problem over consideration. For this reason we should describe elements of W more precisely. Let us introduce an auxiliary space U

$$U = \left\{ \beta \in L^2(R) : \widehat{\beta}(\xi)\xi^2 \cosh(\gamma\sqrt{\xi^2 - k^2}) \in L^2(R) \right\}, \tag{3.4}$$

where $\widehat{\beta}$ denotes the Fourier transform of β.

Proposition 3.3 *Function w belongs to W if and only if $\exists \beta \in U$ such that $\forall (x_1, x_2) \in \Omega_1'$*

$$w(x_1, x_2) = w_\beta(x_1, x_2) := \frac{1}{\sqrt{2\pi}} \int_R \widehat{\beta}(\xi) \cosh(x_2\sqrt{\xi^2 - k^2}) e^{i\xi x_1} d\xi. \tag{3.5}$$

Proof: Applying Fourier transform to $w \in W$ with respect to the variable $x_1 \in R$ we easy state that $\widehat{w}(\xi, x_2)$ is a solution to the following problem

$$\begin{cases} \frac{\partial^2 \widehat{w}}{\partial x_2^2}(\xi, x_2) = (\xi^2 - k^2)\widehat{w}(\xi, x_2) & \text{for } \xi \in R, \; x_2 \in (0,\gamma), \\ \frac{\partial \widehat{w}}{\partial x_2}(\xi, 0) = 0 & \text{for } \xi \in R. \end{cases} \tag{3.6}$$

The general solution of this problem has the form

$$\widehat{w}(\xi, x_2) = h(\xi)\cosh(x_2\sqrt{\xi^2 - k^2}).$$

If $w \in W$, then

$$\widehat{\frac{\partial^2 w}{\partial x_1^2}}(\cdot, x_2) = -\xi^2 \widehat{w}(\cdot, x_2) \in L^2(R),$$

and thus, the function h appearing in the formula above has to be such, that $\xi^2\widehat{w}(\xi, x_2)$belongs to $L^2(R)$ as a function of ξ. So, $h = \widehat{\beta}$ where $\beta \in U$ and w is the inverse Fourier transform of $\widehat{\beta}\cosh(x_2\sqrt{\xi^2 - k^2})$. Inversely, if $\widehat{w}$ is given by (3.5),where $h = \widehat{\beta}$ and $\beta \in U$ then $\widehat{w}$ satisfies the equations (3.6). Since $\xi^2\widehat{w}(\xi, x_2) \in L^2(R)$, $w(\cdot, x_2) \in H^2(R)$ and $Lw = 0$ on $R \times (0, \gamma)$, $\frac{\partial w}{\partial x_2}(x, 0) = 0$ for $x \in R$. This ends the proof. ∎

Remark 3.4 *If $w(x_1, x_2) = w_\beta(x_1, x_2)$ then $w(x_1, 0) = \beta(x_1)$ for almost all $x_1 \in R$.*

Remark 3.5 *The moment problem (2.12) can be now formulated as follows: find $\varphi \in L^2(0,1)$ such that*

$$\int_0^1 \varphi(x)\beta(x)dx = \eta(\beta), \qquad \forall \beta \in U. \tag{3.7}$$

where

$$\eta(\beta) := \mu(w_\beta) = \int_\Gamma \left[f\frac{\partial w_\beta}{\partial \nu} - g w_\beta \right] d\sigma \tag{3.8}$$

and w_β is given by (3.5).

4 Meyer wavelet projections

Following [4], formula (4.2.3), the Meyer wavelet ψ is a function from $C^\infty(R)$ defined by its Fourier transform as follows

$$\widehat{\psi}(\xi) = e^{i\frac{\xi}{2}} b(\xi),$$

where

$$b(\xi) = \begin{cases} \frac{1}{\sqrt{2\pi}} \sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi| - 1\right)\right], & \text{for } \frac{2\pi}{3} \le |\xi| \le \frac{4\pi}{3}, \\ \frac{1}{\sqrt{2\pi}} \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\xi| - 1\right)\right], & \text{for } \frac{4\pi}{3} \le |\xi| \le \frac{8\pi}{3}, \\ 0, & \text{otherwise} \end{cases}$$

and ν is a function satisfying

$$\nu(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x \ge 1, \end{cases}$$

with the additional property

$$\nu(x) + \nu(1 - x) = 1.$$

The corresponding scaling function ϕ is defined by

$$\widehat{\phi}(\xi) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & \text{for } |\xi| \le \frac{2\pi}{3}, \\ \frac{1}{\sqrt{2\pi}} \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi| - 1\right)\right], & \text{for } \frac{2\pi}{3} \le |\xi| \le \frac{4\pi}{3}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Let

$$\psi_{jk}(x) := 2^{\frac{j}{2}}\psi(2^j x - k), \qquad \phi_{jk}(x) := 2^{\frac{j}{2}}\phi(2^j x - k), \quad j, k \in Z, \quad j \ge 0.$$

The set of functions $\{\phi_{0k}, \psi_{jk}\}_{j \in Z^+, k \in Z}$ is the orthonormal basis of $L^2(R)$. To simplify notation, we denote the scaling functions $(\phi_{0,k})$ on the coarse level by $(\psi_{-1,k})$. Moreover,we use index λ concatenating the scale and space indices j and k. Let $\Lambda := \{\lambda = (j,k) : j \ge -1, k \in Z\}$, $\Lambda_J := \{\lambda = (j,k) : j < J, k \in Z\}$ and $|\lambda| = j$.

The set Λ_J determines the subspace $V_J \subset L^2(R)$

$$V_J = \overline{\text{span}\{\psi_\lambda\}}_{\lambda \in \Lambda_J}. \tag{4.2}$$

The corresponding orthogonal projection from $L^2(R)$ on V_J is denoted by P_J.

From the point of view of an application to our problem (3.7), the crucial property of Meyer wavelets is compact support of their Fourier transforms. By the definition

$$\text{supp}(\widehat{\psi}) = \{\xi; \frac{2}{3}\pi \le |\xi| \le \frac{8}{3}\pi\}. \tag{4.3}$$

Moreover, for $j \ge 0$

$$\widehat{\psi}_{jl} = 2^{-j/2} e^{-il\xi 2^{-j}} \widehat{\psi}(2^{-j}\xi). \tag{4.4}$$

Thus, for any $l \in Z$ and $j \geq 0$,

$$\mathrm{supp}(\widehat{\psi}_{jl}) = \{\xi; \frac{2}{3}\pi 2^j \leq |\xi| \leq \frac{8}{3}\pi 2^j\}. \tag{4.5}$$

For the scaling functions,

$$\mathrm{supp}(\widehat{\psi}_{-1,l}) = (-\frac{4}{3}\pi, \frac{4}{3}\pi), \qquad l \in Z. \tag{4.6}$$

Remark 4.1 *From (4.5) and (4.6) it follows, that* $\forall\lambda \in \Lambda$ $\psi_\lambda \in U$. *Moreover, the set* $\{\psi_\lambda\}_{\lambda\in\Lambda}$ *is dense in* U, *because it is dense in* $L^2(R) \supset U$.

Remark 4.2 *Due to Remark 4.1 the moment problem (3.7) can be now formulated as follows: find* $\varphi \in L^2(0,1)$ *such that*

$$\int_0^1 \varphi(x)\psi_\lambda(x)dx = \eta_\lambda, \qquad \forall\lambda \in \Lambda, \tag{4.7}$$

where

$$\eta_\lambda := \eta(\psi_\lambda), \tag{4.8}$$

and $\eta(\psi_\lambda)$ *is given by (3.8).*

Let

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x) & x \in (0,1), \\ 0 & x \in R \setminus (0,1) \end{cases} \tag{4.9}$$

where φ is the exact solution of (4.7). Since $\widetilde{\varphi} \in L^2(R)$ and $\{\psi_\lambda\}_{\lambda\in\Lambda}$ is the basis of $L^2(R)$, from (4.7) it follows that $\widetilde{\varphi}$ has the following wavelet representation

$$\widetilde{\varphi}(x) = \sum_{\lambda\in\Lambda} \eta_\lambda \psi_\lambda. \tag{4.10}$$

As approximate solution let us take the orthogonal projection of $\widetilde{\varphi}$ onto V_J, i.e.:

$$\varphi_J(x) = \sum_{\lambda\in\Lambda_J} \eta_\lambda \psi_\lambda(x). \tag{4.11}$$

It is clear that

$$\|\varphi_J - \varphi\|_{L^2(0,1)} \leq \|\varphi_J - \widetilde{\varphi}\|_{L^2(R)} \longrightarrow 0 \text{ as } J \longrightarrow \infty. \tag{4.12}$$

5 Perturbed data

Let f_δ, and g_δ be perturbed boundary value functions on Γ such that

$$\|f_\delta - f\|_{L^2(\Gamma)} + \|g_\delta - g\|_{L^2(\Gamma)} \le \delta. \tag{5.1}$$

Let

$$\eta_\lambda^\delta := \int_\Gamma \left[f_\delta \frac{\partial w_\lambda}{\partial \nu} - g_\delta w_\lambda \right] d\sigma, \tag{5.2}$$

where $w_\lambda := w_{\psi_\lambda}$, cf. (3.8). The approximate solution for the perturbed data let be defined as follows:

$$\varphi_J^\delta(x) = \sum_{\lambda \in \Lambda_J} \eta_\lambda^\delta \psi_\lambda(x). \tag{5.3}$$

We are going to show, that for any δ it is possible to choose a positive integer $J = J(\delta)$ in such a way that $\|\varphi - \varphi_J^\delta\|_{L^2(0,1)}$ tends to 0 as $\delta \to 0$.

The first step is to show that for fixed J the approximate solution is stable with respect to perturbations of f and g and to derive an error bound. We have

$$\|\varphi_J - \varphi_J^\delta\|^2_{L^2(R)} = \sum_{\lambda \in \Lambda_J} |\eta_\lambda - \eta_\lambda^\delta|^2. \tag{5.4}$$

Let us introduce two auxiliary functions

$$Q_1^\delta(\xi) := \int_\Gamma (f - f_\delta) \frac{\partial}{\partial \nu} \left(\cosh(x_2 \sqrt{\xi^2 - k^2}) e^{i\xi x_1} \right) d\sigma, \tag{5.5}$$

$$Q_2^\delta(\xi) := \int_\Gamma (g - g_\delta) \cosh(x_2 \sqrt{\xi^2 - k^2}) e^{i\xi x_1} d\sigma. \tag{5.6}$$

Lemma 5.1

$$\|\varphi_J - \varphi_J^\delta\|^2_{L^2(R)} \le \sum_{j=-\infty}^{J-1} 2^j \sup_{|\xi| \in (\frac{2}{3}\pi, \frac{8}{3}\pi)} \left| Q_1^\delta(2^j \xi) - Q_2^\delta(2^j \xi) \right|^2.$$

PROOF: For the proof the projection P_J onto V_J in terms of the wavelet basis $\{\psi_{j,k}\}_{j \in (-\infty, J-1), k \in Z}$is used. Applying definitions (3.8) and (5.2) of η_λ and η_λ^δ, and the formula (3.5) we can write (5.4) in the terms of functions Q_i^δ, $i = 1, 2$:

$$\|\varphi_J - \varphi_J^\delta\|^2_{L^2(R)} = \sum_{-\infty}^{J-1} \sum_{l \in Z} \left| \frac{1}{2\pi} \int_R \widehat{\psi_{j,l}}(\xi) \left(Q_1^\delta(\xi) - Q_2^\delta(\xi) \right) d\xi \right|^2$$

Taking into account the compact support of $\widehat{\psi}_\lambda$ and the formula (4.4), and changing the variable under the integral $\zeta = 2^{-j}\xi$ we get

$$\|\varphi_J - \varphi_J^\delta\|^2_{L^2(R)} = \sum_{j=-\infty}^{J-1} 2^j \sum_{l\in Z} \left| \frac{1}{2\pi} \int_{|\zeta|\in(\frac{2}{3}\pi,\frac{8}{3}\pi)} \widehat{\psi}(\zeta)\left(Q_1^\delta(2^j\zeta) - Q_2^\delta(2^j\zeta)\right) e^{-il\zeta} d\zeta \right|^2 .$$

The integral under the sum is the sum of two integrals: one is over the interval $(-\frac{8}{3}\pi, -\frac{2}{3}\pi)$, and the second one over $(\frac{2}{3}\pi, \frac{8}{3}\pi)$. So, each of two integrals can be considered as l-th Fourier coefficient of the function $\widehat{\psi}(\zeta)\left(Q_1^\delta(2^j\zeta) - Q_2^\delta(2^j\zeta)\right)$ with respect to orthogonal set $\{e^{-il\zeta}\}$ of 2π periodic functions in $L^2(-\frac{8}{3}\pi, -\frac{2}{3}\pi)$ and $L^2(\frac{2}{3}\pi, \frac{8}{3}\pi)$, respectively. Thus

$$\|\varphi_J - \varphi_J^\delta\|^2_{L^2(R)} \le \sum_{j=-\infty}^{J-1} 2^j \|\widehat{\psi}(\cdot)\left(Q_1^\delta(2^j\cdot) - Q_2^\delta(2^j\cdot)\right)\|^2_{L^2(R)}.$$

But

$$\|\widehat{\psi}\left(Q_1^\delta(2^j\cdot) - Q_2^\delta(2^j\cdot)\right)\|_{L^2(R)} \le \|\psi\|_{L^2(R)} \sup_{|\xi|\in(\frac{2}{3}\pi,\frac{8}{3}\pi)} \left|Q_1^\delta(2^j\xi) - Q_2^\delta(2^j\xi)\right|$$

and $\|\psi\|_{L^2(R)} = 1$, which ends the proof. ∎

Let k be the positive constant appearing in the Helmholtz equation (1.2). Let

$$c_j := k\frac{3}{8\pi}2^{-j}, \tag{5.7}$$

then $c_j < 1$ if and only if $k < \frac{8}{3}\pi 2^j$. Let

$$j_0 := j_0(k) = max\{j : c_j \ge 1\}. \tag{5.8}$$

Lemma 5.2 *If $|\zeta| \in (\frac{2}{3}\pi, \frac{8}{3}\pi)$ and the error bound (5.1) holds, then*

$$|Q_1^\delta(2^j\zeta)| \le \begin{cases} \delta\sqrt{|\Gamma|}k & \text{for } j \le j_0 \\ \delta\sqrt{|\Gamma|}\left(\frac{k}{c_j}\sqrt{2-c_j^2}\right)\cosh\left(\frac{k}{c_j}\gamma\sqrt{1-c_j^2}\right) & \text{for } j > j_0 \end{cases} \tag{5.9}$$

$$|Q_2^\delta(2^j\zeta)| \le \begin{cases} \delta\sqrt{|\Gamma|} & \text{for } j \le j_0 \\ \delta\sqrt{|\Gamma|}\cosh\left(\frac{k}{c_j}\gamma\sqrt{1-c_j^2}\right) & \text{for } j > j_0 \end{cases} \tag{5.10}$$

where γ and c_j are given by (3.2) and (5.7), respectively and $|\Gamma|$ denotes length of the curve Γ.

PROOF: From (5.5) it follows

$$|Q_1^\delta(\xi)| \le \|f - f_\delta\|_{L^2(\Gamma)} \sqrt{|\Gamma|} q_1(\xi), \tag{5.11}$$

where $q_1^2(\xi)$ is equal to

$$\sup_{(x_1,x_2)\in\Gamma} \left\{ \left| \xi \cosh(x_2\sqrt{\xi^2 - k^2}) e^{i\xi x_1} \right|^2 + \left| \sqrt{\xi^2 - k^2} \sinh(x_2\sqrt{\xi^2 - k^2}) e^{i\xi x_1} \right|^2 \right\}.$$

Similarly, from (5.6) it follows that

$$|Q_2^\delta(\xi)| \le \|g - g_\delta\|_{L^2(\Gamma)} \sqrt{|\Gamma|} \sup_{(x_1,x_2)\in\Gamma} \left| \cosh(x_2\sqrt{\xi^2 - k^2}) e^{i\xi x_1} \right| \tag{5.12}$$

If $j \le j_0$ then for $\xi = 2^j\zeta$, $\sqrt{\xi^2 - k^2}$ is an imaginary number and thus $|\cosh(x_2\sqrt{\xi^2 - k^2})|$ as well as $|\sinh(x_2\sqrt{\xi^2 - k^2})|$ are less or equal to 1. Hence

$$|q_1(\xi)| \le \{\xi^2 + (\sqrt{k^2 - \xi^2})^2\}^{1/2},$$

which proves Lemma 5.2 for $j \le j_0$.

Since $k = c_j \frac{8}{3}\pi 2^j$ and $c_j < 1$ for $j > j_0$, we have for $\xi = 2^j\zeta$

$$\cosh(x_2\sqrt{\xi^2 - k^2}) \le \cosh\left(\frac{8}{3}\pi 2^j \gamma \sqrt{1 - c_j^2}\right) = \cosh\left(\frac{k}{c_j}\gamma\sqrt{1 - c_j^2}\right),$$

which gives (5.10). Taking into account that for $\alpha > 0$, $\sinh^2\alpha = \cosh^2\alpha - 1$, we obtain

$$q_1(\xi) \le \frac{k}{c_j}\sqrt{2 - c_j^2} \cosh\left(\frac{k}{c_j}\gamma\sqrt{1 - c_j^2}\right).$$

∎

Theorem 5.3 *If (5.1) holds, then there exist constants C_1 and C_2 depending on k such that*

$$\|\varphi_J - \varphi_J^\delta\| \le \begin{cases} C_1 \delta 2^{\frac{1}{2}J} & \text{as } J \le j_0 \\ C_2 \delta 2^{\frac{3}{2}J} e^{\frac{4}{3}\pi 2^J \gamma} & \text{as } J > j_0. \end{cases} \tag{5.13}$$

PROOF: Let $b_{k,j} := 1 + \frac{k}{c_j}\sqrt{2 - c_j^2}$. Taking into account (5.9), (5.10) we get for $j \le j_0$

$$\sup_{|\xi|\in(\frac{2}{3}\pi, \frac{8}{3}\pi)} \left| Q_1^\delta(2^j\xi) - Q_2^\delta(2^j\xi) \right| \le$$

$$\leq \sup_{|\xi|\in(\frac{2}{3}\pi,\frac{8}{3}\pi)} \left(|Q_1^\delta(2^j\xi)| + |Q_2^\delta(2^j\xi)|\right) \leq \delta\sqrt{|\Gamma|}(k+1).$$

Similarly, for $j > j_0$

$$\sup_{|\xi|\in(\frac{2}{3}\pi,\frac{8}{3}\pi)} \left|Q_1^\delta(2^j\xi) - Q_2^\delta(2^j\xi)\right| \leq \delta\sqrt{|\Gamma|}b_{k,j}\cosh\left(\frac{k}{c_j}\gamma\sqrt{1-c_j^2}\right).$$

Since $\sum_{l=-\infty}^{j} 2^l = 2^{j+1}$, from Lemma 5.1 it follows

$$\|\varphi_J - \varphi_J^\delta\|^2 \leq \begin{cases} \delta^2|\Gamma|(k+1)^2 2^J & \text{as } J \leq j_0 \\ \delta^2|\Gamma|\left\{(k+1)^2 2^{j_0} + \sum_{j=j_0+1}^{J-1} 2^j b_{k,j}^2 \cosh^2\left(\frac{k}{c_j}\sqrt{1-c_j^2}\gamma\right)\right\} & \text{as } J > j_0 \end{cases}.$$

According to notation (5.8), for $j > j_0$ $c_j \in (0,1)$, and thus $1 < \sqrt{2-c_j} < \sqrt{2}$. Moreover, by (5.7), $\frac{k}{c_j} = \frac{8\pi}{3}2^j$. Thus

$$b_{k,j}^2 < \left(1 + \frac{8\pi}{3}2^j\sqrt{2}\right)^2 \leq \vartheta^2 2^{2j},$$

where $\vartheta = (2^{-j_0} + \frac{8\pi}{3}\sqrt{2})$. Taking into account that

$$\cosh\left(\frac{k}{c_j}\gamma\sqrt{1-c_j^2}\right) < e^{\frac{8\pi}{3}\gamma 2^j}$$

we get

$$\sum_{j=j_0+1}^{J-1} 2^j b_{k,j}^2 \cosh^2\left(\frac{k}{c_j}\gamma\sqrt{1-c_j^2}\right) \leq \vartheta^2 e^{2\frac{8\pi}{3}\gamma 2^{J-1}} \sum_{j=j_0+1}^{J-1} 2^{3j}.$$

Let η be a constant such that $\sum_{j=j_0+1}^{J-1} 2^{3j} = \frac{1}{7}(2^{3(J)} - 2^{3(j_0+1)}) \leq \eta 2^{3J}$. Thus, for $J > j_0$

$$\|\varphi_J - \varphi_J^\delta\|^2 \leq \delta^2|\Gamma|\left\{(k+1)^2 2^{j_0} + \vartheta^2\eta 2^{3J} e^{2\frac{4\pi}{3}\gamma 2^J}\right\} \leq$$

$$\leq \left(C_2\delta 2^{\frac{3}{2}J} e^{\frac{4}{3}\pi 2^J\gamma}\right)^2 \tag{5.14}$$

for a certain constant C_2 depending on k, which ends the proof. ∎

6 Projection-regularized moment method

Now, we can formulate the projection-regularized wavelet moment method for the problem (1.2) in the domain $\Omega_1 \subset R^2$ with the boundary described in Section 3 (cf. (3.1)). Let φ_J^δ be the approximate solution of moment problem given by (5.3) with coefficients (5.2). For given perturbed boundary value functions f_δ, g_δ we define the regularized solution u_J^δ as a solution of the well posed Neumann problem (for $\partial\Omega_1$ sufficiently smooth)

$$\begin{cases} \Delta u_J^\delta + k^2 u_J^\delta = 0, & \text{on } \Omega_1 \\ \frac{\partial u_J^\delta}{\partial \nu} = \varphi_J^\delta & \text{on } \partial\Omega_1 \setminus \Gamma, \\ \frac{\partial u_J^\delta}{\partial \nu} = g_\delta & \text{on } \Gamma, \end{cases} \tag{6.1}$$

with additional condition

$$\int_\Gamma (u_J^\delta - f_\delta) d\sigma = 0. \tag{6.2}$$

Theorem 6.1 *Let α and M be fixed constants and $0 < \alpha < \frac{1}{\gamma}$ for γ given by (3.2). If*

$$J(\delta) := \left[\log_2 \left(\frac{3\alpha}{4\pi} \ln \frac{M}{\delta} \right) \right] \tag{6.3}$$

then

$$\|\varphi - \varphi^\delta_{J(\delta)}\|_{L^2(0,1)} \longrightarrow 0 \text{ as } \delta \longrightarrow 0$$

PROOF: Let $\tau := \frac{4}{3}\pi$. From (6.3)

$$e^{\tau 2^J} \leq (\frac{M}{\delta})^\alpha \text{ and } 2^J \leq \frac{1}{\tau} \ln (\frac{M}{\delta})^\alpha.$$

Thus, according to Theorem 5.3 we have

$$\|\varphi_J - \varphi_J^\delta\| \leq \begin{cases} C_1 \delta \sqrt{\alpha \ln(\frac{M}{\delta})} & \text{as } J(\delta) \leq j_0 \\ C_2 \delta^{1-\gamma\alpha} \left(\alpha \ln(\frac{M}{\delta})\right)^{\frac{3}{2}} M^{\alpha\gamma} & \text{as } J(\delta) > j_0. \end{cases} \tag{6.4}$$

Since $\log_2(-ln(\delta)) \longrightarrow \infty$ as $\delta \longrightarrow 0$, by (6.3)

$$J(\delta) \longrightarrow \infty \text{ as } \delta \longrightarrow 0. \tag{6.5}$$

So, for sufficiently small δ, $J(\delta) > j_0$ and the second part of estimation (6.4) occurs. Using the rule of Bernoulli and L'Hospital we see that

$$\lim_{\delta \to 0} \delta^{1-\gamma\alpha} \ln(\delta) = \frac{1}{\gamma\alpha - 1} \lim_{\delta \to 0} \delta^{1-\gamma\alpha} = 0,$$

therefore,

$$\lim_{\delta \to 0} \delta^{1-\gamma\alpha} \left(\alpha \ln(\frac{M}{\delta}) \right)^{\frac{3}{2}} = 0.$$

Thus

$$\|\varphi_J - \varphi_J^\delta\| \longrightarrow 0 \text{ as } \delta \longrightarrow 0.$$

On the other hand, due to (6.5) and (4.12)

$$\|\varphi - \varphi_{J(\delta)}\| \longrightarrow 0 \text{ as } \delta \longrightarrow 0,$$

which ends the proof. ∎

Finally, from the above theorem and the continuous dependence of the solution of the problem (6.1), (6.2) on the boundary conditions we get an asymptotic convergence of projection-regularized wavelet moment method:

Remark 6.2 *If the assumptions of Theorem 6.1 are satisfied, then*

$$\|u - u^\delta_{J(\delta)}\|_{H^1(\Omega_1)} \longrightarrow 0 \text{ as } \delta \longrightarrow 0.$$

References

[1] A.M. Belsky, T.M. Korneychik, and A.P. Khapalyuk. *Space Structure of laser radiation (in Russian)*. Izd. BGU, Minsk, 1982.

[2] F. Berntsson and L. Eldén. Numerical solution of a Cauchy problem for the Laplace equation. *Inverse Problems*, 17(4):839–853, 2001.

[3] J. Cheng, Y.C. Hon, Wei, and M. T.;Yamamoto. Numerical computation of a Cauchy problem for Laplace's equation. *ZAMM Z. Angew. Math. Mech.*, 81(10):665–674, 2001.

[4] Ingrid Daubechies. *Ten Lectures on Wawelets*. SIAM, 1992.

[5] R. Dautry and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology*. Springer-Verlag, 1990.

[6] T. DeLillo, V. Isakov, N. Valdivia, and L. Wang. The detection of surface vibrations from interior acoustical pressure. *Inverse Problems*, 19:507–524, 2003.

[7] L. Eldén and F. Berntsson. A stability estimate for Cauchy problem for an elliptic partial differential equations. *Inverse Problems*, 21(5):1643–1653, 2005.

[8] H.W. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems*. Kluwer Academic Publishers, 1996.

[9] T. Hrycak and V Isakov. Increased stability in the continuation of solutions to the Helmholtz equation. *Inverse Problems*, 20:697–712, 2004.

[10] V. Isakov. *Inverse Problems for Partial Differential Equations*. Springer, New York, 1998.

[11] R.E. Kleinman and W.L. Wendland. On Neumann method for the exterior Neumann problem for the Helmholtz equation. *J. Math. Anal. Appl.*, 57:170–202, 1977.

[12] T. Regińska and K. Regiński. Approximate solution of a Cauchy problem for the Helmholtz equation. *IM PAN Preprint*, 660, 2005.

[13] H.-J. Reinhardt, Houde Han, and Dinh Nho Hào. Stability and regularization of a discrete approximation to the Cauchy problem for Laplace's equation. *SIAM J. Numer. Anal.*, 36(3):890–905, 1999.

[14] A.E. Siegman. *Lasers*. Oxford University Press, 1986.

[15] A.N. Tikhonov, A.V. Goncharsky, V.V. Stepanov, and A.G. Yagola. *Numerical methods for the solution of Ill-posed Problems*. Kluwer Academic Publishers, 1995.