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The complex symplectic moduli spaces of uni-model parametric plane curve singularities

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The complex symplectic moduli spaces of uni-modal parametric plane curve singularities.

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Abstract

Classification of zero-modal singularities of parametric plane curves under diffeomorphism equivalence is extended to uni-modal singularities. Both the simple and uni-modal singularities of parametric plane curves are classified further under symplectomorphic equivalence. In particular the corresponding cyclic symplectic moduli spaces are reconstructed as a canonical ambient spaces for the diffeomorphism moduli spaces which are no longer Hausdorff spaces.

1 Introduction.

In [3], Bruce and Gaffney classified the simple (0-modal) singularities of parametric plane curves $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ under diffeomorphism equivalence (right-left equivalence) in the complex analytic category into the classes $A_{2\ell}$, $E_{6\ell}$, $E_{6\ell+2}$, W_{12}, W_{18} and $W^{\#}_{1,2\ell-1}$ ($\ell = 1, 2, 3, \dots$). See also [2] and Table 1. The classification is here extended to the uni-modal singularities:

Theorem 1.1 *Under diffeomorphism equivalence the uni-modal singularities of parametric plane curves $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ are classified into the following list:*

$$N_{20}: \quad (t^5, t^6 + t^8 + \lambda t^9)\ (-\lambda \sim \lambda),\ (t^5, t^6 + t^9),\ (t^5, t^6 + t^{14}),\ (t^5, t^6),$$

$$\begin{aligned} N_{24}: \quad & (t^5, t^7 + t^8 + \lambda t^{11}),\ (t^5, t^7 + t^{11} + \lambda t^{13})\ (-\lambda \sim \lambda), \\ & (t^5, t^7 + t^{13}),\ (t^5, t^7 + t^{18}),\ (t^5, t^7), \end{aligned}$$

$$\begin{aligned} N_{28}: \quad & (t^5, t^8 + t^9 + \lambda t^{12}),\ (t^5, t^8 + t^{12} + \lambda t^{14})\ (-\lambda \sim \lambda), \\ & (t^5, t^8 + t^{14} + \lambda t^{17})\ (-\lambda \sim \lambda),\ (t^5, t^8 + t^{17}),\ (t^5, t^8 + t^{22}),\ (t^5, t^8), \end{aligned}$$

$$\begin{aligned} W_{24}: \quad & (t^4, t^9 + t^{10} + \lambda t^{11})\ (\lambda \neq \tfrac{19}{18}),\ (t^4, t^9 + t^{10} + \tfrac{19}{18} t^{11} + \lambda t^{15}), \\ & (t^4, t^9 + t^{11}),\ (t^4, t^9 + t^{15}),\ (t^4, t^9 + t^{19}),\ (t^4, t^9), \end{aligned}$$

$$\begin{aligned} W_{30}: \quad & (t^4, t^{11} + t^{13} + \lambda t^{14})\ (-\lambda \sim \lambda),\ (t^4, t^{11} + t^{14} + \lambda t^{17})\ (\lambda \neq \tfrac{25}{22}), \\ & (t^4, t^{11} + t^{14} + \tfrac{25}{22} t^{17} + \lambda t^{21})\ (\omega\lambda \sim \lambda, \omega^3 = 1), \\ & (t^4, t^{11} + t^{17}),\ (t^4, t^{11} + t^{21}),\ (t^4, t^{11} + t^{25}),\ (t^4, t^{11}), \end{aligned}$$

$$W^{\#}_{2,2\ell-1}: \quad (t^4, t^{10} + t^{2\ell+9} + \lambda t^{2\ell+11})\ (\omega\lambda \sim \lambda, \omega^{2\ell-1} = 1) \quad (\ell = 1, 2, 3, \dots).$$

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In the list, for instance $-\lambda \sim \lambda$ means that $(t^5, t^6 + t^8 + \lambda' t^9)$ is diffeomorphic to $(t^5, t^6 + t^8 + \lambda t^9)$ if and only if $\lambda' = \pm\lambda$.

In [7], motivated by the symplectic bifurcation problem, we gave the symplectic classification of simple singularities of parametric plane curves in the real case. (For the higher dimensional case, see [8]). In this paper, we classify symplectically both the simple and uni-modal singularities of parametric plane curves in the complex case.

We call holomorphic parametric curve-germs $f,\ g : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ *diffeomorphic* (resp. *symplectomorphic*) if there exist a bi-holomorphic diffeomorphism σ of $(\mathbf{C}, 0)$ and a bi-holomorphic diffeomorphism τ (resp. a bi-holomorphic symplectomorphism τ) of $(\mathbf{C}^2, 0)$ (for the holomorphic symplectic form $dx \wedge dy$ on $\mathbf{C}^2$) satisfying $\tau(g(t)) = f(\sigma(t))$.

Let r be a non-negative integer. A curve-germ f is called *r-modal* if a finite number of s-parameter families $(0 \leq s \leq r)$ of diffeomorphism classes form a neighborhood of f in the space of curve-germs. Then we have:

Theorem 1.2 *A simple or uni-modal singularity $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ is symplectomorphic to a germ which belongs to one of the following families (called "symplectic normal forms"):*

$$
\begin{array}{ll}
A_{2\ell}: & (t^2, t^{2\ell+1}),\\
E_{6\ell}: & (t^3, t^{3\ell+1} + \Sigma_{j=1}^{\ell-1}\lambda_j t^{3(\ell+j)-1}),\\
E_{6\ell+2}: & (t^3, t^{3\ell+2} + \Sigma_{j=1}^{\ell-1}\lambda_j t^{3(\ell+j)+1}),\\
W_{12}: & (t^4, t^5 + \lambda t^7),\\
W_{18}: & (t^4, t^7 + \lambda t^9 + \mu t^{13}),\\
W^{\#}_{1,2\ell-1}: & (t^4, t^6 + \lambda t^{2\ell+5} + \mu t^{2\ell+9}), \lambda \neq 0, (\ell = 1, 2, \dots)\\
\\
N_{20}: & (t^5, t^6 + \lambda_1 t^8 + \lambda_2 t^9 + \lambda_3 t^{14}),\\
N_{24}: & (t^5, t^7 + \lambda_1 t^8 + \lambda_2 t^{11} + \lambda_3 t^{13} + \lambda_4 t^{18}),\\
N_{28}: & (t^5, t^8 + \lambda_1 t^9 + \lambda_2 t^{12} + \lambda_3 t^{14} + \lambda_4 t^{17} + \lambda_5 t^{22}),\\
W_{24}: & (t^4, t^9 + \lambda_1 t^{10} + \lambda_2 t^{11} + \lambda_3 t^{15} + \lambda_4 t^{19}),\\
W_{30}: & (t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17} + \lambda_4 t^{21} + \lambda_5 t^{25}),\\
W^{\#}_{2,2\ell-1}: & (t^4, t^{10} + \lambda_1 t^{2\ell+9} + \lambda_2 t^{2\ell+11} + \lambda_3 t^{2\ell+13} + \lambda_4 t^{2\ell+17} + \lambda_5 t^{2\ell+21}), \lambda_1 \neq 0 (\ell = 1, 2, \dots)
\end{array}
$$

Moreover we determine their symplectic moduli spaces as listed in Tables 1 and 2:

Theorem 1.3 *Let $f_\lambda(t) = (t^m, t^n + \lambda_1 t^{r_1} + \lambda_2 t^{r_2} + \cdots + \lambda_s t^{r_s})$ be one of the symplectic normal forms of simple or uni-modal singularities. Then two curve-germs f_λ and $f_{\lambda'}$ belonging to the same family are symplectomorphic if and only if there exists an $(m+n)$-th root $\zeta \in \mathbf{C}$ of unity satisfying*

$$\lambda'_1 = \zeta^{r_1 - n}\lambda_1,\ \lambda'_2 = \zeta^{r_2 - n}\lambda_2,\ \dots,\ \lambda'_s = \zeta^{r_s - n}\lambda_s.$$

In particular each symplectic moduli space of a family is a Hausdorff space in the natural topology and it is extended to a cyclic quotient singularity.

In his lecture notes [18], O. Zariski studied the moduli space of parametric plane curve-germs, under diffeomorphism equivalence, for a given topological type, or the equi-singularity class $(m, \beta_1, \dots, \beta_g)$. (See §2). In particular, Zariski determined the moduli spaces for the classes

$(2, 2\ell+1), (3, 3\ell+1), (4, 5), (4, 6, 2\ell+5), (5, 6)$ and $(6, 7)$. He did not mention symplectomorphic equivalence at all, but surprisingly, he used, as pre-normal forms, several symplectic normal forms given in Theorem 1.2. For instance, in [18] page 68, he started with

$$x = t^5, y = t^6 + a_8 t^8 + a_9 t^9 + a_{14} t^{14}$$

in the concrete classification of the case $(5, 6)$.

In this paper, clarifying the role of symplectomorphism equivalence, we proceed Zariski's classification via modality: By Bruce-Gaffney's classification and by Theorem 1.1, we determine the moduli spaces for the classes $(4, 7), (5, 7), (5, 8), (4, 9), (4, 11)$ and $(4, 10, 2\ell+9)$ beyond Zariski's result, except for the class $(6, 7)$ which is actually bi-modal. Moreover we can treat the case $(6, 7)$ by the same method developed in this paper.

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The classification of plane curve singularities is closely related to the classification of Legendre curve singularities and the classification of Goursat distributions ([12][19][13]). Actually P. Mormul has predicted several forms in Theorem 1.1 from his classification results for uni-modal singularities of Goursat distributions (private communication to the first author). Note however, that these classification problems have different features, and therefore, to get the exact classification, we need a detailed analysis in each case.

In the next section we give an outline of the proofs of Theorem 1.2 and Theorem 1.3. In the last section we outline the proof of Theorem 1.1.

2 Symplectic normal forms.

Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ be a germ of parametric holomorphic plane curve. Then the following conditions on f are known to be equivalent ([16][7]):

(i) f has an injective representative.

(ii) f is a normalization onto the image.

(iii) The diffeomorphism class of f is determined by a finite jet of f.

(iv) The symplectomorphism class of f is determined by a finite jet of f.

(v) The quotient vector space $\mathcal{O}_1/f^*\mathcal{O}_2$ is finite dimensional.

We assume f satisfies one (and therefore all) of the above conditions. Here $f^* : \mathcal{O}_2 = \mathbf{C}\{x, y\} \to \mathcal{O}_1 = \mathbf{C}\{t\}$ is defined by composition: $f^*(h) = h \circ f$. Recall that *the number of double points* $\delta(f) = \dim_{\mathbf{C}} \mathcal{O}_1/f^*\mathcal{O}_2$ ([11][17]) also has the meaning of *the symplectic codimension* of f, that is, the number of parameters needed to produce its versal unfolding via symplectomorphism equivalence ([7]).

We briefly recall the theory developed in [7] §7: The symplectic codimension of f is defined by

$$\text{sp-cod}(f) = \dim_{\mathbf{C}} \frac{V_f}{tf(V_1) + wf(VH_2)}$$

as an infinitesimal symplectic invariant of Mather's type. Here V_f is the space of germs of holomorphic vector fields $v : (\mathbf{C}, 0) \to T\mathbf{C}^2$ along f, the space of infinitesimal deformations of

	DIFF. NORMAL FORM	SYM. NORMAL FORM	SYM. MODULI SPACE
$A_{2\ell}$	$(t^2, t^{2\ell+1})$	$(t^2, t^{2\ell+1})$	
$E_{6\ell}$ $(\ell \geq 1)$	$(t^3, t^{3\ell+1} + t^{3(\ell+p)+2})$ $(0 \leq p \leq \ell - 2)$ $(t^3, t^{3\ell+1})$	$(t^3, t^{3\ell+1} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1})$	$\mathbf{C}^{\ell-1}/G,\ G = \mathbf{Z}/(3\ell+4)\mathbf{Z}$ $(\lambda_1, \ldots, \lambda_{\ell-1}) \mapsto$ $(\zeta\lambda_1, \ldots, \zeta^{3j-2}\lambda_j, \ldots, \zeta^{3\ell-5}\lambda_{\ell-1})$ $(\zeta^{3\ell+4} = 1, \text{primitive})$
$E_{6\ell+2}$ $(\ell \geq 1)$	$(t^3, t^{3\ell+2} + t^{3(\ell+p)+4})$, $(0 \leq p \leq \ell - 2)$ $(t^3, t^{3\ell+2})$	$(t^3, t^{3\ell+2} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1})$	$\mathbf{C}^{\ell-1}/G,\ G = \mathbf{Z}/(3\ell+5)\mathbf{Z}$ $(\lambda_1, \ldots, \lambda_{\ell-1}) \mapsto$ $(\zeta^2\lambda_1, \ldots, \zeta^{3j-1}\lambda_j, \ldots, \zeta^{3\ell-4}\lambda_{\ell-1})$ $(\zeta^{3\ell+5} = 1, \text{primitive})$
W_{12}	$(t^4, t^5 + t^7)$ (t^4, t^5)	$(t^4, t^5 + \lambda t^7)$	$\mathbf{C}/G,\ G = \mathbf{Z}/9\mathbf{Z}$ $\lambda \mapsto \zeta\lambda,\ (\zeta^9 = 1)$
W_{18}	$(t^4, t^7 + t^9)$ $(t^4, t^7 + t^{13})$ (t^4, t^7)	$(t^4, t^7 + \lambda t^9 + \mu t^{13})$	$\mathbf{C}^2/G,\ G = \mathbf{Z}/11\mathbf{Z}$ $(\lambda, \mu) \mapsto (\zeta\lambda, \zeta^3\mu),\ (\zeta^{11} = 1)$
$W^{\#}_{1,2\ell-1}$ $(\ell \geq 1)$	$(t^4, t^6 + t^{2\ell+5})$	$(t^4, t^6 + \lambda t^{2\ell+5} + \mu t^{2\ell+9})$ $(\lambda \neq 0)$	$(\mathbf{C}^* \times \mathbf{C})/G,\ G = \mathbf{Z}/10\mathbf{Z}$, $(\lambda, \mu) \mapsto (\zeta^{2\ell-1}\lambda, \zeta^{2\ell+3}\mu)$, $(\zeta^{10} = 1, \text{primitive})$

Table 1: The complex symplectic moduli spaces of simple parametric plane curve singularities.

	SYM. NORMAL FORM	SYM. MODULI SPACE
N_{20}	$(t^5, t^6 + \lambda_1 t^8 + \lambda_2 t^9 + \lambda_3 t^{14})$	$\mathbf{C}^3/G,\ G = \mathbf{Z}/11\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3) \mapsto (\zeta^2\lambda_1, \zeta^3\lambda_2, \zeta^8\lambda_3)$, $(\zeta^{11} = 1)$
N_{24}	$(t^5, t^7 + \lambda_1 t^8 + \lambda_2 t^{11} + \lambda_3 t^{13} + \lambda_4 t^{18})$	$\mathbf{C}^4/G,\ G = \mathbf{Z}/12\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto$ $(\zeta\lambda_1, \zeta^4\lambda_2, \zeta^6\lambda_3, \zeta^{11}\lambda_4)$, $(\zeta^{12} = 1, \text{primitive})$
N_{28}	$(t^5, t^8 + \lambda_1 t^9 + \lambda_2 t^{12} + \lambda_3 t^{14} + \lambda_4 t^{17} + \lambda_5 t^{22})$	$\mathbf{C}^5/G,\ G = \mathbf{Z}/13\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \mapsto$ $(\zeta\lambda_1, \zeta^4\lambda_2, \zeta^6\lambda_3, \zeta^9\lambda_4, \zeta^{14}\lambda_5)$, $(\zeta^{13} = 1, \text{primitive})$
W_{24}	$(t^4, t^9 + \lambda_1 t^{10} + \lambda_2 t^{11} + \lambda_3 t^{15} + \lambda_4 t^{19})$	$\mathbf{C}^4/G,\ G = \mathbf{Z}/13\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto$ $(\zeta\lambda_1, \zeta^2\lambda_2, \zeta^6\lambda_3, \zeta^{10}\lambda_4)$, $(\zeta^{13} = 1)$
W_{30}	$(t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17}$ $+\lambda_4 t^{21} + \lambda_5 t^{25})$	$\mathbf{C}^5/G,\ G = \mathbf{Z}/15\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \mapsto$ $(\zeta^2\lambda_1, \zeta^5\lambda_2, \zeta^6\lambda_3, \zeta^{10}\lambda_4, \zeta^{14}\lambda_5)$, $(\zeta^{15} = 1, \text{primitive})$
$W^{\#}_{2,2\ell-1}$	$(t^4, t^{10} + \lambda_1 t^{2\ell+9} + \lambda_2 t^{2\ell+11} + \lambda_3 t^{2\ell+13}$ $+\lambda_4 t^{2\ell+17} + \lambda_5 t^{2\ell+21}),\quad (\lambda_1 \neq 0)$.	$(\mathbf{C}^* \times \mathbf{C}^4)/G,\ G = \mathbf{Z}/14\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \mapsto$ $(\zeta^{2\ell-1}\lambda_1, \zeta^{2\ell+1}\lambda_2, \zeta^{2\ell+3}\lambda_3, \zeta^{2\ell+7}\lambda_4, \zeta^{2\ell+11}\lambda_5)$, $(\zeta^{14} = 1, \text{primitive})$

Table 2: The complex symplectic moduli spaces of uni-modal parametric plane curve singularities.

f, V_1 the space of germs of holomorphic vector fields over $(\mathbf{C}, 0)$ and VH_2 the space of germs of holomorphic Hamiltonian vector fields over $(\mathbf{C}^2, 0)$. The homomorphisms $tf : V_1 \to V_f$ and $wf : VH_2 \to V_f$ are defined by $tf(\xi) := f_*(\xi)$, $\xi \in V_1$ and $wf(\eta) := \eta \circ f$ respectively. An unfolding $F : (\mathbf{C} \times \mathbf{C}^r, (0, 0)) \to (\mathbf{C}^2 \times \mathbf{C}^r, (0, 0))$ of f, $F(t, u) = (f_u(t), u)$, is *symplectically versal* if $\dfrac{\partial f_u}{\partial u_1}(t, 0), \ldots, \dfrac{\partial f_u}{\partial u_r}(t, 0)$ generate V_f, over $\mathbf{R}$, up to the space $tf(V_1) + wf(VH_2)$ of deformations which are covered by symplectomorphisms ([7], Proposition 7.1).

Set $f(t) = (x(t), y(t))$. For an infinitesimal deformation $v(t) = a(t)\dfrac{\partial}{\partial x} + b(t)\dfrac{\partial}{\partial y} \in V_f$, we define a *generating function* $e(t) \in \mathcal{O}_1$ of v by $d(e(t)) = b(t)d(x(t)) - a(t)d(y(t))$, or $e'(t) = b(t)x'(t) - a(t)y'(t)$ up to the constant term. The generating function of $tf(\xi) + wf(H_k)$ is equal to f^*k, where k is the Hamiltonian function of the Hamiltonian vector field H_k. Then there exists an exact sequence of vector spaces:

$$0 \to \frac{V_f'}{tf(V_1)} \to \frac{V_f}{tf(V_1) + wf(VH_2)} \to \frac{R_f}{f^*\mathcal{O}_2} \to 0,$$

where $R_f = \{e(t) \in \mathcal{O}_1 \mid \mathrm{ord}(e'(t)) \geq \mathrm{ord}(f) - 1\}$ and

$$V_f' = \{v(t) = a(t)\partial/\partial x + b(t)\partial/\partial y \in V_f \mid b(t)x'(t) - a(t)y'(t) = 0\}.$$

Then we see that $\dim_{\mathbf{C}} V_f'/tf(V_1) = \mathrm{ord}(f) - 1 = \dim_{\mathbf{C}} \mathcal{O}_1/R_f$. Therefore we have

$$\text{sp-cod}(f) = \dim_{\mathbf{C}} V_f'/tf(V_1) + \dim_{\mathbf{C}} R_f/f^*\mathcal{O}_2 = \dim_{\mathbf{C}} \mathcal{O}_1/f^*\mathcal{O}_2 = \delta(f).$$

Some of parameters of the symplectically versal unfolding correspond to deformations into less singular germs, and the remaining parameters provide the symplectic normal form within a given *equi-singular class* up to discrete symplectomorphism equivalence classes. We recall a basic fact from the textbook [17] in our context: Set $m = \mathrm{ord}(f)$. Then f is symplectomorphic to a germ of the form $(t^m, \sum_{k=m}^{\infty} a_k t^k)$. Suppose $m \geq 2$, that is, f is not an immersion. Set $\beta_1 = \min\{k \mid a_k \neq 0, m \nmid k\}$ and let e_1 be the greatest common divisor of m and β_1, and inductively set $\beta_q = \min\{k \mid a_k \neq 0, e_{q-1} \nmid k\}$, and let e_q be the greatest common divisor of β_q and e_{q-1}, $q \geq 2$. Then $e_g = 1$ for sufficiently large g, and we call $(m = \beta_0, \beta_1, \beta_2, \ldots, \beta_g)$ *the Puiseux characteristic* of f, which is a basic diffeomorphism invariant. Setting $e_0 = m$, we have $\delta(f) = \frac{1}{2}\sum_{q=1}^{g}(\beta_q - 1)(e_{q-1} - e_q)$ ([11][17]). Moreover the Puiseux characteristic determines the homeomorphism equivalence class of f ([10][18]). We call a deformation of plane curve singularities *equi-singular* if the Puiseux characteristic is preserved. Under an equi-singular deformation of f, we can take a common monomial basis of $\mathcal{O}_1/f^*\mathcal{O}_2$.

First we have:

Lemma 2.1 *f is symplectomorphic to a germ of the form $(t^m, t^{\beta_1} + \sum_{k=\beta_1+1}^{\infty} b_k t^k)$.*

Proof: Set $\psi(x) = \sum_{k=m}^{\beta_1 - 1} a_k x^{k/m}$ and $\tau_1(x, y) = (x, y - \psi(x))$. Then $\tau_1(f(t)) = (t^m, \sum_{k=\beta_1}^{\infty} a_k t^k)$ with $a_{\beta_1} \neq 0$. Define $\alpha \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ by $\alpha^{m+\beta_1} a_{\beta_1} = 1$, and set $\sigma(t) = \alpha t$ and $\tau_2(x, y) = (\alpha^{-m}x, \alpha^m y)$. Then τ_1, τ_2 are both symplectomorphisms and we see that $\tau_2(\tau_1(f(\sigma(t))))$ has the required form. $\square$

To get symplectic normal forms, we first remark the following:

Lemma 2.2 *Suppose $\mathcal{O}_1/f^*\mathcal{O}_2$ has a monomial basis*

$$t,\ t^2,\dots,\ t^{m-1},\ t^{m+1},\dots,\ t^{r_1+m},\dots,\ t^{r_s+m}$$

where $r_1+m,\dots,r_s+m$ are all exponents greater than $\beta_1+m (j=1,\dots,s)$. Then the family

$$f_c(t)=(t^m,\ t^{\beta_1}+\sum_{k=\beta_1+1}^{\infty}b_kt^k+\sum_{j=1}^{s}c_jt^{r_j}),$$

$c=(c_1,\dots,c_s)\in\mathbf{C}^s$, gives a transversal to the symplectomorphism orbit.

Proof: Let $v=\psi(t)\left(\frac{\partial}{\partial y}\circ f\right)$, $\psi(t)=\sum_{k=\beta_1+1}^{\infty}u_kt^k$, be an infinitesimal deformation of f among the forms given in Lemma 2.1. Take the generating function e of v satisfying $de(t)=\psi(t)d(t^m)$, $e(0)=0$. Then there exist $\tilde{b}_1,\tilde{b}_2,\dots,\tilde{b}_{m-1},\tilde{b}_{m+1},\dots,\tilde{b}_{r_1+m},\dots,\tilde{b}_{r_s+m}\in\mathbf{C}$ such that, setting

$$\varphi(t)=\tilde{b}_1t+\tilde{b}_2t^2+\cdots+\tilde{b}_{m-1}t^{m-1}+\tilde{b}_{m+1}t^{m+1}+\cdots+\tilde{b}_{r_1+m}t^{r_1+m}+\cdots+\tilde{b}_{r_s+m}t^{r_s+m},$$

we have $e-\varphi\in f^*\mathcal{O}_2$. Set $e-\varphi=h\circ f$. Since $\mathrm{ord}(e)\geq\beta_1+m$, we see $\mathrm{ord}(h)\geq 2$. On the other hand, $\varphi(t)=\tilde{b}_{r_1+m}t^{r_1+m}+\cdots+\tilde{b}_{r_s+m}t^{r_s+m}$. Then we have

$$\begin{aligned} d\varphi(t) &= \{(r_1+m)\tilde{b}_{r_1+m}t^{r_1+m-1}+\cdots+(r_s+m)\tilde{b}_{r_s+m}t^{r_s+m-1}\}dt \\ &= \left(\tfrac{r_1+m}{m}\tilde{b}_{r_1+m}t^{r_1}+\cdots+\tfrac{r_s+m}{m}\tilde{b}_{r_s+m}t^{r_s}\right)d(t^m). \end{aligned}$$

Set $w=\left(\frac{r_1+m}{m}\tilde{b}_{r_1+m}t^{r_1}+\cdots+\frac{r_s+m}{m}\tilde{b}_{r_s+m}t^{r_s}\right)\left(\frac{\partial}{\partial y}\circ f\right)$. Consider the Hamiltonian vector field X_h. Then, the field $(v-w)-wf(X_h)$ has the zero as a generating function, that is, $(v-w)-wf(X_h)\in V'_f$. Then there exists $\xi\in V_1$ with $\xi(0)=0$ satisfying $tf(\xi)=(v-w)-wf(X_h)$, that is, $v=w+tf(\xi)+wf(X_h)$ (cf. Lemma 8.2 and Theorem 8.7 of [7]). This means that the above family is transversal to the symplectomorphism orbit through f. □

A monomial basis of $\mathcal{O}_1/f^*\mathcal{O}_2$ can be calculated by considering *the semigroup* $S(f)=\{\mathrm{ord}(h)\mid h\in f^*\mathcal{O}_2\}\subseteq\mathbf{N}$. In fact $\{t^r\mid r\in\mathbf{N}\setminus S(f),\ r>0\}$ forms a monomial basis of $\mathcal{O}_1/f^*\mathcal{O}_2$. Note that a system of generators for the semigroup $S(f)$ is calculated explicitly from the Puiseux characteristic. Moreover there exists a number N depending only on the Puiseux characteristic of f such that if $\phi\in\mathcal{O}_1$ has order $\geq N$, then $\phi\in f^*\mathcal{O}_2$ ([17]).

Example 2.3 (1) (W_{30}) Let $m=4,\ \beta_1=11$. Then the semigroup $S(f)$ is generated by 4 and 11. A monomial basis of $\mathcal{O}_1/f^*\mathcal{O}_2$ is given by $t,t^2,t^3,t^5,t^6,t^7,t^9,t^{10},t^{13},t^{14},t^{17},t^{18},t^{21},t^{25},t^{29}$.

(2) ($W^{\#}_{1,2\ell-1}$) Let $m=4,\ \beta_1=6$ and $\beta_2=2\ell+5$. Then $S(f)$ is generated by $4,6$ and $2\ell+11$. The complement $\mathbf{N}\setminus S(f)$ consists of $1,2,3,5,7,9,11,\dots,2\ell+9,2\ell+13$.

(3) ($W^{\#}_{2,2\ell-1}$) Let $m=4,\ \beta_1=10$ and $\beta_2=2\ell+9$. Then $S(f)$ is generated by $4,10$ and $2\ell+19$. The complement $\mathbf{N}\setminus S(f)$ consists of $1,2,3,5,7,9,11,13,15,17,19,21,23,\dots,2\ell+13,2\ell+15,2\ell+17,2\ell+21,2\ell+25$.

Proof of Theorem 1.2 : Under the notations of Lemma 2.2, consider the infinitesimal deformation $v=\kappa(t)\left(\frac{\partial}{\partial y}\circ f\right)$, $\kappa(t)=-\sum b_kt^k$, where the summation runs over k different from $r_1,\dots,r_s$. Then the Puiseux characteristics are preserved under the deformation

$$f_u=(t^m,\ t^{\beta_1}+\sum_{k=\beta_1+1}^{\infty}b_kt^k-u\kappa(t))$$

($u \in [0,1]$) corresponding to v. This is clear when the greatest common divisor e_1 of m and β_1 is equal to 1. From Example 2.3 (2)(3), it also holds for $W^{\#}_{1,2\ell-1}$ and $W^{\#}_{2,2\ell-1}$. Then there exist $w_u = \sum_{j=1}^{s} c_{j,u} t^{r_j},\ c_{j,u} \in \mathbf{C},\ \xi_u \in V_1,\ \xi_u(0) = 0$, and $\eta_u \in VH_2,\ \eta_u(0) = 0$, smoothly depending on u and satisfying $v = w_u + tf_u(\xi_u) + wf_u(\eta_u)$. By integrating from $u = 0$ to $u = 1$ we see that f is symplectomorphic to

$$f_\lambda(t) = (t^m,\ t^{\beta_1} + \lambda_1 t^{r_1} + \lambda_2 t^{r_2} + \cdots + \lambda_s t^{r_s}),$$

for some $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbf{C}^s$. In all cases except $W^{\#}_{1,2\ell-1}$ and $W^{\#}_{2,2\ell-1}$, there are no restrictions on λ and we get the symplectic normal forms given in Theorem 1.2. In the case of $W^{\#}_{1,2\ell-1}$, f is symplectomorphic to

$$f_c = (t^4, t^6 + c_1 t^7 + \cdots + c_\ell t^{2\ell+5} + c_{\ell+1} t^{2\ell+7}).$$

Since the Puiseux characteristic of f is $(4; 6, 2\ell+5)$, we necessarily have $c_1 = 0, \ldots, c_{\ell-1} = 0$ and $c_\ell \neq 0$. Setting $\lambda = c_\ell,\ \mu = c_{\ell+1}$, we get the symplectic normal form. In the case of $W^{\#}_{2,2\ell-1}$, f is symplectomorphic to

$$f_c = (t^4, t^{10} + c_1 t^{11} + c_2 t^{13} + c_3^{15} + \cdots + c_\ell t^{2\ell+9} + c_{\ell+1} t^{2\ell+11} + c_{\ell+2} t^{2\ell+13} + c_{\ell+3} t^{2\ell+17} + c_{\ell+4} t^{2\ell+21}).$$

Since the Puiseux characteristic of f is $(4; 10, 2\ell+9)$, we have $c_1 = 0, \ldots, c_{\ell-1} = 0$ and $c_\ell \neq 0$, which gives the symplectic normal form. $\square$

In the process of symplectic classification, we observe a kind of rigidity. Let f_λ and $f_{\lambda'}$, with $\lambda \neq \lambda'$, be germs belonging to one of the symplectic normal forms of simple or uni-modal parametric plane curve singularities. Then f_λ and $f_{\lambda'}$, are not isotopic by symplectomorphisms. Moreover we have the following strong rigidity which implies Theorem 1.3 in each case:

Proposition 2.4 *Let f_λ and $f_{\lambda'}$ be germs belonging to one of the symplectic normal forms of simple or uni-modal parametric plane curve singularities. If f_λ and $f_{\lambda'}$ are symplectomorphic, then they are linearly symplectomorphic: If there exists a symplectomorphism equivalence (σ, τ) satisfying $\tau \circ f_{\lambda'} = f_\lambda \circ \sigma$, then there exists a symplectomorphism equivalence (Σ, T) such that $T \circ f_{\lambda'} = f_\lambda \circ \Sigma$, $\Sigma : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ is a complex linear transformation, and $T : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ is a complex linear symplectic transformation.*

Proof: We give the calculation in the case W_{30}. Other cases can be treated similarly. Set $f_\lambda = (t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17} + \lambda_4 t^{21} + \lambda_5 t^{25})$, and suppose f_λ and $f_{\lambda'}$ are symplectomorphic for $\lambda = (\lambda_1, \ldots, \lambda_5)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_5)$.

Set $\sigma(t) = a_1 t + a_2 t^2 + \cdots$ and, as components of $\tau(x, y)$,

$$\begin{aligned} x \circ \tau(x,y) &= ax + by + h_1 x^2 + h_2 xy + h_3 y^2 + \ell_1 x^3 + \ell_2 x^2 y + \ell_3 x y^2 + \ell_4 y^3 + \cdots, \\ y \circ \tau(x,y) &= cx + ey + k_1 x^2 + k_2 xy + k_3 y^2 + m_1 x^3 + m_2 x^2 y + m_3 x y^2 + m_4 y^3 + \cdots. \end{aligned}$$

Consider the equation $f_\lambda(\sigma(t)) = \tau(f_{\lambda'}(t))$:

$$\begin{aligned} &\sigma(t)^m = x \circ \tau(t^4, t^{11} + \lambda'_1 t^{13} + \lambda'_2 t^{14} + \lambda'_3 t^{17} + \lambda'_4 t^{21} + \lambda'_5 t^{25}) \quad \cdots\cdots\cdots (*), \\ &\sigma(t)^{11} + \lambda_1 \sigma(t)^{13} + \lambda_2 \sigma(t)^{14} + \lambda_3 \sigma(t)^{17} + \lambda_4 \sigma(t)^{21} + \lambda_5 \sigma(t)^{25} \\ &\qquad = y \circ \tau(t^4, t^{11} + \lambda'_1 t^{13} + \lambda'_2 t^{14} + \lambda'_3 t^{17} + \lambda'_4 t^{21} + \lambda'_5 t^{25}) \quad \cdots\cdots\cdots (**). \end{aligned}$$

Now we are going to determine the coefficients of σ and τ of lower degree terms, using the equations (*) and (**) in a zigzag manner. We denote for comparison of terms in (*) (resp. (**)) of degree i by (*i) (resp. (**i)). First by (*4), we have $a_1^4 = a$. By (*5)(*6)(*7), we have $a_2 = 0,\ a_3 = 0,\ a_4 = 0$. By (**4), $c = 0$. By (**8), $k_1 = 0$. By (**11), we have $a_1^{11} = e$. Since τ is a symplectomorphism, we see that $ae = 1$, so we have $a_1^{15} = 1$. By (**12), $m_1 = 0$. By (**13), we get $\lambda_1 a^{13} = e\lambda'_1$ and therefore $\lambda_1 a^2 = \lambda'_1$. By (**14), $\lambda_2 a^{14} = e\lambda'_2$ and therefore $\lambda_1 a^3 = \lambda'_1$. By (**15), $11a_1^{10}a_5 = k_2$. From (*8), we have $4a_3a_5 = h_1$. Since τ is a symplectomorphism, we have $2h_1e + ak_2 = 0$. Thus we see that $a_5 = 0$. Then $k_2 = 0,\ h_1 = 0$. By (*9), $4a_3a_6 = 0$ so $a_6 = 0$. By (*10), $a_7 = 0$. By (*11), we have $4a_1^3a_8 = b$. Then by (**17), we have $\lambda_3 a^{17} = e\lambda'_3$, thus $\lambda_3 a^6 = \lambda'_3$. By (**18), we have $a_8 = 0$. Therefore we have $b = 0$. By (*12), $4a_1^3a_8 = \ell_1$. By (**19), $11a_1^{10}a_9 = m_2$. Since τ is a symplectomorphism we have $6\ell_1 e + 2am_2 = 0$. Thus we have $a_9 = 0$. Then we have $\ell_1 = 0,\ m_2 = 0$. By (*13)(*14), we have $a_{10} = 0,\ a_{11} = 0$. By (**21), we have $\lambda_4 a^{21} = e\lambda'_4$ so $\lambda_4 a^{10} = \lambda'_4$. By (**22), $11a_1^{10}a_{12} = k_3$. By (*15), we have $4a_1^3a_{12} = h_2$. Since τ is a symplectomorphism we have $h_2e + 2ak_3 = 0$. Therefore $a_{12} = 0$, and $k_3 = 0,\ h_2 = 0$. Then, by (**23), we have $a_{13} = 0$, and by (*17)(*18), $a_{14} = 0,\ a_{15} = 0$. Finally, by (**25),we have $\lambda_5 a^{25} = e\lambda'_5$, and $\lambda_5 a^{13} = \lambda'_5$. Therefore, setting T and Σ as the linear parts of τ and σ respectively, we have $T \circ f_{\lambda'} = f_\lambda \circ \Sigma$. $\square$

Remark 2.5 If two curve-germs $f,\ g : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ are symplectomorphic, then they are *symplectically isotopic*, that is, there exist C^∞ families of bi-holomorphic diffeomorphisms σ_s and bi-holomorphic symplectomorphisms τ_s ($s \in [0,1]$) on $(\mathbf{C}, 0)$ and $(\mathbf{C}^2, 0)$ respectively such that $\sigma_0(t) = t,\ \tau_0(x,y) = (x,y)$ and $\tau_1(g(t)) = f(\sigma_1(t))$. This fact is a feature of the complex case and it is proved by using the fact that $\mathrm{SL}(2,\mathbf{C})$ is arcwise connected and the group of symplectomorphisms with identity linear part is arcwise connected (cf. [6]). Thus our symplectic moduli space in Tables 1 and 2 are also moduli spaces for the symplectic isotopy equivalence.

3 Differential normal forms.

The proof of Theorem 1.1 is similar to the one in [3]. We note that the symplectic normal forms (Proposition 2.2) can play the role of an intermediate classification, which also makes the diffeomorphic classification easier and clearer.

First we have

Lemma 3.1 *Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ have the Puiseux characteristic $(m, \beta_1, \dots)$. If $m \geq 4$ and $\beta_1 \geq 13$, or $m \geq 5$ and $\beta_1 \geq 9$, or $m \geq 6$ then the modality of f is at least 2.*

Proof: For instance, assume $m = 4,\ \beta_1 = 13$. Then, in any neighborhood of f, there exists a two-parameter family of germs at 0 which are diffeomorphic to $g_\lambda = (t^4, t^{13} + t^{14} + \lambda_1 t^{15} + \lambda_2 t^{19})$. We find this family by an infinitesimal calculation: First, for each $\alpha \in \mathbf{N}$, we try to find $\xi \in \mathcal{O}_1$ and $\eta_1, \eta_2 \in \mathcal{O}_2$ with $\xi(0) = 0, \eta_1(0,0) = 0, \eta_2(0,0) = 0$, satisfying the equation

$$\begin{pmatrix} 0 \\ t^\alpha \end{pmatrix} \equiv \begin{pmatrix} 4t^3\xi \\ (13t^{12} + \cdots)\xi \end{pmatrix} + \begin{pmatrix} \eta_1(t^4, t^{13} + \cdots) \\ \eta_2(t^4, t^{13} + \cdots) \end{pmatrix} \mod \begin{pmatrix} 0 \\ t^{\alpha+1}\mathcal{O}_1 \end{pmatrix}.$$

Then we see that the equation is not solvable for $\alpha = 15$ and $\alpha = 19$. Second, by a formal calculation, we verify that g_λ and $g_{\lambda'}$ are diffeomorphic if and only if $\lambda' = \lambda$. From this observation we see that, if $m \geq 4,\ \beta_1 \geq 13$, then the modality of f is ≥ 2. Other cases can be treated in a similar way. $\square$

Thus Theorem 1.1 will be proved if we check all remaining cases. Here we will treat only the class W_{30} with the Puiseux characteristic $(4, 11)$.

Consider the symplectic normal form $f_\lambda(t) = (t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17} + \lambda_4 t^{21} + \lambda_5 t^{25})$. Suppose $\lambda_1 \neq 0$. Consider, for given $\rho(t)$, the equation

$$\begin{pmatrix} 0 \\ \rho(t) \end{pmatrix} = \xi \begin{pmatrix} 4t^3 \\ 11t^{10} + 13\lambda_1 t^{12} + \cdots \end{pmatrix} + \begin{pmatrix} \eta_1(f_\lambda(t)) \\ \eta_2(f_\lambda(t)) \end{pmatrix},$$

and try to find $\xi(t), \eta_1(x, y), \eta_2(x, y)$ with $\xi(0) = 0,\ \eta_1(0, 0) = 0,\ \eta_2(0, 0) = 0$. The equation is solvable for $\rho(t) = t^{13}$, up to higher order terms, and solvable for any $\rho(t)$ with $\mathrm{ord}\rho(t) \geq 15$. Then, by the homotopy method, we see that, if $\lambda_1 \neq 0$, then f is diffeomorphic to $(t^4, t^{11} + t^{13} + \lambda t^{14})$ for some $\lambda \in \mathbf{C}$. If $\lambda_1 = 0, \lambda_2 \neq 0$, then f is diffeomorphic to $(t^4, t^{11} + t^{14} + \lambda t^{17})$ for some $\lambda \in \mathbf{C}, \lambda \neq \frac{25}{22}$, or to $(t^4, t^{11} + t^{14} + \frac{25}{22} t^{17} + \lambda t^{21})$ for some $\lambda \in \mathbf{C}$. If $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 \neq 0$ (resp. $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 \neq 0$; $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 \neq 0$), then f is diffeomorphic to $(t^4, t^{11} + t^{17})$ (resp. $(t^4, t^{11} + t^{21}); (t^4, t^{11} + t^{25})$). The exact determination of the moduli space is completed by direct formal calculations. The other cases are classified in a similar way.

Remark 3.2 In general, for each equi-singularity class, the symplectic moduli space is mapped canonically onto the differential moduli space, i.e. the ordinary moduli space. The dimension of the fiber over a diffeomorphism class $[f]$ is called the *symplectic defect* and denoted by $\mathrm{sd}(f)$ in [7]. It is known that $\mathrm{sd}(f) = \mu(f) - \tau(f)$, where $\mu(f) = 2\delta(f)$ is the Milnor number of f and $\tau(f)$ is the Tyurina number of f ([15][9][4]). Let $s(f)$ (resp. $c(f)$) be the symplectic modality, that is, the number of parameters in the symplectic normal form of f (resp. the codimension of the locus in the parameter space corresponding to germs diffeomorphic to f). Then $s(f) - c(f) = \mathrm{sd}(f)$. Thus we have the formula for the Tyurina number (by means of Varchenko-Lando's formula) as

$$\tau(f) = 2\delta(f) + c(f) - s(f).$$

For example, for $f = (t^4, t^{11} + t^{21})$ in the case of W_{30}, we have $\delta(f) = 15,\ c(f) = 3,\ s(f) = 5$ and in fact $\tau(f) = 28$.

Note that the differential moduli space is not a Hausdorff space, while the symplectic moduli space is, at least for 0-modal and 1-modal cases, as we clearly observe in Theorems 1.1 and 1.3. Therefore the symplectic moduli space can be called a *Hausdorffication* of the differential moduli space.

Remark 3.3 The adjacency of simple and uni-modal singularities of parametric plane curves is generated (as an ordering) by $A_{2\ell} \leftarrow A_{2\ell+2}, E_{6\ell} \leftarrow E_{6\ell+2} \leftarrow E_{6\ell+6}$ $(\ell = 1, 2, \dots), A_{6s-2} \leftarrow E_{12s-6}, A_{6s} \leftarrow E_{12s}, A_{6s-2} \leftarrow E_{12s-4}, A_{6s+2} \leftarrow E_{12s+2}$ $(s = 1, 2, \dots), E_8 \leftarrow W_{12} \leftarrow W_{18}, W_{12} \leftarrow W^{\#}_{1,1}, E_{12} \leftarrow W^{\#}_{1,1} \leftarrow W_{18}, W^{\#}_{1,2\ell-1} \leftarrow W^{\#}_{1,2\ell+1}$ $(\ell = 1, 2, \dots), W^{\#}_{1,1} \leftarrow N_{20} \leftarrow N_{24} \leftarrow N_{28}, W_{18} \leftarrow N_{24}, W_{24} \leftarrow N_{28}, W_{18} \leftarrow W_{24} \leftarrow W_{30}, E_{18} \leftarrow W_{24} \leftarrow W^{\#}_{2,1}, E_{20} \leftarrow W_{30}, W^{\#}_{2,2\ell-1} \leftarrow W^{\#}_{2,2\ell+1}$ $(\ell = 1, 2, \dots)$.

References

[1] V.I. Arnold, *First steps of local symplectic algebra*, Adv. in Soviet Math., D. Fuchs birthday volume, Providence, Amer. Math. Soc., 1999.

[2] V.I. Arnold, V.V. Goryunov, O.V. Lyashko, V.A. Vasil'ev, *Singularity Theory II: Classification and Applications,* Encyclopaedia of Math. Sci., Vol. 39, Dynamical System VIII, Springer-Verlag (1993).

[3] J.W. Bruce, T. Gaffney, *Simple singularities of mappings* $\mathbf{C}, 0 \to \mathbf{C}^2, 0$, J. London Math. Soc., **26** (1982), 465–474.

[4] W. Domitrz, S. Janeczko, M. Zhitomirskii, *Relative Pincaré lemma, contractibility, quasi-homogeneity and vector fields tangent to a singular variety,* Illinois J. Math. **48–3** (2004), 803–835.

[5] A. B. Givental', *Singular Lagrangian manifolds and their Lagrangian mappings*, Itogi Nauki Tekh., Ser. Sovrem. Prob. Mat., (Contemporary Problems of Mathematics) **33**, VINITI, 1988, pp. 55–112. English transl., J. Soviet Math. **52** (1990), no. 4, 3246–3278.

[6] G. Ishikawa, *Determinacy, transversality and Lagrange stability,* Banach Center Publ. **50** (1999), 123–135.

[7] G. Ishikawa, S. Janeczko, *Symplectic bifurcations of plane curves and isotropic liftings,* Quarterly J. Math., **54** (2003), 73–102.

[8] G. Ishikawa, S. Janeczko, *Symplectic singularities of isotropic mappings,* Geometric Singularity Theory, Banach Center Publications **65**, (2004), 85-106.

[9] S.K. Lando, *Normal forms of the degrees of a volume form,* Funct. Anal. Appl. **19–2** (1984), 146–148.

[10] Lê Dũng Tráng, *Sur un critère d'équisingularité,* C. R. Acad. Sci. Paris Sér. A, **272** (1971), 138–140.

[11] J. Milnor, *Singular Points of Complex Hypersurfaces,* Annals of Math. Studies, **61**, Princeton Univ. Press, 1986.

[12] R. Montgomery, M. Zhitomirskii, *Geometric approach to Goursat flags,* Ann. Inst. H. Poincaré, **18–4** (2001), 459–493.

[13] P. Mormul, *Real moduli in codimension 2 in local classification of Goursat flags,* Preprint.

[14] P. Orlik, L. Solomon, *Singularities. II. Automorphisms of forms,* Math. Ann. **231–3** (1977/78), 229–240.

[15] A.N. Varchenko, *Local classification of volume forms in the presence of a hypersurface,* Funct. Anal. Appl. **19–4** (1984), 269–276.

[16] C.T.C. Wall, *Finite determinacy of smooth map-germs,* Bull. London Math. Soc. **13** (1981), 481–539.

[17] C.T.C. Wall, *Singular Points of Plane Curves,* Cambridge Univ. Press (2004).

[18] O. Zariski, *Le problème des modules pour les branches planes,* Cours donné au Centre de mathématiques de l'École Polytechnique, 1973, (ed. F. Kmety, M. Merle, with an appendix of B. Tessier), Hermann, Paris (1987).

[19] M. Zhitomirskii, *Germs of integral curves in contact 3-space, plane and space curves,* Isaac Newton Inst. Preprint NI00043-SGT, December 2000.

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