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Abstract

A one-to-one correspondence is established between Fourier transforms of ultradistribution semigroups in the sense of Beurling and some class of pseudoresolvents characterized by conditions concerning their domains of existence and growth.

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1 Introduction

The Laplace transformation determines a one-to-one correspondence between the class $\mathcal{S}'_+(\xi_0, \infty; X)$ of distributions of exponential growth on $\mathbb{R}$ with values in a complex Banach space X and support in $\mathbb{R}^+$, and some set $\mathcal{H}_{\xi_0}(X)$ of X-valued functions holomorphic on an open right halfplane. This implies that if $\mathcal{A}$ is a Banach algebra, then each *distribution semigroup* belonging to $\mathcal{S}'_+(\xi_0, \infty; \mathcal{A})$ is the inverse Laplace transform of an $\mathcal{A}$-valued *pseudoresolvent* belonging to $\mathcal{H}_{\xi_0}(\mathcal{A})$. In the present paper it is proved that a similar representation by pseudoresolvents also holds for ultradistribution semigroups in the sense of Beurling. In this case the proof depends in an essential way on algebraic properties of ultradistribution semigroups and does not work for general ultradistributions with support in $\mathbb{R}^+$. Sections 1.1 and 1.2 describe the situation for $\mathcal{S}'_+(\xi_0, \infty; X)$ and distribution semigroups in $\mathcal{S}'_+(\xi_0, \infty; \mathcal{A})$. A comparison with these sections elucidates the main result concerning ultradistribution semigroups presented in Section 1.3, and discussed in Sections 1.4 and 1.5.

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1.1 Laplace transforms of tempered distributions with support in $\mathbb{R}^+$

Let $\mathcal{D} = C_c^\infty(\mathbb{R})$, $\mathbb{R}^+ = [0, \infty[$, $\mathcal{D}^+ = \{\mathbb{1}_{\mathbb{R}^+}\varphi : \varphi \in \mathcal{D}\}$ where $\mathbb{1}_{\mathbb{R}^+}$ denotes the characteristic function of the subset $\mathbb{R}^+$ of $\mathbb{R}$. Denote by $\mathcal{D}'(X)$ the space of distributions of L. Schwartz on $\mathbb{R}$ with values in a complex Banach space X, and let $\mathcal{D}'_+(X) = \{S \in \mathcal{D}'(X) : \operatorname{supp} S \subset \mathbb{R}^+\}$. The R. T. Seeley extension theorem [S] implies that:

$\mathcal{D}^+$ is equal to the set of functions φ defined on $\mathbb{R}$ such that $\varphi|_{]-\infty,0[} \equiv 0$ and $\varphi|_{\mathbb{R}^+} \in C_c^\infty(\mathbb{R}^+)$. Hence $\mathcal{D}^+$ may be endowed with the topology inherited from $C_c^\infty(\mathbb{R}^+)$. (1.1)

A distribution $S \in \mathcal{D}'(X)$ belongs to $\mathcal{D}'_+(X)$ if and only if there is a (unique) continuous linear operator $S^+ \in L(\mathcal{D}^+, X)$ such that $S(\varphi) = S^+(\mathbb{1}_{\mathbb{R}^+}\varphi)$ for every $\varphi \in \mathcal{D}$. (1.2)

For any $\lambda \in \mathbb{C}$ denote by $e_{-\lambda}$ the exponential function

$$e_{-\lambda}(t) = \exp(-\lambda t), \qquad t \in \mathbb{R}.$$

Let $\mathcal{S}'(X)$ be the space of tempered distributions on $\mathbb{R}$ with values in X. For any $\xi_0 \in \mathbb{R}$ define

$$\mathcal{S}'_+(\xi_0, \infty; X) = \{S \in \mathcal{D}'_+(X) : e_{-\xi}S \in \mathcal{S}'(X) \text{ for every } \xi > \xi_0\}.$$

Denote by $\mathcal{H}_{\xi_0}(X)$ the set of X-valued functions holomorphic on the open halfplane $\mathbb{C}^+_{\xi_0} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \xi_0\}$ whose growth is at most polynomial on every closed halfplane contained in $\mathbb{C}^+_{\xi_0}$. If $S \in \mathcal{S}'_+(\xi_0, \infty; X)$, then the X-valued function

$$R : \mathbb{C}_{\xi_0} \ni \lambda \mapsto S(e_{-\lambda}) \in X \tag{1.3}$$

makes sense and is called the *Laplace transform* of S.

This definition is compatible with the one given in a more general setting in Chapter VIII of [Sch]. See also Sec. 7.4 of [H]. The following proposition is closely related to the statement in Remark 2 on p. 310 of [Sch].

Proposition 1.1. *The Laplace transformation $\mathcal{L}$ maps $\mathcal{S}'_+(\xi_0, \infty; X)$ in one-to-one manner onto $\mathcal{H}_{\xi_0}(X)$. If $R \in \mathcal{H}_{\xi_0}(X)$, then the inverse Laplace transform of R is the distribution $S = \mathcal{L}^{-1}R \in \mathcal{S}'_+(\xi_0, \infty; X)$ such that*

$$S(\varphi) = \frac{1}{2\pi}\int_{\xi-i\infty}^{\xi+i\infty} \widehat{\varphi}(i\lambda)\, d\lambda \quad \textit{for every } \varphi \in \mathcal{D} \textit{ and } \xi > \xi_0. \tag{1.4}$$

Here $\widehat{\varphi}(i\lambda) = \int_{-\infty}^{\infty} e^{\lambda t}\varphi(t)\, dt$ is the value of the Fourier–Laplace transform of $\varphi \in \mathcal{D}$ at $i\lambda$.

1.2 A corollary for tempered distribution semigroups

Let $*$ denote convolution on $\mathbb{R}$. From (1.1) it follows that $\mathcal{D}^+ * \mathcal{D}^+ \subset \mathcal{D}^+$, so that $\mathcal{D}^+$ is a convolution algebra. Let $\mathcal{A}$ be a complex Banach algebra. A distribution $S \in \mathcal{D}'_+(\mathcal{A})$ will be called a *distribution semigroup* if the operator $S^+ \in L(\mathcal{D}^+, X)$, $X = \mathcal{A}$, appearing in (1.2) is a homomorphism of the convolution algebra $(\mathcal{D}^+, *)$ into the Banach algebra $\mathcal{A}$ (i.e. $S^+(\phi * \psi) = S^+(\phi)S^+(\psi)$ for every $\phi, \psi \in \mathcal{D}^+$). The set of $\mathcal{A}$-valued distribution semigroups will be denoted by $\mathcal{D}'\mathcal{S}(\mathcal{A})$. Whenever $S \in \mathcal{S}'_+(\xi_0, \infty; \mathcal{A})$, $R \in \mathcal{H}_{\xi_0}(\mathcal{A})$ and $R = \mathcal{L}S$, then $S \in \mathcal{D}'\mathcal{S}(\mathcal{A})$ if and only if R is a *pseudoresolvent*. The last means that R satisfies on $\mathbb{C}^+_{\xi_0}$ the Hilbert equality

$$R(\lambda_1) - R(\lambda_2) = (\lambda_2 - \lambda_1)R(\lambda_1)R(\lambda_2) \quad \text{for every } \lambda_1, \lambda_2 \in \mathbb{C}^+_{\xi_0}.$$

The "if" may be deduced from (1.2) and Proposition 2.1 of the subsequent section. The "only if" is a consequence of the equality

$$\mathbb{1}_{\mathbb{R}^+}e_{-\lambda_1} - \mathbb{1}_{\mathbb{R}^+}e_{-\lambda_2} = (\lambda_2 - \lambda_1)(\mathbb{1}_{\mathbb{R}^+}e_{-\lambda_1}) * (\mathbb{1}_{\mathbb{R}^+}e_{-\lambda_2}).$$

Corollary 1.1. *The Laplace transformation $\mathcal{L}$ defined by* (1.1) *maps the set of $\mathcal{A}$-valued distribution semigroups belonging to $\mathcal{S}'_+(\xi_0, \infty; \mathcal{A})$ in a one-to-one manner onto the set of $\mathcal{A}$-valued pseudoresolvents belonging to $\mathcal{H}_{\xi_0}(\mathcal{A})$. The mapping inverse to $\mathcal{L}$ is determined by* (1.4).

1.3 Analogue of Corollary 1.1 for ultradistribution semigroups in the sense of A. Beurling

The purpose of the present paper is to prove an analogue of Corollary 1.1 for the class $\mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$-valued ultradistribution semigroups corresponding to any Beurling's test function space $\mathcal{D}_\omega = \{\varphi \in C_c(\mathbb{R}) : \widehat{\varphi}\exp(n\omega) \in L^1(\mathbb{R})$ for every $n \in \mathbb{N}\}$. Here $\widehat{}$ denotes the Fourier transformation, and ω is a non-negative continuous subadditive function defined on $\mathbb{R}$ such that

$$\omega(0) = 0, \qquad \sup_{x\in\mathbb{R}} \frac{\log(1 + |x|)}{1 + \omega(x)} < \infty,$$

and

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1 + x^2}\, dx < \infty. \tag{1.5}$$

By a theorem of Beurling, if ω is a non-negative continuous subadditive function on $\mathbb{R}$ such that $\omega(0) = 0$, then (1.5) holds if and only if $\mathcal{D}_\omega \neq \{0\}$, and these equivalent conditions imply that $\mathcal{D}_\omega$ admits partitions of unity. The conditions imposed on ω imply that $\mathcal{D}_\omega \subset C_c^\infty(\mathbb{R})$. If $\omega(x) = \log(1 + |x|)$, then $\mathcal{D} = C_c^\infty(\mathbb{R})$. If $\omega(x) \equiv |x|^{1/s}$, $s = \text{const} > 1$, then $\mathcal{D}_\omega$ is equal to the M. Gevrey space $\mathcal{D}^{(n^{ns})}$.

In the definition of the ultradistribution semigroup in the sense of Beurling we use the convolution algebra $\mathcal{D}^+_\omega = \{\mathbb{1}_{\mathbb{R}^+}\varphi : \varphi \in \mathcal{D}_\omega\}$, an analogue of $\mathcal{D}^+$ from Sections 1.1 and 1.2. The formula (1.3) does not make sense for general distributions, and we do not try to generalize Proposition 1.1. However, if $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$, then, by means of a special construction going back to

J. Chazarain [C], it is possible to determine an $\mathcal{A}$-valued pseudoresolvent R representing the Laplace transform of S. For this pseudoresolvent R there are $\mathfrak{a}, \mathfrak{b} \geq 0$ such that

$$R \text{ is defined on a set containing } \Lambda_{\mathfrak{a}\omega+\mathfrak{b}} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \mathfrak{a}\omega(-\operatorname{Im}\lambda) + \mathfrak{b}\} \quad (1.6)$$

and R satisfies on $\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$ the growth condition: there is $\kappa \in \mathbb{R}$ such that

$$\sup_{\lambda \in \Lambda_{\mathfrak{a}\omega+\mathfrak{b}}} \|R(\lambda)\|_{\mathcal{A}} \exp(-\varepsilon \operatorname{Re}\lambda - \kappa\omega(-\operatorname{Im}\lambda)) < \infty \quad \text{for every } \varepsilon > 0. \quad (1.7)$$

Conversely, if an $\mathcal{A}$-valued pseudoresolvent R satisfies (1.6) and (1.7) for some $\mathfrak{a}, \mathfrak{b} \geq 0$, then, by means of a formula similar to (1.2) but involving an integral along a path running inside $\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$, R determines an ultradistribution semigroup $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$ which will be called the *inverse Laplace transform* of R. The correspondence between the ultradistribution semigroups S and maximal pseudoresolvents R satisfying (1.6) and (1.7) is one-to-one.

The sets $\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$ will be called *ω-regions*. For the first time they appeared in connection with distribution semigroups of normal operators in a Hilbert space in the paper of C. Foiaş [Fo], where $\omega(x) \equiv \log(1+|x|)$, and then in the paper of E. Larsson [La], where $\omega(x) \equiv |x|^{1/s}$, $s = \text{const} > 1$. In connection with applications of the inverse Laplace transformation to differential equations in Banach spaces, the ω-regions appeared later in the articles of J. Chazarain [C] and R. Beals [Be].

1.4 Operator-valued distribution semigroups

If $\mathcal{A} = L(X)$ is the Banach algebra of continuous linear operators on a complex Banach space X, then every distribution semigroup $S \in \mathcal{D}'_\omega\mathcal{S}(L(X))$ is uniquely determined by its generator, every maximal $L(X)$-valued pseudoresolvent R is uniquely determined by its generator, and S is the inverse Laplace transform of R if and only if the generator of S is equal to the generator of R. The details are similar to those presented in [K] for $\omega(x) \equiv \log(1+|x|)$. If $S \in \mathcal{D}'_\omega\mathcal{S}(L(X))$, $\mathfrak{I} = \{\sum_{k=1}^n S(\varphi_k)x_k : n \in \mathbb{N},\ \varphi_k \in \mathcal{D}_\omega,\ x_k \in X$ for $k = 1, \ldots, n\}$ and $\mathcal{N} = \{x \in X : S(\varphi)x = 0$ for every $\varphi \in \mathcal{D}_\omega\}$, then the *generator* of S is defined as the algebraic linear operator $G : \mathfrak{I} \to X/\mathcal{N}$ such that

$$GS(\varphi)x = -S(D\varphi)x - \varphi(0)x + \mathcal{N} \quad \text{for every } \varphi \in \mathcal{D}_\omega \text{ and } x \in X.$$

Correctness of this definition is a consequence of Proposition 2.1.

If R is an $L(X)$-valued pseudoresolvent, then the operators $R(\lambda)$ have range $\mathfrak{I}$ and null space $\mathcal{N}$ independent of λ and the *generator* of R is defined as the linear operator $G : \mathfrak{I} \to X/\mathcal{N}$ such that, for some λ,

$$GR(\lambda)x = \lambda R(\lambda)x - x + \mathcal{N} \quad \text{for every } x \in X.$$

From the Hilbert equality it follows that this definition is independent of the choice of λ. *Theorem* 2.1 *yields a characterization of the generator of a distribution semigroup* $S \in \mathcal{D}^+_\omega\mathcal{S}(L(X))$ *as the generator of an* $L(X)$*-valued pseudoresolvent which satisfies* (1.6) *and* (1.7) *for some* $\mathfrak{a} > 0$ and $\mathfrak{b} \in \mathbb{R}$. The above result extends Theorem 6.1 from the pioneering paper [L] of J.-L. Lions and Theorem 5.1 from [C] to distribution semigroups which, as in [W] and [Ku], need not satisfy the denseness assumption (iv) from Definition 1.1 in [L], nor the "non-degeneracy" assumption (v) from that definition. The connection between "degenerate" distribution semigroups (for which $\mathcal{N} \neq \{0\}$) and "degenerate" differential equations in Banach spaces is briefly explained in Section 6.

1.5 Indispensability of the condition (1.5)

By Beurling's Lemma I in [B, p. 16], whenever ω is a non-negative continuous subadditive function defined on $\mathbb{R}$, vanishing at zero and satisfying (1.5), then there is a function $\tilde{\omega}$ satisfying the same conditions (including (1.5)) such that $\omega(x) \leq \tilde{\omega}(x)$ for every $x \in \mathbb{R}$, $\tilde{\omega}$ is even, and $\tilde{\omega}|_{\mathbb{R}^+}$ is concave. In the present paper the condition (1.5) appears in connection with the framework of Beurling's ultradistributions and, by means of (1.6) and (1.7), imposes restrictions on the pseudoresolvents R. In particular (1.5) & (1.6) says that the ω-region $\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$ in which R has to exist must be a "sufficiently wide" subset of the halfplane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \mathfrak{b}\}$. This last restriction on R is indispensable in possible theorems similar to the one discussed in Section 1.3, whether or not the Beurling ultradistributions are used. Namely an example of a homogeneous ACP without non-trivial solutions given in Theorem 2 of the paper [Be] of R. Beals implies the following

Corollary 1.2. *Suppose that* ω *is a non-negative, even, continuous function on* $\mathbb{R}$ *such that* $\omega|_{\mathbb{R}^+}$ *is concave and*

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2}\,dx = \infty.$$

Then there is a closed densely defined linear operator A *from a Hilbert space* X *into* X *whose resolvent* $R(\lambda) = (\lambda - A)^{-1} \in L(X)$ *exists in* $\Lambda_{\omega+\mathfrak{b}} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \omega(\operatorname{Im}\lambda) + \mathfrak{b}\}$ *for some* $\mathfrak{b} \in \,]0, \infty[$ *and satisfies the condition*

$$\sup_{\lambda \in \Lambda_{\omega+\mathfrak{b}}} \|R(\lambda)\|_{L(X)} \operatorname{Re}\lambda < \infty,$$

but A *is not the generator of a* $\mathbb{D}$*-distribution semigroup for any test function space* $\mathbb{D}$ *satisfying the conditions* (2.1)–(2.4) *from the next section.*

2 The setting and the main result

2.1 The convolution algebras $\mathbb{D}$ and $\mathbb{D}^+$

Let $\mathbb{D}$ be an l.c.v.s. whose elements are infinitely differentiable complex-valued functions on $\mathbb{R}$ with compact supports. For any $\varphi \in \mathbb{D}$ and $t \in \mathbb{R}$ denote by φ_t the translate of φ by t, that is, $\varphi_t(u) = \varphi(u+t)$, $u \in \mathbb{R}$. Let $\mathbb{R}^- =]-\infty, 0]$, $\mathbb{R}^+ = [0, \infty[$. Assume that:

$\mathbb{D}$ is sequentially complete and $\mathbb{D}_- = \{\varphi \in \mathbb{D} : \operatorname{supp}\varphi \subset \mathbb{R}^-\}$ is a closed subspace of $\mathbb{D}$, (2.1)

$\mathbb{D}$ is translation-invariant and for every $\varphi \in \mathbb{D}$ the $\mathbb{D}$-valued function $t \mapsto \varphi_t$ is continuous on $\mathbb{R}$, (2.2)

if $0 \neq K \subset U \subset \mathbb{R}$, K is compact and U is open, then there is a function $\varphi \in \mathbb{D}$ with values in $[0,1]$ such that $\varphi = 1$ on K and $\varphi = 0$ outside U, (2.3)

if $\varphi \in \mathbb{D}$ and $\operatorname{supp}\varphi \subset U_1 \cup \dots \cup U_k$ where $U_1, \dots, U_k$ are non-empty open subsets of $\mathbb{R}$, then there are $\varphi_1, \dots, \varphi_k \in \mathbb{D}$ such that $\operatorname{supp}\varphi_i \subset U_i$ for $i = 1, \dots, k$ and $\varphi = \varphi_1 + \dots + \varphi_k$. (2.4)

If $\mathbb{D}$ is an algebra with respect to pointwise multiplication, then (2.3) implies (2.4), by the argument in the proof of Theorem 1.4.4 in [H]. From (2.1) and (2.2) it follows that

$$\varphi * \psi = \int \varphi(u)\psi_{-u}\, du \qquad \text{for every } \varphi, \psi \in \mathbb{D},$$

where the integral of the $\mathbb{D}$-valued compactly supported continuous function $u \mapsto \varphi(u)\psi_{-u}$ is computed in the sense of Riemann. Hence $\mathbb{D}$ is a convolution algebra.

Examples of spaces $\mathbb{D}$ satisfying the conditions (2.1)–(2.4) are: the space $\mathcal{D} = C_c^\infty(\mathbb{R})$ of test functions of L. Schwartz, the test function spaces $\mathcal{D}_\omega$ of A. Beurling which will be discussed and applied in what follows, the spaces $\mathcal{D}^{\{M_n\}}$ of C. Roumieu type, and the spaces $\mathcal{D}^{(M_n)}$ of A. Beurling type used by I. Cioranescu in [Ci]. The spaces $\mathcal{D}^{\{M_n\}}$ and $\mathcal{D}^{(M_n)}$, both discussed in [Ko], consist of compactly supported functions ultradifferentiable in the Denjoy–Carleman sense. $\mathcal{D}$ is a particular case of $\mathcal{D}_\omega$. The M. Gevrey spaces $\mathcal{D}_{\{s\}} = \mathcal{D}^{\{n^{ns}\}}$ and $\mathcal{D}_{(s)} = \mathcal{D}^{(n^{ns})}$, $s = \text{const} > 1$, are particular cases of $\mathcal{D}^{\{M_n\}}$ and $\mathcal{D}^{(M_n)}$. At the same time, if $s > 1$ and $\omega(x) \equiv |x|^{1/s}$, then $\mathcal{D}_{(s)} = \mathcal{D}_\omega$.

Following Sheng Wang Wang [W] and P. C. Kunstmann [Ku], for any $\varphi, \psi \in \mathbb{D}$ put

$$\varphi *_0 \psi = \mathbb{1}_{\mathbb{R}^+}\varphi * \mathbb{1}_{\mathbb{R}^+}\psi - \mathbb{1}_{\mathbb{R}^-}\varphi * \mathbb{1}_{\mathbb{R}^-}\psi. \tag{2.5}$$

The assumptions (2.1) and (2.2) imply that if $\varphi, \psi \in \mathbb{D}$, then

$$\varphi *_0 \psi = \int_0^\infty [\varphi(u)\psi_{-u} - \psi(-u)\varphi_u]\, du \in \mathbb{D}, \tag{2.6}$$

so that

$$\mathbb{D} *_0 \mathbb{D} \subset \mathbb{D}. \tag{2.7}$$

Indeed, the integrand in (2.6) is a $\mathbb{D}$-valued function of u continuous on $\mathbb{R}$ and vanishing if $u \geq a = \max(0, \max \operatorname{supp} \varphi, -\min \operatorname{supp} \psi)$. Since $\mathbb{D}$ is a sequentially complete l.c.v.s., the integral in (2.6), taken in fact only over $[0, a]$ and computed in the sense of Riemann, makes sense and is equal to a function $\eta \in \mathbb{D}$ such that

$$\begin{aligned} \eta(t) &= \int_0^\infty \varphi(u)\psi(t-u)\,du - \int_0^\infty \psi(-u)\varphi(t+u)\,du \\ &= \left(\int_0^\infty - \int_t^\infty\right)\varphi(u)\psi(t-u)\,du = \int_0^t \varphi(u)\psi(t-u)\,du \\ &= (\mathbb{1}_{\mathbb{R}^+}\varphi * \mathbb{1}_{\mathbb{R}^+}\psi)(t) - (\mathbb{1}_{\mathbb{R}^-}\varphi * \mathbb{1}_{\mathbb{R}^-}\psi)(t) \quad \text{for every } t \in \mathbb{R}. \end{aligned}$$

Let

$$\mathbb{D}^+ = \{\mathbb{1}_{\mathbb{R}^+}\varphi : \varphi \in \mathbb{D}\}. \tag{2.8}$$

Since $\mathbb{1}_{\mathbb{R}^+}\varphi * \mathbb{1}_{\mathbb{R}^+}\psi = \mathbb{1}_{\mathbb{R}^+}(\varphi *_0 \psi)$, from (2.7) it follows that

$$\mathbb{D}^+ * \mathbb{D}^+ \subset \mathbb{D}^+. \tag{2.9}$$

The space $\mathbb{D}^+$ defined by (2.8) is algebraically isomorphic to $\mathbb{D}/\mathbb{D}_-$, which, by (2.1), is an l.c.v.s. equipped with the quotient topology. In the following $\mathbb{D}^+$ will be treated as an l.c.v.s. whose topology is inherited from $\mathbb{D}/\mathbb{D}_-$. The inclusion (2.9) means that $(\mathbb{D}^+, *)$ *is a convolution algebra*. Let $\mathbb{D}_0^+ = \{\varphi \in \mathbb{D} : \operatorname{supp} \varphi \subset \mathbb{R}^+\}$. Then $\mathbb{D}_0^+ \subset \mathbb{D}^+$ and $\mathbb{D}_0^+$ *is an ideal in* $(\mathbb{D}^+, *)$. Indeed, if $\varphi \in \mathbb{D}_0^+$ and $\psi \in \mathbb{D}$, then $\varphi * \mathbb{1}_{\mathbb{R}^+}\psi \in \mathbb{D}_0^+$, because $\varphi * \mathbb{1}_{\mathbb{R}^+}\psi = \varphi *_0 \psi \in \mathbb{D}$ and $\operatorname{supp}(\varphi * \mathbb{1}_{\mathbb{R}^+}\psi) \subset \mathbb{R}^+$.

2.2 Distribution semigroups

The space of continuous linear maps of $\mathbb{D}$ into a Banach space will be denoted by $\mathbb{D}'(X)$, and the maps belonging to $\mathbb{D}'(X)$ will be called *X-valued $\mathbb{D}$-distributions*. The condition (2.4) implies that every $S \in \mathbb{D}'(X)$ has a well defined *support*. Sometimes it is convenient to use the fact that, by (2.2) and (2.4), $S(\varphi) = 0$ whenever $S \in \mathbb{D}'(X)$, $\varphi \in \mathbb{D}$, and there is an open interval I (finite or infinite) such that $\operatorname{supp} S \subset \overline{I}$ and $\operatorname{supp} \varphi \subset \mathbb{R} \setminus I$.

There is a one-to-one map of $L(\mathbb{D}^+, X)$ onto $\{S \in \mathbb{D}'(X) : \operatorname{supp} S \subset \mathbb{R}^+\}$ which to any $S^+ \in L(\mathbb{D}^+, X)$ assigns the distribution $S \in \mathbb{D}'(X)$ such that $S(\varphi) = S^+(\mathbb{1}_{\mathbb{R}^+}\varphi)$ for every $\varphi \in \mathbb{D}$.

Let $\mathcal{A}$ be a *Banach algebra*. A distribution $S \in \mathbb{D}'(\mathcal{A})$ is called a *distribution semigroup* if

$$S(\varphi) = S^+(\mathbb{1}_{\mathbb{R}^+}\varphi) \quad \text{for every } \varphi \in \mathbb{D}, \tag{2.10}$$

where $S^+ \in L(\mathbb{D}^+, \mathcal{A})$ satisfies the condition

$$S^+(\phi)S^+(\psi) = S^+(\phi * \psi) \quad \text{for every } \phi, \psi \in \mathbb{D}^+. \tag{2.11}$$

The condition (2.11) makes sense thanks to (2.9) and means that S^+ is a homomorphism of the convolution algebra $(\mathbb{D}^+, *)$ into the Banach algebra $\mathcal{A}$. The set of $\mathcal{A}$-valued $\mathbb{D}$-distribution semigroups will be denoted by $\mathbb{D}'\mathcal{S}(\mathcal{A})$. From (2.5) it follows that a distribution $S \in \mathbb{D}'(\mathcal{A})$ is a distribution semigroup if and only if

$$\operatorname{supp} S \subset \mathbb{R} \quad \text{and} \quad S(\varphi)S(\psi) = S(\varphi *_0 \psi) \quad \text{for every } \varphi, \psi \in \mathbb{D}. \tag{2.12}$$

Let D denote the differentiation: $(D\varphi)(t) = \frac{d}{dt}\varphi(t)$ for every $\varphi \in \mathbb{D}$ and $t \in \mathbb{R}$.

Proposition 2.1. *Suppose that $\mathcal{A}$ is a Banach algebra and $S \in \mathbb{D}'(\mathcal{A})$. Then $S \in \mathbb{D}'\mathcal{S}(\mathcal{A})$ if and only if*

$$\operatorname{supp} S \subset \mathbb{R}^+ \quad \textit{and} \quad S(D\varphi)S(\psi) + \varphi(0)S(\psi) = S(\varphi)S(D\psi) + \psi(0)S(\varphi) \tag{2.13}$$

for every $\varphi, \psi \in \mathbb{D}$.

Proof. If $S \in \mathbb{D}'(\mathcal{A})$, then (2.13) is a consequence of (2.12) and the equalities $D\varphi *_0 \psi + \varphi(0)\psi = D(\varphi *_0 \psi) = \varphi *_0 D\psi + \psi(0)\varphi$. Conversely, suppose that (2.13) holds. Fix any $\varphi, \psi \in \mathbb{D}$ and let $a = \max(0, \max \operatorname{supp} \varphi, -\min \operatorname{supp} \psi)$. By (2.6), one has

$$\begin{aligned} S(\varphi *_0 \psi) &= S\left(\int_0^a [\varphi(u)\psi_{-u} - \psi(-u)\varphi_u]\, du \right) \\ &= \int_0^a [\varphi_u(0) S(\psi_{-u}) - \psi_{-u}(0) S(\varphi_u)]\, du. \end{aligned}$$

Applying (1.13) to φ_u and ψ_{-u} one concludes that

$$\begin{aligned} S(\varphi *_0 \psi) &= \int_0^a [S(\varphi_u)S(D\psi_{-u}) - S(D\varphi_u)S(\psi_{-u})]\, du \\ &= -\int_0^a \frac{d}{du}[S(\varphi_u)S(\psi_{-u})]\, du = S(\varphi)S(\psi) - S(\varphi_a)S(\psi_{-a}). \end{aligned}$$

Since $\operatorname{supp} \varphi_a = \operatorname{supp} \varphi - a \subset \mathbb{R}^-$, one has $S(\varphi_a) = 0$, and hence $S(\varphi *_0 \psi) = S(\varphi)S(\psi)$. □

2.3 The test function spaces $\mathcal{D}_\omega$ of A. Beurling

Throughout this paper the one-dimensional Fourier transformations $\mathcal{F}$ and $\mathcal{F}^{-1}$ are defined by

$$\mathcal{F}\varphi(x) = \int_{-\infty}^{\infty} e^{-itx}\varphi(t)\, dt, \qquad \mathcal{F}^{-1}f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx} f(x)\, dx.$$

Consider the spaces $L^1(\mathbb{R})$ and $C_0(\mathbb{R})$ of complex-valued functions. For any non-negative continuous function ω on $\mathbb{R}$ define

$$\mathcal{L}_\omega = \{f \in L^1(\mathbb{R}) : f e^{n\omega} \in L^1(\mathbb{R}) \text{ for } n = 1, 2, \ldots\}, \qquad \mathcal{A}_\omega = \mathcal{F}^{-1}\mathcal{L}_\omega.$$

For any $\varphi \in \mathcal{A}_\omega$ denote by $\widehat{\varphi}$ the unique element of $\mathcal{L}_\omega$ such that $\varphi = \mathcal{F}^{-1}\widehat{\varphi}$. Equipped with the topology determined by the system of norms

$$\|\varphi\|_{n\omega} = \int_{-\infty}^{\infty} |\widehat{\varphi}(x)| \exp(n\omega(x))\, dx, \quad n = 0, 1, \ldots, \tag{2.14}$$

$\mathcal{A}_\omega$ is a Fréchet space, densely and continuously embedded in $C_0(\mathbb{R})$. If in addition ω is subadditive, i.e.

$$\omega(x_1 + x_2) \le \omega(x_1) + \omega(x_2) \quad \text{for every } x_1, x_2 \in \mathbb{R},$$

then $\mathcal{L}_\omega$ is a convolution algebra, and hence $\mathcal{A}_\omega$ is an algebra with respect to pointwise multiplication.

Theorem of Beurling ([B, Theorem I]; [Bj, Theorem 1.3.7]) *Suppose that*

ω is a non-negative continuous subadditive function on $\mathbb{R}$ such that $\omega(0) = 0$. (α)

Then

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1 + x^2}\, dx < \infty \tag{β}$$

if and only if the condition (2.3) *is satisfied with $\mathcal{D}_\omega$ in place of $\mathbb{D}$.*

Henceforth we assume that ω satisfies (α) and (β). For any compact interval $[a, b] \subset \mathbb{R}$ let

$$\mathcal{D}_\omega[a, b] = \{\varphi \in \mathcal{A}_\omega : \operatorname{supp} \varphi \subset [a, b]\}.$$

Then $\mathcal{D}_\omega[a, b] \subset C_c[a, b]$, and if $-\infty < a < b < \infty$, then $\mathcal{D}_\omega[a, b] \ne \{0\}$ by Beurling's theorem. Let

$$\mathcal{D}_\omega = \bigcup_{-\infty < a < b < \infty} \mathcal{D}_\omega[a, b].$$

Then $\mathcal{D}_\omega \subset C_c(\mathbb{R})$, and since each $\mathcal{D}_\omega[a, b]$ is a closed subalgebra of $\mathcal{A}_\omega$, it follows that, equipped with the inductive topology, $\mathcal{D}_\omega$ *is an* LF-*space and an algebra with respect to pointwise multiplication*. Since (2.3) holds for $\mathbb{D} = \mathcal{D}_\omega$, by an argument mentioned in Section 2.1 it follows that $\mathcal{D}_\omega$ *admits partitions of unity* in the sense that (2.4) holds for $\mathbb{D} = \mathcal{D}_\omega$. Furthermore, by another argument of Section 2.1, $\mathcal{D}_\omega$ *is a convolution algebra*, because $\mathbb{D} = \mathcal{D}_\omega$ satisfies (2.1) and (2.2). Pointwise multiplication and convolution are both continuous bilinear maps of $\mathcal{D}_\omega \times \mathcal{D}_\omega$ into $\mathcal{D}_\omega$.

By Theorem 1.3.18 of G. Björck's paper [Bj], whenever both ω and $\widetilde{\omega}$ satisfy (α) and (β), then $\sup_{x\in\mathbb{R}} \frac{\widetilde{\omega}(x)}{1+\omega(x)} < \infty$ if and only if $\mathcal{D}_\omega$ is densely and continuously embedded in $\mathcal{D}_{\widetilde{\omega}}$. If $\omega(x) = \log(1 + |x|)$, then ω satisfies (α) and (β), and $\mathcal{D}_\omega$ is equal to the L. Schwartz test function space $\mathcal{D} = C_c^\infty(\mathbb{R})$. The condition

$$\sup_{x\in\mathbb{R}} \frac{\log(1 + |x|)}{1 + \omega(x)} < \infty \tag{γ}$$

holds if and only if $\mathcal{D}_\omega$ is densely and continuously embedded in $\mathcal{D}$.

In the following, $\Omega(\alpha, \beta)$ and $\Omega(\alpha, \beta, \gamma)$ will denote the sets of functions ω defined on $\mathbb{R}$ and satisfying respectively the Beurling conditions (α) and (β) or all the three conditions (α), (β) and (γ). If $\omega \in \Omega(\alpha, \beta, \gamma)$, then $\mathbb{D} = \mathcal{D}_\omega$ satisfies the assumptions of Section 2.1, and all the statements of Sections 2.1 and 2.2 remain valid for $\mathcal{D}_\omega$.

2.4 The M. Gevrey test function spaces $\mathcal{D}_{(s)}$

If $\omega(x) \equiv |x|^{1/s}$ where $s = \text{const} > 1$, then $\omega \in \Omega(\alpha, \beta, \gamma)$. The corresponding $\mathcal{D}_\omega$ will be denoted by $\mathcal{D}_{(s)}$. It follows from [H, Lemma 12.7.4] that if $-\infty < a < b < \infty$ and $s > 1$, then

$$\mathcal{D}_{(s)}[a,b] = \left\{ \varphi \in \mathcal{D} : \operatorname{supp} \varphi \subset [a,b], \sup_{t \in [a,b],\, n \in \mathbb{N}} \frac{|D^n \varphi(t)|}{n^{ns} \varepsilon^n} < \infty \text{ for all } \varepsilon > 0 \right\}.$$

In the terminology of [EDM 2, p. 650] this equality means that $\mathcal{D}_{(s)}[a,b]$ coincides with the set of those M. Gevrey functions of the class (s) whose supports are contained in $[a,b]$.

2.5 The Fourier transforms of elements of $\mathcal{D}_\omega$ *and* $\mathcal{D}'_\omega(X)$

For any compact interval $[a,b] \subset \mathbb{R}$ denote its *support function* by $H_{[a,b]}$:

$$H_{[a,b]}(y) = \sup_{t \in [a,b]} ty = \begin{cases} ay & \text{if } y < 0, \\ by & \text{if } y \geq 0. \end{cases}$$

Let $\omega \in \Omega(\alpha, \beta, \gamma)$. By the equivalence (ii)$\Leftrightarrow$(iii) in Theorem 1.4.1 of [Bj], proved in the present paper as Corollary 3.8, if $-\infty < a < b < \infty$, then the Fourier transformation is an isomorphism of $\mathcal{D}_\omega[a,b]$ onto the space $\mathcal{Z}_\omega[a,b]$ of entire functions f such that

$$\sup_{x+iy \in \mathbb{C}} |f(x+iy)| \exp(n\omega(x) - H_{[a,b]}(y) - \varepsilon|y|) < \infty \quad \text{for every } n \in \mathbb{N} \text{ and } \varepsilon > 0.$$

Consequently, the Fourier transformation is an isomorphism of the union $\mathcal{D}_\omega = \bigcup_{-\infty<a<b<\infty} \mathcal{D}_\omega[a,b]$ onto $\mathcal{Z}_\omega = \bigcup_{-\infty<a<b<\infty} \mathcal{Z}_\omega[a,b]$.

Let X be a complex Banach space. The space $\mathcal{D}'_\omega(X)$ of X-valued $\mathcal{D}_\omega$-distributions of Beurling is defined as the space of continuous linear maps of $\mathcal{D}_\omega$ into X. Let $^\vee$ denote the reflection in 0. The Fourier transform S of a distribution $S \in \mathcal{D}'_\omega(X)$ is defined as the unique linear map $\widehat{S} : \mathcal{Z}_{\omega^\vee} \to X$ satisfying the Parseval equality

$$S(\varphi) = \frac{1}{2\pi} \widehat{S}(\widehat{\varphi}^\vee) \quad \text{for every } \varphi \in \mathcal{D}_\omega.$$

Let $\mathcal{A}$ be an algebra over a commutative ring $\mathbb{K}$. A map R of a non-empty subset $D(R)$ of $\mathbb{K}$ into $\mathcal{A}$ is called a *pseudoresolvent* if it satisfies the Hilbert equality

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) \quad \text{for every } \lambda, \mu \in D(R).$$

An $\mathcal{A}$-valued pseudoresolvent is called *maximal* if it has no proper extension to an $\mathcal{A}$-valued pseudoresolvent.

Proposition 2.2. *Every $\mathcal{A}$-valued pseudoresolvent extends uniquely to a maximal $\mathcal{A}$-valued pseudoresolvent.*

Proof. Consider the binary relations $\rightarrow$ and $\leftrightarrow$ on $\mathbb{K} \times \mathcal{A}$ defined as follows: whenever $(\lambda, A), (\mu, B) \in \mathbb{K} \times \mathcal{A}$, then

$$(\lambda, A) \rightarrow (\mu, B) \quad \text{means that} \quad A - B = (\mu - \lambda)AB$$

and

$$(\lambda, A) \leftrightarrow (\mu, B) \quad \text{means that} \quad (\lambda, A) \rightarrow (\mu, B) \text{ and } (\mu, B) \rightarrow (\lambda, A).$$

(It follows that $(\lambda, A) \leftrightarrow (\mu, B)$ if and only if $A - B = (\mu - \lambda)AB$ and $AB = BA$.) Suppose that $(\lambda, A) \rightarrow (\mu, B)$ and $(\mu, B) \rightarrow (\nu, C)$. Then

$$B = A + (\lambda - \mu)AB = C + (\nu - \mu)BC,$$

whence

$$\begin{aligned} A - C &= (\mu - \lambda)AB + (\nu - \mu)BC \\ &= (\mu - \lambda)A[C + (\nu - \mu)BC] + (\nu - \mu)[A + (\lambda - \mu)AB]C \\ &= (\nu - \lambda)AC, \end{aligned}$$

so that $(\lambda, A) \rightarrow (\nu, C)$. Thus the relation $\rightarrow$ is transitive, and hence so is $\leftrightarrow$. Since obviously $\leftrightarrow$ is reflexive and symmetric, it follows that it is an equivalence relation. Therefore $\mathbb{K} \times \mathcal{A}$ splits into the equivalence classes of $\leftrightarrow$. Proposition 2.2 is a consequence of two obvious facts:

1° the graph of every $\mathcal{A}$-valued pseudoresolvent is contained in some equivalence class of $\leftrightarrow$,

2° a subset of $\mathbb{K} \times \mathcal{A}$ is an equivalence class of $\leftrightarrow$ if and only if it is the graph of a maximal $\mathcal{A}$-valued pseudoresolvent. □

If $\mathcal{A}$ is a Banach algebra, then Proposition 2.2 coincides with Theorem 5.8.6 of [H-P]. The proof given in [H-P] is non-elementary. An application of the C. Neumann series shows that every maximal pseudoresolvent, with values in a Banach algebra over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, is an analytic function defined on an open subset of $\mathbb{K}$.

Let $\mathcal{A}$ be a complex Banach algebra. Let $\omega \in \Omega(\alpha,\beta,\gamma)$. For any $\mathfrak{a},\mathfrak{b} \ge 0$ let

$$\Lambda_{\mathfrak{a}\omega+\mathfrak{b}} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \ge \mathfrak{a}\omega(-\operatorname{Im}\lambda) + \mathfrak{b}\}.$$

As in Section 2.6, for any $\mathcal{A}$-valued pseudoresolvent R denote by $D(R)$ the subset of $\mathbb{C}$ on which R is defined.

Definition 2.1. For any $\mathfrak{a},\mathfrak{b} \ge 0$, $\varepsilon > 0$ and $\kappa \in \mathbb{R}$ denote by $\mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A};\varepsilon,\kappa)$ the set of $\mathcal{A}$-valued pseudoresolvents R such that $\Lambda_{\mathfrak{a}\omega+\mathfrak{b}} \subset D(R)$ and

$$\sup_{\lambda\in\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}} \|R(\lambda)\|_{\mathcal{A}} \exp(-\varepsilon \operatorname{Re}\lambda - \kappa\omega(-\operatorname{Im}\lambda)) < \infty. \tag{2.15}$$

Let

$$\widetilde{\mathcal{R}}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A}) = \bigcap_{\varepsilon>0}\bigcup_{\kappa\in\mathbb{R}} \mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A};\varepsilon,\kappa), \qquad \mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A}) = \bigcup_{\kappa\in\mathbb{R}}\bigcap_{\varepsilon>0} \mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A};\varepsilon,\kappa),$$
$$\widetilde{\mathfrak{R}}_\omega(\mathcal{A}) = \bigcup_{\mathfrak{a},\mathfrak{b}\ge 0} \widetilde{\mathcal{R}}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A}), \qquad \mathfrak{R}_\omega(\mathcal{A}) = \bigcup_{\mathfrak{a},\mathfrak{b}\ge 0} \mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A}).$$

Definition 2.2. For any $\omega \in \Omega(\alpha,\beta,\gamma)$ such that $\omega^\vee = \omega$, and any $\mathfrak{a},\mathfrak{b} \ge 0$ and $\kappa \in \mathbb{R}$ denote by $\mathcal{R}^0_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A};\kappa)$ the set of $\mathcal{A}$-valued pseudoresolvents R such that $\Lambda_{\mathfrak{a}\omega+\mathfrak{b}} \subset D(R)$ and

$$\sup_{\lambda\in\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}} \|R(\lambda)\|_{\mathcal{A}} \exp(-\kappa\omega(|\lambda|)) < \infty.$$

Let

$$\mathcal{R}^0_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A}) = \bigcup_{\kappa\in\mathbb{R}} \mathcal{R}^0_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A};\kappa), \qquad \mathfrak{R}^0_\omega(\mathcal{A}) = \bigcup_{\mathfrak{a},\mathfrak{b}\ge 0} \mathcal{R}^0_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A}).$$

Proposition 2.3. *If $\omega \in \Omega(\alpha,\beta,\gamma)$, then $\widetilde{\mathfrak{R}}_\omega(\mathcal{A}) = \mathfrak{R}_\omega(\mathcal{A})$. If either $\omega(x) \equiv \log(1+|x|)$ or $\omega(x) \equiv |x|^{1/s}$, $s = \mathrm{const} > 1$, then $\widetilde{\mathfrak{R}}_\omega(\mathcal{A}) = \mathfrak{R}^0_\omega(\mathcal{A})$.*

Denote by $\mathfrak{R}^{\max}_\omega(\mathcal{A})$ the set of maximal pseudoresolvents belonging to $\mathfrak{R}_\omega(\mathcal{A})$, and by $\widetilde{\Omega}_\omega$ the set of real uniformly lipschitzian functions $\widetilde{\omega}$ on $\mathbb{R}$ such that $\widetilde{\omega} - \omega$ is non-negative and $\frac{\widetilde{\omega}}{1+\omega}$ is bounded on $\mathbb{R}$. For any $\mathfrak{a},\mathfrak{b} \ge 0$ and $\widetilde{\omega} \in \Omega_\omega$ denote by $\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}}$ the path $\mathbb{R} \ni x \mapsto x - i(\mathfrak{a}\widetilde{\omega}(-x) + \mathfrak{b}) \in \mathbb{C}$ directed by the natural order in $\mathbb{R}$.

Theorem 2.1. *Assume that $\mathcal{A}$ is a complex Banach algebra and $\omega \in \Omega(\alpha,\beta,\gamma)$. Then there is a one-to-one map of $\mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$ onto $\mathfrak{R}^{\max}_\omega(\mathcal{A})$ which to any distribution semigroup $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$ assigns the unique pseudoresolvent $R \in \mathfrak{R}^{\max}_\omega(\mathcal{A})$ such that*

$$\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}}} \|\widehat{\varphi}(-z)R(iz)\|_{\mathcal{A}}|dz| < \infty \quad \text{and} \quad S(\varphi) = \frac{1}{2\pi}\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}}} \widehat{\varphi}(-z)R(iz)\,dz \tag{2.16}$$

for every $\varphi \in \mathcal{D}_\omega$, $\widetilde{\omega} \in \widetilde{\Omega}_\omega$ and every $\mathfrak{a},\mathfrak{b} \ge 0$ such that $R \in \widetilde{\mathcal{R}}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A})$.

Equality (2.16) means that, in the sense of the definition formulated in Section 2.5, the map

$$\mathcal{Z}_{\omega^\vee} \ni f \mapsto \int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}} f(z)R(iz)\, dz \in \mathcal{A}$$

is the Fourier transform of the distribution semigroup $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$. One can rewrite (2.16) in the equivalent form

$$S(\varphi) = \frac{1}{2\pi i}\int_{i\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}} \widehat{\varphi}(i\lambda)R(\lambda)\, d\lambda, \qquad \varphi \in \mathcal{D}_\omega,$$

which resembles (1.4) and means that S is equal to the inverse Laplace transform of R.

3 Paley–Wiener estimations of Fourier transforms of Beurling test functions and distributions

A thorough exposition of the theory of Beurling test functions and distributions is presented in [Bj]. We limit ourselves to a concise selfcontained presentation of the Paley–Wiener type results needed in the following.

For any $y > 0$ denote by $\mathcal{P}_y$ the probability measure on $\mathbb{R}$ such that

$$\mathcal{P}_y(B) = \frac{y}{\pi}\int_B \frac{dx}{x^2+y^2}$$

for every Lebesgue measurable subset B of $\mathbb{R}$. Let $\mathcal{P}_0$ be the Dirac measure on $\mathbb{R}$. The family $(\mathcal{P}_y)_{y\geq 0}$ is a convolution semigroup of probability measures on $\mathbb{R}$ called the *Poisson semigroup.*

Proposition 3.1. *Let $-\infty < a < b < \infty$ and $f \in L^1(\mathbb{R})$. Then the following three conditions are equivalent:*

(i) *$\mathcal{F}^{-1}f$ vanishes outside $[a, b]$,*

(ii) *there is an entire function F such that $F|_\mathbb{R} = f$ a.e. on $\mathbb{R}$ and*

$$|F(x + iy)| \leq \exp([\mathcal{P}_{|y|} * \log|f|](x) + H_{[a,b]}(y)) \quad \text{for every } x + iy \in \mathbb{C}, \tag{3.1}$$

(iii) *there is an entire function F such that $F|_\mathbb{R} = f$ a.e. on $\mathbb{R}$ and*

$$\sup_{y\in\mathbb{R}}\left(\int_{-\infty}^{\infty} |F(x + iy)|\, dx\right)\exp(-H_{[a,b]}(y)) < \infty. \tag{3.2}$$

Note that $\log|f|$ appearing in (3.1) is a function with values in $[-\infty, \infty[$ continuous and bounded from above on $\mathbb{R}$, so that the convolution makes sense and defines a function of $z = x + iy$ with values in $[-\infty, \infty[$. It is understood that $\exp(-\infty) = 0$.

Proof. (i)⇒(ii). Suppose that $\varphi \in C(\mathbb{R})$, $\operatorname{supp}\varphi \subset [a, b]$ and $f = \widehat{\varphi}$ a.e. on $\mathbb{R}$. For any $z \in \mathbb{C}$ let $F(z) = \int_a^b e^{-itz}\varphi(t)\, dt$. Then F is an entire function such that $F|_{\mathbb{R}} = f$ a.e. on $\mathbb{R}$ and we have to prove that F satisfies (3.1). To this end notice first of all that

$$F(x+iy) = \widehat{\varphi}_y(x)\exp(H_{[a,b]}(y)) \quad \text{for every } x+iy \in \mathbb{C}, \tag{3.3}$$

where

$$\varphi_y(t) = \varphi(t)\exp(ty - H_{[a,b]}(y))$$

for every $t, y \in \mathbb{R}$. If $y \in \mathbb{R}$ is fixed, then $\varphi_y \in C(\mathbb{R})$, $\operatorname{supp}\varphi_y \subset [a,b]$ and $|\varphi_y| \le |\varphi|$. If $\varepsilon \in]0, \frac{1}{2}(b-a)]$ and $t \in [a+\varepsilon, b-\varepsilon]$, then

$$|\varphi_y(t)| \le |\varphi(t)|\exp(\inf_{s\in[a,b]}(t-s)y) \le |\varphi(t)|\exp(-\varepsilon|y|).$$

By the Lebesgue dominated convergence theorem, it follows that the map $\mathbb{R} \ni y \mapsto \varphi_y \in L^1(\mathbb{R})$ is continuous and $\lim_{|y|\to\infty}\|\varphi_y\|_{L^1(\mathbb{R})} = 0$. Consequently, the map $\mathbb{R} \ni y \mapsto \widehat{\varphi}_y \in C_0(\mathbb{R})$ is continuous and $\lim_{|y|\to\infty}\|\widehat{\varphi}_y\|_{C_0(\mathbb{R})} = 0$, which, by (3.3), implies that

$$\lim_{|z|\to\infty} F(z)\exp(-H_{[a,b]}(\operatorname{Im} z)) = 0. \tag{3.4}$$

The inequality (3.1) is an immediate consequence of the fact that both the entire functions

$$\Phi(z) = F(z)e^{ibz} \quad \text{and} \quad \Phi(z) = \overline{F(\overline{z})}e^{-iaz} \tag{3.5}$$

satisfy the inequality

$$\log|\Phi(x+iy)| \le (\mathcal{P}_y * \log|f|)(x) \quad \text{for every } x \in \mathbb{R} \text{ and } y \in \mathbb{R}^+. \tag{3.6}$$

In the proof of (3.6) we shall use (3.4), and we shall follow the idea indicated in the proof of Lemma 1.4.2 of [Bj]. By (3.4), for both the functions (3.5) one has

$$\lim_{\operatorname{Im} z\ge 0,\, |z|\to\infty} \Phi(z) = 0,$$

and hence $g(z) = \log|\Phi(z)|$ is a continuous function of $z \in \mathbb{C}$ with values in $[-\infty, \infty[$ such that

$$\lim_{\operatorname{Im} z\ge 0,\, |z|\to\infty} g(z) = -\infty. \tag{3.7}$$

Since Φ is holomorphic, the functions g and $g_k = \max(g, -k)$, $k = 1, 2, \ldots$, are subharmonic on $\mathbb{C}$. It follows that for every $k = 1, 2, \ldots$ the function

$$h_k(z) = [\mathcal{P}_y * g_k|_{\mathbb{R}}](x), \quad z = x+iy,\ y \ge 0,$$

is bounded and continuous on $\{\operatorname{Im} z \ge 0\}$ and harmonic in $\{\operatorname{Im} z > 0\}$, and the function $g - h_k$ with values in $[-\infty, \infty[$ is continuous on $\{\operatorname{Im} z \ge 0\}$ and subharmonic in $\{\operatorname{Im} z > 0\}$. Furthermore

$$g(x) - h_k(x) = g(x) - g_k(x) \le 0 \quad \text{for every } x \in \mathbb{R}$$

and, by (3.7),

$$\lim_{\operatorname{Im} z\geq 0,\,|z|\to\infty}(g(z)-h_k(z))=-\infty.$$

Hence from the maximum principle for subharmonic functions it follows that

$$g(z)-h_k(z)\leq 0 \quad \text{whenever } \operatorname{Im} z\geq 0 \text{ and } k=1,2,\ldots.$$

This means that

$$g(x+iy)\leq (\mathcal{P}_y * g_k|_{\mathbb{R}})(x) \quad \text{for every } x\in\mathbb{R},\ y\in\mathbb{R}^+ \text{ and } k=1,2,\ldots.$$

By the Lebesgue monotone convergence theorem, passing to the limit as $k\to\infty$, one concludes that

$$g(x+iy)\leq (\mathcal{P}_y * g|_{\mathbb{R}})(x) \quad \text{for every } x\in\mathbb{R} \text{ and } y\in\mathbb{R}^+,$$

so that (3.6) holds.

(ii)⇒(iii). If (ii) holds, then, by Jensen's inequality,

$$|F(x+iy)|\leq (\mathcal{P}_{|y|} * |f|)(x)\cdot\exp(H_{[a,b]}(y)) \quad \text{for every } x+iy\in\mathbb{C}.$$

This implies (3.2), because the operators $\mathcal{P}_{|y|}\,*,\ y\in\mathbb{R}$, are contractions in $L^1(\mathbb{R})$.

(iii)⇒(i). Suppose that (iii) holds and take any $c\in\,]0,\infty[$. Then, by Fubini's theorem,

$$\iint_{\substack{-\infty<x<\infty\\ -c\leq y\leq c}} |F(x+iy)|\,dx\,dy<\infty$$

and hence

$$\liminf_{x\to-\infty}\int_{-c}^{c}|F(x+iy)|\,dy=\liminf_{x\to\infty}\int_{-c}^{c}|F(x+iy)|\,dy=0. \tag{3.8}$$

By the Cauchy integral theorem, from (iii) and (3.8) it follows that

$$\int_{-\infty}^{\infty}e^{it(x+iy)}F(x+iy)\,dx=\int_{-\infty}^{\infty}e^{itx}f(x)\,dx \quad \text{for all } t,y\in\mathbb{R}. \tag{3.9}$$

Let

$$\varphi(t)=\mathcal{F}^{-1}f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{itx}f(x)\,dx, \qquad t\in\mathbb{R}.$$

Then $\varphi\in C_0(\mathbb{R})$ and we have to prove that $\operatorname{supp}\varphi\subset[a,b]$. From (3.9) and (3.2) it follows that there is $C\in\,]0,\infty[$ such that

$$|\varphi(t)|\leq\frac{e^{-ty}}{2\pi}\int_{-\infty}^{\infty}|F(x+iy)|\,dx\leq\frac{C}{2\pi}\exp(H_{[a,b]}(y)-ty)=r(t,y)$$

for every $t,y\in\mathbb{R}$. If $t\in\,]-\infty,a[$, then $\lim_{y\to\infty}r(t,y)=0$, and if $t\in\,]b,\infty[$, then $\lim_{y\to-\infty}r(t,y)=0$. Hence $\operatorname{supp}\varphi\subset[a,b]$.

Lemma 3.2 ([Bj, Lemma 1.3.11]). *If $\omega \in \Omega(\alpha,\beta)$, then for every $\delta \in]0,\infty[$ there is $C_\delta \in]0,\infty[$ such that*

$$|(\mathcal{P}_y * \omega)(x) - \omega(x)| \le \delta y + C_\delta \quad \textit{for every } x \in \mathbb{R} \textit{ and } y \in]0,\infty[. \tag{3.10}$$

Proof. By (β), there is $r \in]0,\infty[$ such that

$$\frac{1}{\pi}\left(\int_{-\infty}^{-r} + \int_r^\infty\right)\frac{\omega(u)}{u^2}\,du \le \delta.$$

Since $\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{du}{u^2+y^2} = 1$, it follows that

$$\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\omega(u)}{u^2+y^2}\,du \le \delta y + C_\delta, \quad \text{where} \quad C_\delta = \max_{-r\le u\le r}\omega(u).$$

This implies (3.10), because

$$(\mathcal{P}_y * \omega)(x) - \omega(x) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\omega(x-u)-\omega(x)}{u^2+y^2}\,dy$$

and $-\omega(u) \le \omega(x-u) - \omega(x) \le \omega(-u)$, by (α). □

Proposition 3.3. *If $\omega \in \Omega(\alpha,\beta)$, then for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there is $C_{n,\varepsilon} \in]0,\infty[$ such that*

$$\|e_{-iz}\varphi\|_{n\omega} \le C_{n,\varepsilon}\|\varphi\|_{n\omega}\exp(n\omega(-\operatorname{Re} z) + H_{[a,b]}(\operatorname{Im} z) + \varepsilon|\operatorname{Im} z|) \tag{3.11}$$

whenever $-\infty < a < b < \infty$, $\varphi \in \mathcal{D}_\omega[a,b]$, and $z \in \mathbb{C}$.

Proof. Fix $\omega \in \Omega(\alpha,\beta)$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $\delta = \varepsilon/n$. We will prove that (3.11) holds for $C_{n,\varepsilon} = \exp(nC_\delta)$ where $C_\delta \in]0,\infty[$ is the constant from Lemma 3.2. The equality

$$\widehat{e_{-iz}\varphi}(x) = \widehat{\varphi}(x+z)$$

implies that

$$\begin{aligned}\|e_{-iz}\varphi\|_{n\omega} &= \int_{-\infty}^{\infty}|\widehat{\varphi}(x+i\operatorname{Im} z)|\exp(n\omega(x-\operatorname{Re} z))\,dx \\ &\le \exp(n\omega(-\operatorname{Re} z))\int_{-\infty}^{\infty}|\widehat{\varphi}(x+i\operatorname{Im} z)|\exp(n\omega(x))\,dx.\end{aligned}$$

Therefore (3.11) will follow once it is shown that

$$\int_{-\infty}^{\infty}|\widehat{\varphi}(x+iy)|\exp(n\omega(x))\,dx \le C_{n,\varepsilon}\|\varphi\|_{n\omega}\exp(H_{[a,b]}(y) + \varepsilon|y|) \tag{3.12}$$

for any $-\infty < a < b < \infty$, $\varphi \in \mathcal{D}_\omega[a,b]$ and $y \in \mathbb{R}$. In the proof of (3.12) we repeat the argument which may be found on pp. 366–367 of [Bj]. Let

$\varphi \in \mathcal{D}_\omega[a,b]$ and $x, y \in \mathbb{R}$. Then, by (i)⇒(ii) of Proposition 3.1,

$$\begin{aligned}
|\widehat{\varphi}(x+iy)| \exp([\mathcal{P}_{|y|} * n\omega](x)) \\
&\leq \exp([\mathcal{P}_{|y|} * (\log |\widehat{\varphi}|_{\mathbb{R}}| + n\omega)](x) + H_{[a,b]}(y)) \\
&\leq \exp([\mathcal{P}_{|y|} * \log(|\widehat{\varphi}|_{\mathbb{R}}| \exp(n\omega))](x)) \cdot \exp(H_{[a,b]}(y)),
\end{aligned}$$

whence, by Jensen's inequality,

$$|\widehat{\varphi}(x+iy)| \exp([\mathcal{P}_{|y|} * n\omega](x)) \leq [\mathcal{P}_{|y|} * (|\widehat{\varphi}|_{\mathbb{R}}| \exp(n\omega)](x) \cdot \exp(H_{[a,b]}(y)).$$

By (3.10), it follows that

$$\begin{aligned}
C_{n,\varepsilon}^{-1} |\widehat{\varphi}(x+iy)| \exp(n\omega(x) - \varepsilon|y|) &\leq |\widehat{\varphi}(x+iy)| \exp(n\omega(x) - n\delta|y| - nC_\delta) \\
&\leq [\mathcal{P}_{|y|} * (|\widehat{\varphi}|_{\mathbb{R}}| \exp(n\omega))](x) \cdot \exp(H_{[a,b]}(y)),
\end{aligned}$$

and so

$$|\widehat{\varphi}(x+iy)| \exp(n\omega(x)) \leq C_{n,\varepsilon} [\mathcal{P}_{|y|} * (|\widehat{\varphi}|_{\mathbb{R}}| \exp(n\omega))](x) \cdot \exp(H_{[a,b]} + \varepsilon|y|).$$

Since the operators $\mathcal{P}_{|y|}*$ are contractions in $L^1(\mathbb{R})$, the last inequality implies (3.12), by integration with respect to x. □

Corollary 3.4. *If $\omega \in \Omega(\alpha,\beta)$, $-\infty < a < b < \infty$, and $\varphi \in \mathcal{D}_\omega[a,b]$, then $e_{-iz}\varphi$ is an entire $\mathcal{D}_\omega[a,b]$-valued function of z.*

For any $\omega \in \Omega(\alpha,\beta)$, $\varphi \in \mathcal{D}_\omega$ and $n \in \mathbb{N}$ put

$$|||\varphi|||_{n\omega} = \sup_{x\in\mathbb{R}} |\widehat{\varphi}(x)| \exp(n\omega(x)). \tag{3.13}$$

Proposition 3.5. (A) *If $\omega \in \Omega(\alpha,\beta)$, then for every $n \in \mathbb{N}$ and every $a, b \in \mathbb{R}$ with $a < b$ there is $C_{n,a,b} \in]0,\infty[$ such that*

$$|||\varphi|||_{n\omega} \leq C_{n,a,b} \|\varphi\|_{n\omega} \quad \textit{for every } \varphi \in \mathcal{D}_\omega[a,b].$$

(B) ([Bj, Proposition 1.3.26]) *If $\omega \in \Omega(\alpha,\beta,\gamma)$, then there are $m \in \mathbb{N}$ and $D \in]0,\infty[$ such that*

$$\|\varphi\|_{n\omega} \leq D |||\varphi|||_{(m+n)\omega} \quad \textit{for every } \varphi \in \mathcal{D}_\omega \textit{ and every } n \in \mathbb{N}.$$

Proof. (A) ([Bj, pp. 365–366]) Fix $r > 0$ and $\varepsilon > 0$. For every $\varphi \in \mathcal{D}_\omega[a,b]$ and $x \in \mathbb{R}$ one has

$$\widehat{\varphi}(x) = \frac{1}{\pi r^2} \iint_{|u+iv|\leq r} \widehat{\varphi}(x+u+iv)\, du\, dv,$$

because $\widehat{\varphi}$ is an entire function. Hence, by (3.12),

$$
\begin{aligned}
|\widehat{\varphi}(x)| \exp(n\omega(x)) \\
&\le \frac{1}{\pi r^2} \int_{-r}^{r} \left[\int_{-r}^{r} |\widehat{\varphi}(x+u+iv)| \exp(n\omega(x+u)) \exp(n\omega(-u))\, du \right] dv \\
&\le \frac{2}{\pi r} M(r) \max_{-r\le v\le r} \int_{-\infty}^{\infty} |\widehat{\varphi}(x+iv)| \exp(n\omega(x))\, dx \\
&\le \frac{2}{\pi r} M(r) N(r,a,b) C_{n,\varepsilon} \|\varphi\|,
\end{aligned}
$$

where $M(r) = \max_{-r\le u\le r} \exp(n\omega(-u))$, $N(r,a,b) = \max_{-r\le v\le r} \exp(H_{[a,b]}(v) + \varepsilon|v|)$.

(B) Let $K = \sup_{x\in\mathbb{R}} \frac{\log(1+|x|)}{1+\omega(x)}$. Then $K < \infty$, by (γ), and

$$
\int_{-\infty}^{\infty} \exp(-m\omega(x))\, dx \le e^m \int_{-\infty}^{\infty} (1+|x|)^{-m/K}\, dx < \infty
$$

whenever $m > K$. By (α), for any such m one has

$$
\|\varphi\|_{n\omega} \le |||\varphi|||_{(n+m)\omega} \int_{-\infty}^{\infty} \exp(-m\omega(x))\, dx
$$

for every $\varphi \in \mathcal{D}_\omega$ and $n \in \mathbb{N}$. □

Corollary 3.6. *If $\omega \in \Omega(\alpha,\beta,\gamma)$ and $-\infty < a < b < \infty$, then both the systems of norms,* (2.14) *and* (3.13)*, determine on $\mathcal{D}_\omega[a,b]$ the same topology of a Fréchet space.*

Proposition 3.7 ([Bj, Theorem 1.4.1, (iii)⇒(ii)]). *Let $\omega \in \Omega(\alpha,\beta)$. Then for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there is $C_{n,\varepsilon} \in]0,\infty[$ such that*

$$
|\widehat{\varphi}(x+iy)| \le C_{n,\varepsilon} |||\varphi|||_{n\omega} \exp(-n\omega(x) + H_{[a,b]}(y) + \varepsilon|y|) \tag{3.14}
$$

for any $-\infty < a < b < \infty$, $\varphi \in \mathcal{D}_\omega[a,b]$, and $x, y \in \mathbb{R}$.

Proof. Let $\varphi \in \mathcal{D}_\omega[a,b]$. By Lemma 3.2, for every $\delta > 0$ there is $C_\delta \in]0,\infty[$ such that

$$
\begin{aligned}
[\mathcal{P}_{|y|} * \log|\widehat{\varphi}|_{\mathbb{R}}|](x) &\le [\mathcal{P}_{|y|} * (\log |||\varphi|||_{n\omega} - n\omega)](x) \\
&\le \log |||\varphi|||_{n\omega} - n\omega(x) + n\delta|y| + nC_\delta
\end{aligned}
$$

for every $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence, by Proposition 3.1,

$$
\begin{aligned}
|\widehat{\varphi}(x+iy)| &\le \exp([\mathcal{P}_{|y|} * \log|\widehat{\varphi}|_{\mathbb{R}}|](x) + H_{[a,b]}(y)) \\
&\le \exp(nC_\delta) |||\varphi|||_{n\omega} \exp(-n\omega(x) + H_{[a,b]}(y) + n\delta|y|),
\end{aligned}
$$

so that (3.14) holds for any $\varepsilon > 0$ with $C_{n,\varepsilon} = \exp(nC_\delta)$, $\delta = \varepsilon/n$. □

Corollary 3.8. *If $\omega \in \Omega(\alpha,\beta,\gamma)$ and $-\infty < a < b < \infty$, then the Fourier transformation $\mathcal{F}$ is an isomorphism of $\mathcal{D}_\omega[a,b]$ onto the space $\mathcal{Z}_\omega[a,b]$ of entire functions defined in Section* 2.5. □

Proof. Proposition 3.7 says that $\mathcal{F}$ maps $\mathcal{D}_\omega[a,b]$ into $\mathcal{Z}_\omega[a,b]$. Since $\mathcal{F}|_{\mathcal{D}_\omega[a,b]}$ is invertible, it remains to prove that for every $f \in \mathcal{Z}_\omega[a,b]$ there is $\varphi \in \mathcal{D}_\omega[a,b]$ such that $\mathcal{F}\varphi = f$. So, take any $f \in \mathcal{Z}_\omega[a,b]$. Then estimations similar to those used in the proof of Proposition 3.5(B) show that

$$\sup_{x+iy\in\mathbb{C}} \left(\int_{-\infty}^{\infty} |f(x+iy)| \exp(n\omega(x))\, dx \right) \exp(-H_{[a-\varepsilon,b+\varepsilon]}(y)) < \infty$$

for every $n \in \mathbb{N}$ and $\varepsilon > 0$. Hence, for $\mathcal{L}_\omega$ and $\mathcal{A}_\omega$ defined in Section 2.3, one has $f|_\mathbb{R} \in \mathcal{L}_\omega$ and $\varphi := \mathcal{F}^{-1} f|_\mathbb{R} \in \mathcal{A}_\omega$. Moreover, by (iii)$\Rightarrow$(i) of Proposition 3.1, $\operatorname{supp}\varphi \subset \bigcap_{\varepsilon>0}[a-\varepsilon, b+\varepsilon] = [a,b]$, so that $\varphi \in \mathcal{D}_\omega[a,b]$. $\square$

Assume that $\omega \in \Omega(\alpha,\beta,\gamma)$, X is a complex Banach space, and $-\infty < a < b < \infty$. Let

$$\mathcal{D}'_\omega(a,b;X) = \{S \in \mathcal{D}'_\omega(X) : \operatorname{supp} S \subset [a,b]\},$$

and denote by $\mathcal{U}_\omega(a,b;X)$ the set of X-valued entire functions U such that there is $m_0 = m_0(U) \in \mathbb{N}$ for which

$$\sup_{z\in\mathbb{C}} \|U(z)\|_X \exp(-m_0\omega(-\operatorname{Re} z) - H_{[a,b]}(\operatorname{Im} z) - \varepsilon|\operatorname{Im} z|) < \infty \tag{3.15}$$

for every $\varepsilon > 0$. Let $S \in \mathcal{D}'_\omega(a,b;X)$. Take any $\chi \in \mathcal{D}_\omega$ such that $\chi = 1$ on $[a,b]$. Then, by Corollary 3.4, the formula

$$U(z) = S(e_{-iz}\chi), \qquad z \in \mathbb{C}, \tag{3.16}$$

defines an entire X-valued function U. Since $\operatorname{supp} S \subset [a,b]$, another choice of $\chi \in \mathcal{D}_\omega$ does not affect U provided that $\chi = 1$ on $[a,b]$. By (2.3) for any $\varepsilon > 0$ one can choose $\chi \in \mathcal{D}_\omega$ so that $\chi = 1$ on $[a,b]$ and χ vanishes outside $[a-\varepsilon, b+\varepsilon]$. Then applying Proposition 3.3 one concludes that the transformation $S \mapsto U$ maps $\mathcal{D}'_\omega(a,b;X)$ into $\mathcal{U}_\omega(a,b;X)$.

The following theorem is equivalent to a combination of Theorem 1.8.11 and (b)$\Leftrightarrow$(c) from Theorem 1.8.14 of [Bj]. A complete proof of this theorem is given because of its crucial role in the subsequent section.

Theorem 3.9. *Assume that $\omega \in \Omega(\alpha,\beta,\gamma)$, $a,b \in \mathbb{R}$, $a < b$, and X is a Banach space.*

(A) *Let $U \in \mathcal{U}_\omega(a,b;X)$, and let υ be a real function uniformly lipschitzian on $\mathbb{R}$ such that*

$$\sup_{x\in\mathbb{R}} \frac{|\upsilon(x)|}{1+\omega(x)} < \infty. \tag{3.17}$$

Denote by $\mathcal{C}_\upsilon$ the oriented path $\mathbb{R} \ni x \mapsto x - i\upsilon(-x) \in \mathbb{C}$. Then for every $\varphi \in \mathcal{D}_\omega$ the integral $\int_{\mathcal{C}_\upsilon} \widehat{\varphi}(-z)U(z)\,dz$ is absolutely convergent in the sense of the norm in X, and the value of this integral is independent of the choice of a uniformly lipschitzian υ satisfying (3.17)*. Furthermore, the formula*

$$S(\varphi) = \frac{1}{2\pi} \int_{\mathcal{C}_\upsilon} \widehat{\varphi}(-z)U(z)\,dz, \qquad \varphi \in \mathcal{D}_\omega, \tag{3.18}$$

determines a distribution $S \in \mathcal{D}'_\omega(a,b;X)$.

(B) *The map* $\mathcal{D}'_\omega(a,b;X) \ni S \mapsto U \in \mathcal{U}_\omega(a,b;X)$ *determined by* (3.16) *is an isomorphism of* $\mathcal{D}'_\omega(a,b;X)$ *onto* $\mathcal{U}_\omega(a,b,X)$*, and its inverse is the map* $\mathcal{U}_\omega(a,b;X) \ni U \mapsto S \in \mathcal{D}'_\omega(a,b;X)$ *determined by* (3.18).

Taking $\upsilon \equiv 0$ and comparing (3.18) with the Parseval equality from Section 2.5, one concludes that for every distribution $S \in \mathcal{D}'_\omega(a,b;X)$ its Fourier transform $\widehat{S} : \mathcal{Z}_{\omega^\vee} \to X$ is represented by the X-valued entire function U determined by (3.16).

Proof. (A) By Propositions 3.5(A) and 3.7, if $-\infty < c < d < \infty$, $\varphi \in \mathcal{D}_\omega[c,d]$, $n \in \mathbb{N}$, $\varepsilon > 0$ and $z \in \mathbb{C}$, then

$$|\widehat{\varphi}(-z)| \le C_{c,d,n,\varepsilon}\|\varphi\|_{n\omega} \exp(-n\omega(-\operatorname{Re} z) + H_{[c,d]}(-\operatorname{Im} z) + \varepsilon|\operatorname{Im} z|) \quad (3.19)$$

where $C_{c,d,n,\varepsilon}$ is a function of (c,d,n,ε) with values in $]0,\infty[$. Suppose now that $U \in \mathcal{U}_\omega(a,b;X)$ is fixed, and υ is a real function uniformly lipschitzian on $\mathbb{R}$, satisfying (3.17). By (3.15) and (3.19), if $-\infty < c < d < \infty$, $\varphi \in \mathcal{D}_\omega[c,d]$, $n \in \mathbb{N}$, $\varepsilon > 0$ and $z = x + iy \in \mathbb{C}$, then

$$\begin{aligned}\|\widehat{\varphi}(-z)U(z)\|_X \le \widetilde{C}_{c,d,n,\varepsilon}\|\varphi\|_{n\omega} \exp((m_0(U) - n)\omega(-x) \\ + H_{[a,b]}(y) + H_{[c,d]}(-y) + 2\varepsilon|y|). \quad (3.20)\end{aligned}$$

Consequently,

$$\begin{aligned}&\|\widehat{\varphi}(-z)U(z)\|_X \\ &\le \widetilde{C}_{c,d,n,\varepsilon}\|\varphi\|_{n\omega} \exp((m_0(U) + 2K(M+\varepsilon) - n)\omega(-x) + 2K(M+\varepsilon)) \quad (3.21)\end{aligned}$$

for every $z = x+iy \in \mathcal{C}_\upsilon$, where $K = \sup_{x\in\mathbb{R}} \frac{|\upsilon(x)|}{1+\omega(x)}$ and $M = \max(|a|,|b|,|c|,|d|)$. If $n \in \mathbb{N}$ is sufficiently large, then, by (γ), the right side of the last inequality is a function of x belonging to $L^1(\mathbb{R})$. Since υ is uniformly lipschitzian, it follows that the integral $\int_{\mathcal{C}_\upsilon} \widehat{\varphi}(-z)U(z)\,dz$ is absolutely convergent in the sense of the norm in X. Furthermore, by (3.20), (3.21) and by Cauchy's integral theorem,

$$\int_{\mathcal{C}_\upsilon} \widehat{\varphi}(-z)U(z)\,dz = \int_{-\infty}^{\infty} \widehat{\varphi}(-x-iy)U(x+iy)\,dx \quad (3.22)$$

for every $\varphi \in \mathcal{D}_\omega[c,d]$ and $y \in \mathbb{R}$. Consequently, the formula (3.18) defines a distribution $S \in \mathcal{D}'_\omega(X)$ which is independent of the choice of υ. It remains to prove that $\operatorname{supp} S \subset [a,b]$. To this end it is sufficient to show that if a compact integral $[c,d]$ is disjoint from $[a,b]$ and $\varphi \in \mathcal{D}_\omega[c,d]$, then $S(\varphi) = 0$. So, suppose that $[a,b]$ and $[c,d]$ are disjoint and $\varphi \in \mathcal{D}_\omega[c,d]$. Then there is an $\varepsilon > 0$ such that

$$\text{either} \quad c - b - 2\varepsilon = A > 0, \quad \text{or} \quad a - d - 2\varepsilon = B > 0.$$

By (3.20) and (3.22) one has

$$\begin{aligned}\|S(\varphi)\|_X &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\|\widehat{\varphi}(-x-iy)U(x+iy)\|_X\,dx\\ &\le L\exp(H_{[a,b]}(y)+H_{[c,d]}(-y)+2\varepsilon|y|)\end{aligned}$$

and hence

$$\|S(\varphi)\|_X \le L\begin{cases}\exp(-Ay) & \text{if } A>0 \text{ and } y>0,\\ \exp(By) & \text{if } B>0 \text{ and } y<0,\end{cases}$$

where $L=\frac{1}{2\pi}\widetilde{C}_{c,d,n,\varepsilon}\|\varphi\|_{n\omega}\int_{-\infty}^{\infty}\exp((m_0(U)-n)\omega(-x))\,dx$ is finite provided that n is sufficiently large. Letting $y\to\infty$ if $A>0$, and $y\to-\infty$ if $B>0$, one concludes that $S(\varphi)=0$.

(B) We have to prove that if $S\in\mathcal{D}'_\omega(a,b;X)$, $\varphi,\chi\in\mathcal{D}_\omega$, and $\chi=1$ on $[a,b]$, then

$$S(\varphi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\varphi}(-x)S(e_{-ix}\chi)\,dx, \tag{3.23}$$

and if $U\in\mathcal{U}_\omega(a,b;X)$, $\chi\in\mathcal{D}_\omega$, and $\chi=1$ on $[a,b]$, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{e_{-ix_0}\chi}(-x)U(x)\,dx=U(x_0)\quad \text{for every } x_0\in\mathbb{R}. \tag{3.24}$$

Indeed, (3.23) and (3.24) mean respectively that the map of $\mathcal{U}(a,b;X)$ into $\mathcal{D}'_\omega(a,b;X)$ determined by (3.18) is a left and right inverse for the map of $\mathcal{D}'_\omega(a,b;X)$ into $\mathcal{U}_\omega(a,b;X)$ determined by (3.16).

For the proof of (3.23) notice that if c and d are chosen so that $-\infty<c<a<b<d<\infty$ and $\chi\in\mathcal{D}_\omega[c,d]$, then, by Propositions 3.3, 3.7 and Corollary 3.4, $\widehat{\varphi}(-z)e_{-iz}\chi$ is an entire function of z with values in the Fréchet space $\mathcal{D}_\omega[c,d]$, such that

$$\begin{aligned}\|\widehat{\varphi}(-x)e_{-ix}\chi\|_{n\omega} &\le |\widehat{\varphi}(-x)|\|e_{-ix}\chi\|_{n\omega}\\ &\le |||\varphi|||_{(m+n)\omega}\exp(-(m+n)\omega(-x))\cdot C_{n,1}\|\chi\|_{n\omega}\exp(n\omega(-x))\\ &= C_{n,1}|||\varphi|||_{(m+n)\omega}\|\chi\|_{n\omega}\exp(-m\omega(-x))\end{aligned}$$

for every $m,n\in\mathbb{N}$ and $x\in\mathbb{R}$. Since, by (γ), $\int_{-\infty}^{\infty}\exp(-m\omega(-x))\,dx<\infty$ for sufficiently large m, and since there are $n\in\mathbb{N}$ and $C\in\,]0,\infty[$ such that $\|S(\psi)\|_X\le C\|\psi\|_{n\omega}$ for every $\psi\in\mathcal{D}_\omega[c,d]$, it follows that the vector-valued integrals $\int_{-\infty}^{\infty}\widehat{\varphi}(-x)e_{-ix}\chi\,dx$ and $\int_{-\infty}^{\infty}\widehat{\varphi}(-x)S(e_{-ix}\chi)\,dx$ make sense and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\varphi}(-x)S(e_{-ix}\chi)\,dx=S\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\varphi}(-x)e_{-ix}\chi\,dx\right).$$

For every $x,t\in\mathbb{R}$ one has $\widehat{\varphi}(-x)(e_{-ix}\chi)(t)=\widehat{\varphi}(-x)e^{-itx}\chi(t)$, so that by the Fourier inversion formula, the $\mathcal{D}_\omega[c,d]$-valued integral $\int_{-\infty}^{\infty}\widehat{\varphi}(-x)e_{-ix}\chi\,dx$ is equal to $2\pi\chi\varphi$. Hence

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\varphi}(-x)S(e_{-ix}\chi)\,dx=S(\chi\varphi)=S(\varphi),$$

where the last equality follows from (2.2), (2.4) and the fact that $\chi=1$ on $\operatorname{supp}S$. The assertion (3.23) is proved.

Now we prove (3.24). By Beurling's Lemma I from [B], quoted in Section 1.5, there is an even function $\widetilde{\omega} \in \Omega(\alpha, \beta, \gamma)$ such that $\widetilde{\omega}|_{\mathbb{R}^+}$ is concave and $\omega(x) \leq \widetilde{\omega}(x)$ for every $x \in \mathbb{R}$. By Beurling's Theorem I from [B], quoted in Section 2.3, there is $\chi_1 \in \mathcal{D}_{\widetilde{\omega}}$ equal to one on the interval $[-c, c]$, $c = \max(|a|, |b|)$. Obviously $\mathcal{D}_{\widetilde{\omega}} \subset \mathcal{D}_\omega$ and $\mathcal{U}_\omega(a, b; X) \subset \mathcal{U}_{\widetilde{\omega}}(a, b; X)$. Let $\chi_k(t) = \chi_1(k^{-1}t)$. Then $\chi_k \in \mathcal{D}_{\widetilde{\omega}}$ and $\chi_k = 1$ on $[a, b]$ for every $k = 1, 2, \ldots$. If $U \in \mathcal{U}_{\omega(a,b;X)}$, then, by the already proved part (A) of Theorem 3.9, the formula $S(\varphi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\varphi}(-x)U(x)\,dx$, $\varphi \in \mathcal{D}_\omega$, determines a distribution $S \in \mathcal{D}'_\omega(a, b; X)$. Therefore for U and χ appearing in (3.24) and for every $x_0 \in \mathbb{R}$ and $k = 1, 2, \ldots$ one has

$$\begin{aligned}\int_{-\infty}^{\infty} \widehat{e_{-ix_0}\chi}(-x)U(x)\,dx &= \int_{-\infty}^{\infty} \widehat{e_{-ix_0}\chi_k}(-x)U(x)\,dx = \int_{-\infty}^{\infty} \widehat{\chi}_k(x_0 - x)U(x)\,dx \\ &= k\int_{-\infty}^{\infty} \widehat{\chi}_1(k(x_0 - x))U(x)\,dx \\ &= \int_{-\infty}^{\infty} \widehat{\chi}_1(x)U(x_0 - k^{-1}x)\,dx.\end{aligned}$$

Since $\int_{-\infty}^{\infty} \widehat{\chi}_1(x)\,dx = 2\pi\chi_1(0) = 2\pi$, for the proof of (3.24) it is sufficient to show that

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \widehat{\chi}_1(x)U(x_0 - k^{-1}x)\,dx = \int_{-\infty}^{\infty} \widehat{\chi}_1(x)U(x_0)\,dx \tag{3.25}$$

for every $x_0 \in \mathbb{R}$. To this end, fix $x_0 \in \mathbb{R}$. By continuity of U, for every $x \in \mathbb{R}$ one has

$$\lim_{k\to\infty} \widehat{\chi}_1(x)U(x_0 - k^{-1}x) = \widehat{\chi}_1(x)U(x_0) \tag{3.26}$$

in the sense of the norm in X. Since $U \in \mathcal{U}_\omega(a, b; X) \subset \mathcal{U}_{\widetilde{\omega}}(a, b; X)$ and $\widetilde{\omega}$ is even, subadditive and non-decreasing on $\mathbb{R}^+$, it follows from (3.15) that there are $C \in]0, \infty[$ and $m_0 \in \mathbb{N}$ such that

$$\begin{aligned}\|U(x_0 - k^{-1}x)\|_X &\leq C\exp(m_0\widetilde{\omega}(x_0 - k^{-1}x)) \\ &\leq C\exp(m_0\widetilde{\omega}(x_0) + m_0\widetilde{\omega}(k^{-1}x)) \\ &\leq C\exp(m_0\widetilde{\omega}(x_0) + m_0\widetilde{\omega}(x))\end{aligned}$$

for every $x \in \mathbb{R}$ and $k \in \mathbb{N}$. Since $\chi_1 \in \mathcal{D}_{\widetilde{\omega}}$, one concludes that

$$\begin{aligned}\|\widehat{\chi}_1(x)U(x_0 - k^{-1}x)\|_X &\leq |\widehat{\chi}_1(x)| \cdot \|U(x_0 - k^{-1}x)\|_X \\ &\leq |||\chi_1|||_{n\widetilde{\omega}} \exp(-n\widetilde{\omega}(x))C\exp(m_0\widetilde{\omega}(x) + m_0\widetilde{\omega}(x)) \\ &\leq C\exp(m_0\widetilde{\omega}(x_0))|||\chi_1|||_{n\widetilde{\omega}} \cdot \exp((m_0 - n)\widetilde{\omega}(x))\end{aligned} \tag{3.27}$$

for every $x \in \mathbb{R}$ and $k, n \in \mathbb{N}$. By (γ), one has $\int_{-\infty}^{\infty} \exp((m_0 - n)\widetilde{\omega}(x))\,dx < \infty$ for sufficiently large n. Therefore (3.26) and (3.27) imply (3.25) in view of the Lebesgue dominated convergence theorem. □

4 Proofs of Theorem 2.1 and Proposition 2.3

Throughout the present section it is assumed that $\omega \in \Omega(\alpha, \beta, \gamma)$, $\mathcal{A}$ is a complex Banach algebra, and X is a complex Banach space. $\widetilde{\mathfrak{R}}_\omega^{\max}(\mathcal{A})$ and $\widetilde{\mathcal{R}}_{\mathfrak{a}\omega+\mathfrak{b}}^{\max}(\mathcal{A})$ denote the sets of maximal $\mathcal{A}$-valued pseudoresolvents belonging respectively to $\widetilde{\mathfrak{R}}_\omega(\mathcal{A})$ or $\widetilde{\mathcal{R}}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A})$. Theorem 2.1 and Proposition 2.3 are consequences of the following three statements:

(A) *For every $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$ there are $\mathfrak{a}, \mathfrak{b} \geq 0$ and a pseudoresolvent $R \in \mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A})$ such that* (2.16) *holds for every $\varphi \in \mathcal{D}_\omega$ and $\widetilde{\omega} \in \widetilde{\Omega}_\omega$.*

(A)^0 *If either $\omega(x) \equiv \log(1 + |x|)$ or $\omega(x) \equiv |x|^{1/s}$, $s = \mathrm{const} > 1$, and $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$, then there are $\mathfrak{a}, \mathfrak{b} \geq 0$ and a pseudoresolvent $R \in \mathcal{R}^0_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A})$ such that* (2.16) *holds for every $\varphi \in \mathcal{D}_\omega$ and $\widetilde{\omega} \in \widetilde{\Omega}_\omega$.*

(B) *Whenever $\mathfrak{a}, \mathfrak{b} \geq 0$, $R \in \widetilde{\mathcal{R}}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A})$, $\varphi \in \mathcal{D}_\omega$ and $\widetilde{\omega} \in \widetilde{\Omega}_\omega$, then $\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}}} \|\widehat{\varphi}(-z)R(iz)\|_{\mathcal{A}}\,|dz| < \infty$. If $R \in \widetilde{\mathfrak{R}}_\omega$ then the value of the integral $\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}}} \widehat{\varphi}(-z)R(iz)\,dz$ is independent of $\mathfrak{a}, \mathfrak{b} \geq 0$ and $\widetilde{\omega} \in \widetilde{\Omega}_\omega$ provided that $R \in \widetilde{\mathcal{R}}_{\mathfrak{a}\omega+\mathfrak{b}}$. There is an invertible map*

$$\widetilde{\mathfrak{R}}_\omega^{\max}(\mathcal{A}) \ni R \mapsto S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A}) \tag{4.1}$$

which to any pseudoresolvent $R \in \widetilde{\mathfrak{R}}_\omega^{\max}(\mathcal{A})$ assigns the distribution semigroup $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$ satisfying (2.16).

4.1 Proof of (A)

Given $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$, in order to construct a pseudoresolvent $R \in \mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}$ satisfying (2.16) we will follow J. Chazarain. Namely, we will use a cut-off function $\vartheta \in \mathcal{D}_\omega$, analogous to the one introduced on p. 394 of [C], such that $\vartheta = 1$ on some interval $[0, a]$, $a > 0$. The existence of ϑ is a consequence of the Theorem of Beurling quoted in Section 2.3. Recall that $D = \frac{d}{dt}$.

Lemma 4.1. *Suppose that $\vartheta \in \mathcal{D}_\omega$ is a cut-off function as above. Fix any $a \in \,]0, \infty[$ such that $\vartheta = 1$ on $[0, a]$, and let $b = \sup \operatorname{supp} \vartheta$. Then $a < b < \infty$,*

$$\begin{gathered} \mathbb{1}_{\mathbb{R}^+}\vartheta \in \mathcal{D}^+_\omega[0, b], \qquad \mathbb{1}_{\mathbb{R}^+}D\vartheta \in \mathcal{D}_\omega[a, b], \\ \mathbb{1}_{\mathbb{R}^+}\vartheta * (-\mathbb{1}_{\mathbb{R}^+}D\vartheta)^{*,k} \in \mathcal{D}_\omega[ka, (k+1)b] \quad \text{for } k = 1, 2, \ldots, \end{gathered} \tag{4.2}$$

and

$$\mathbb{1}_{\mathbb{R}^+}\vartheta + \sum_{k \geq 1} \mathbb{1}_{\mathbb{R}^+}\vartheta * (-\mathbb{1}_{\mathbb{R}^+}D\vartheta)^{*,k} = \mathbb{1}_{\mathbb{R}^+}, \tag{4.3}$$

the sum being locally finite.

Proof. The first of the relations (4.2) is obvious, the second follows from (2.4), and the third is a consequence of the fact that $\mathcal{D}^0_\omega = \{\varphi \in \mathcal{D}_\omega : \operatorname{supp}\varphi \subset \mathbb{R}^+\}$

is an ideal in $(\mathcal{D}^+_\omega, *)$. Let

$$\psi = \mathbb{1}_{\mathbb{R}^+}\vartheta + \sum_{k\geq 1} \mathbb{1}_{\mathbb{R}^+}\vartheta * (-\mathbb{1}_{\mathbb{R}^+}D\vartheta)^{*,k}.$$

Then $\operatorname{supp}\psi \subset \mathbb{R}^+$, $\psi|_{\mathbb{R}^+} \in C^\infty(\mathbb{R}^+)$, and

$$\psi + \psi * \mathbb{1}_{\mathbb{R}^+}D\vartheta = \mathbb{1}_{\mathbb{R}^+}\vartheta.$$

On the other hand,

$$\mathbb{1}_{\mathbb{R}^+} + \mathbb{1}_{\mathbb{R}^+} * \mathbb{1}_{\mathbb{R}^+}D\vartheta = \mathbb{1}_{\mathbb{R}^+}\vartheta,$$

and hence

$$(\psi - \mathbb{1}_{\mathbb{R}^+}) * (\delta + \mathbb{1}_{\mathbb{R}^+}D\vartheta) = 0.$$

Both the factors in the last equality belong to the convolution algebra $\mathcal{D}'_+$ of distributions of L. Schwartz on $\mathbb{R}$ with supports in $\mathbb{R}^+$. By Theorem XIV on p. 173 of [Sch], $\mathcal{D}'_+$ is a convolution algebra without zero divisors. Since $\delta + \mathbb{1}_{\mathbb{R}^+}D\vartheta \neq 0$, it follows that $\psi - \mathbb{1}_{\mathbb{R}^+} = 0$, proving (4.3). □

Lemma 4.2. *Suppose that $\vartheta \in \mathcal{D}_\omega$ is a cut-off function, $\mathcal{S} \in \mathcal{D}'_\omega(X)$ and $\operatorname{supp} S \subset \mathbb{R}^+$. Then there is $\kappa \in \mathbb{N}$ such that*

$$\sup_{\operatorname{Im} z\leq 0} \|S(e_{-iz}\vartheta)\|_X \exp(-\kappa\omega(-\operatorname{Re} z) + \varepsilon \operatorname{Im} z) < \infty \quad \textit{for every } \varepsilon > 0, \quad (4.4)$$

and there are $\mathfrak{a} > 0$ and $\mathfrak{b} \geq 0$ such that

$$\|S(e_{-iz}D\vartheta)\|_X \leq \frac{1}{2} \quad \textit{whenever } \operatorname{Im} z \leq -\mathfrak{a}\omega(-\operatorname{Re} z) - \mathfrak{b}. \quad (4.5)$$

Proof. Let $b = \sup\operatorname{supp}\vartheta$. By continuity of S there are $\kappa \in \mathbb{N}$ and $K \in]0,\infty[$ such that

$$\|S(\varphi)\|_X \leq K\|\varphi\|_{\kappa\omega} \quad \text{for every } \varphi \in \mathcal{D}_\omega[-1, b].$$

Hence, by Proposition 3.3, for every $\varepsilon > 0$ there is $C_\varepsilon \in]0,\infty[$ such that whenever $\varphi \in \mathcal{D}_\omega[-1, b]$, $a_\varphi = \inf\operatorname{supp}\varphi$ and $\operatorname{Im} z \leq 0$, then

$$\begin{aligned}\|S(e_{-iz}\varphi)\|_X &\leq K\|e_{-iz}\varphi\|_{\kappa\omega} \\ &\leq KC_\varepsilon\|\varphi\|_{\kappa\omega} \exp(\kappa\omega(-\operatorname{Re} z) + (a_\varphi - \varepsilon/2)\operatorname{Im} z). \end{aligned} \quad (4.6)$$

By (2.4), for every $\varepsilon > 0$ there is $\varphi \in \mathcal{D}_\omega[-1, b]$ such that $-\varepsilon/2 \leq a_\varphi < 0$ and $\varphi = \vartheta$ on $\mathbb{R}^+$. Since $\operatorname{supp} S \subset \mathbb{R}^+$, it follows that $S(e_{-iz}\vartheta) = S(e_{-iz}\varphi)$ and hence, by (4.6),

$$\begin{aligned}\|S(e_{-iz}\vartheta)\|_X &= \|S(e_{-iz}\varphi)\|_X \\ &\leq KC_\varepsilon\|\varphi\|_{\kappa\omega} \exp(\kappa\omega(-\operatorname{Re} z) - \varepsilon \operatorname{Im} z) \quad \text{whenever } \operatorname{Im} z \leq 0,\end{aligned}$$

which proves (4.4).

To prove (4.5) take $a \in]0, b]$ such that $\vartheta = 1$ on $[0, a]$. By (2.4) one has $D\vartheta = \psi + \varphi$ where $\psi \in \mathcal{D}_\omega$, $\operatorname{supp}\psi \subset]-\infty, 0]$, and $\varphi \in \mathcal{D}_\omega[a, b]$. Since

$\operatorname{supp} S \subset \mathbb{R}^+$, one has $S(e_{-iz}D\vartheta) = S(e_{-iz}\varphi)$ and, by (4.6),

$$\|S(e_{-iz}D\vartheta)\|_X \le KC_a\|\varphi\|_{\kappa\omega} \exp\left(\kappa\omega(-\operatorname{Re} z) + \frac{1}{2}a \operatorname{Im} z\right) \quad \text{whenever } \operatorname{Im} z \le 0.$$

Letting $\mathfrak{a} = 2a^{-1}\kappa$, $\mathfrak{b} = 2a^{-1}\log(2KC_a)$, one concludes that (4.5) holds. $\square$

Completion of the proof of (A). Suppose that $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$. Take any cut-off function $\vartheta \in \mathcal{D}_\omega$ and fix $\mathfrak{a} > 0$ and $\mathfrak{b} \ge 0$ for which (4.5) holds. For $z \in \mathbb{C}$ one has $\operatorname{Im} z \le -\mathfrak{a}\omega(-\operatorname{Re} z) - \mathfrak{b}$ if and only if $z \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$. Hence the formula

$$V(z) = \sum_{k=1}^{\infty}(-S(e_{-iz}D\vartheta))^k, \qquad z \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}, \tag{4.7}$$

defines on $-i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$ an $\mathcal{A}$-valued function V such that $\sup_{z\in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}} \|V(z)\| \le 1$. Let

$$U(z) = S(e_{-iz}\vartheta) + S(e_{-iz}\vartheta)V(z), \qquad z \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}. \tag{4.8}$$

Then U is an $\mathcal{A}$-valued function on $-i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$ such that

$$\|U(z)\|_{\mathcal{A}} \le 2\|S(e_{-iz}\vartheta)\|_{\mathcal{A}} \tag{4.9}$$

and

$$\begin{aligned} -izU(z) &= S(D(e_{-iz}\vartheta)) - S(e_{-iz}D\vartheta) + [S(D(e_{-iz}\vartheta)) - S(e_{-iz}D\vartheta)]V(z) \\ &= S(D(e_{-iz}\vartheta)) + S(D(e_{-iz}\vartheta))V(z) - S(e_{-iz}D\vartheta) - S(e_{-iz}D\vartheta)V(z) \\ &= S(D(e_{-iz}\vartheta)) + S(D(e_{-iz}\vartheta))V(z) + V(z) \end{aligned} \tag{4.10}$$

for every $z \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$. If $\mathcal{A}$ is unital, then let $\mathbb{1}$ denote the unit of $\mathcal{A}$; otherwise denote by $\mathbb{1}$ the unit of the unitization of $\mathcal{A}$. One can rewrite (4.10) in the equivalent form

$$\mathbb{1} - izU(z) = [\mathbb{1} + S(D(e_{-iz}\vartheta))] \cdot [\mathbb{1} + V(z)], \qquad z \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}.$$

Since all the elements of $\mathcal{A}$ appearing here belong to the commutative subalgebra generated by $\{S(\varphi) : \varphi \in \mathcal{D}_\omega\}$, it follows that

$$U(z_2) - iz_1U(z_1)U(z_2) = [S(D(e_{-iz_1}\vartheta)) + \mathbb{1}]S(e_{-iz_2}\vartheta)(\mathbb{1} + V(z_1))(\mathbb{1} + V(z_2))$$

for every $z_1, z_2 \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$. Since, by Proposition 2.1, the right side of the last equality is a symmetric function of $(z_1, z_2) \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}} \times -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$, one concludes that

$$U(z_2) - iz_1U(z_1)U(z_2) = U(z_1) - iz_2U(z_1)U(z_2) \quad \text{for every } z_1, z_2 \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}.$$

It follows that the formula

$$R(\lambda) = U(-i\lambda), \qquad \lambda \in \Lambda_{\mathfrak{a}\omega+\mathfrak{b}}, \tag{4.11}$$

determines an $\mathcal{A}$-valued pseudoresolvent R defined on $\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$. From (4.4) and (4.9) it follows that $R \in \mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A})$.

To complete the proof of (A) it remains to show that if $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$, $\vartheta \in \mathcal{D}_\omega$ is a cut-off function, $\mathfrak{a} > 0$ and $\mathfrak{b} \geq 0$ are chosen so that (4.5) holds, and $\widetilde{\omega} \in \widetilde{\Omega}_\omega$, then the pseudoresolvent $R \in \mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A})$, determined by (4.7), (4.8) and (4.11), satisfies the equality

$$S(\varphi) = \frac{1}{2\pi}\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}}} \widehat{\varphi}(-z)R(iz)\,dz \qquad \text{for every } \varphi \in \mathcal{D}_\omega. \tag{2.16}$$

To this end, remembering (4.2), consider the sequence of compactly supported distributions $S_k \in \mathcal{D}'_\omega(\mathcal{A})$, $k = 0, 1, \ldots$, such that

$$\begin{aligned} &S_0(\varphi) = S(\vartheta\varphi) \quad \text{for every } \varphi \in \mathcal{D}_\omega, \\ &S_k(\varphi) = S((\mathbb{1}_{\mathbb{R}^+}\vartheta * (-\mathbb{1}_{\mathbb{R}^+}D\vartheta)^{*,k})\varphi) \quad \text{for every } 1, 2, \ldots \text{ and } \varphi \in \mathcal{D}_\omega. \end{aligned} \tag{4.12}$$

If $a \in \,]0, \infty[$ is such that $\vartheta = 1$ on $[0, a]$ and $b = \sup\operatorname{supp}\vartheta$, then by (4.2) one has

$$\operatorname{supp} S_k \subset [ka, (k+1)b] \qquad \text{for } k = 0, 1, \ldots. \tag{4.13}$$

Hence, by (4.3) and (4.12),

$$S(\varphi) = \sum_{k=0}^{k(\varphi)-1} S_k(\varphi) \qquad \text{for every } \varphi \in \mathcal{D}_\omega, \tag{4.14}$$

where

$$k(\varphi) = \inf\{k \in \mathbb{N} : \operatorname{supp}\varphi \subset \,]-\infty, ka]\}$$

is finite for every $\varphi \in \mathcal{D}_\omega$. For every $k = 0, 1, \ldots$ take any $\chi_k \in \mathcal{D}_\omega$ such that $\chi_k = 1$ on $[ka, (k+1)b]$. From (4.13) it follows that the formula

$$U_k(z) = S_k(e_{-iz}\chi_k), \qquad z \in \mathbb{C}, \tag{4.15}$$

defines a function $U_k \in \mathcal{U}_\omega(ka, (k+1)b; \mathcal{A})$ such that

$$S_k(\varphi) = \frac{1}{2\pi}\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}}} \widehat{\varphi}(-z)U_k(z)\,dz \qquad \text{for every } \varphi \in \mathcal{D}_\omega. \tag{4.16}$$

This last equality coincides with (3.18) applied to $S_k \in \mathcal{D}'_\omega(ka, (k+1)b; \mathcal{A})$ and $\upsilon = \mathfrak{a}\widetilde{\omega} + \mathfrak{b}$. From (4.12), (4.15) and (4.11) it follows that

$$U_0(z) = S(e_{-iz}\vartheta) \tag{4.17}$$

and

$$\begin{aligned} U_k(z) &= S(e_{-iz}(\mathbb{1}_{\mathbb{R}^+}\vartheta * (-\mathbb{1}_{\mathbb{R}^+}D\vartheta)^{*,k}) \\ &= S^+(\mathbb{1}_{\mathbb{R}^+}e_{-iz}\vartheta * (-\mathbb{1}_{\mathbb{R}^+}e_{-iz}D\vartheta)^{*,k}) \\ &= (-1)^k S^+(\mathbb{1}_{\mathbb{R}^+}e_{-iz}\vartheta)S^+(\mathbb{1}_{\mathbb{R}^+}e_{-iz}D\vartheta)^k \\ &= (-1)^k S(e_{-iz}\vartheta)S(e_{-iz}D\vartheta)^k \end{aligned} \tag{4.18}$$

for $k = 1, 2, \dots$, where $S^+ \in L(\mathcal{D}^+_\omega, \mathcal{A})$ is the homomorphism of $(\mathcal{D}^+_\omega, *)$ into $\mathcal{A}$ such that $S(\varphi) = S^+(\mathbb{1}_{\mathbb{R}^+}\varphi)$ for every $\varphi \in \mathcal{D}_\omega$. Hence, by (4.7), (4.8) and (4.11),

$$\sum_{k=0}^{\infty} U_k(z) = S(e_{-iz}\vartheta) + S(e_{-iz}\vartheta)V(z) = U(z) = R(iz) \tag{4.19}$$

for every $z \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$, the series being absolutely convergent in the sense of the norm in $\mathcal{A}$. Furthermore, from (4.17), (4.18), (4.4) and (4.5) it follows that there are $\kappa \in \mathbb{N}$ and $C \in]0, \infty[$ such that

$$\|U_k(z)\|_{\mathcal{A}} \le \frac{1}{2^k}\|S(e_{-iz}\vartheta)\|_{\mathcal{A}} \le \frac{1}{2^k} C \exp(\kappa\omega(-\operatorname{Re} z) - \operatorname{Im} z)$$

for every $k = 0, 1, \dots$ and $z \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$. Consequently, by Proposition 3.7 for every $n \in \mathbb{N}$ there is $C_n \in]0, \infty[$ such that

$$\|\widehat{\varphi}(-z)U_k(z)\|_{\mathcal{A}} \le \frac{1}{2^k} C C_n |||\varphi|||_{n\omega} \exp((\kappa - n)\omega(-\operatorname{Re} z) - (b+2)\operatorname{Im} z)$$

for every $k = 0, 1, \dots,$ $z \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$, $b \in]0, \infty[$ and $\varphi \in \mathcal{D}_\omega$ such that $\sup\operatorname{supp}\varphi \le b$. Let $b \in]0, \infty[$ and choose $\mathfrak{d} \in [\max(0, \mathfrak{b}), \infty]$ so large that $\operatorname{Im} z \ge -\mathfrak{a}\omega(-\operatorname{Re} z) - \mathfrak{d}$ for every $z \in \mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}$. If $z \in \mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}$, $\varphi \in \mathcal{D}_\omega$ and $\sup\operatorname{supp}\varphi \le b$, then

$$\|\widehat{\varphi}(-z)U_k(z)\|_{\mathcal{A}} \le \frac{1}{2^k} C C_n |||\varphi|||_{n\omega} \exp(\mathfrak{d}(b+2)) \exp((\kappa + \mathfrak{a}(b+2) - n)\omega(-\operatorname{Re} z)).$$

By (γ), one can choose n so large that

$$m(x) = C C_n \exp(\mathfrak{d}(b+2)) \exp((\kappa + \mathfrak{a}(b+2) - n)\omega(-x))$$

is a function of x belonging to $L^1(\mathbb{R})$. On $\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}$ one has

$$\left\| \sum_{k=0}^{l} \widehat{\varphi}(-z)U_k(z) \right\|_{\mathcal{A}} |dz| \le 2(1 + \mathfrak{a}^2 L^2)^{1/2} m(\operatorname{Re} z)\, d\operatorname{Re} z$$

for every $l = 0, 1, \dots$, where L is the Lipschitz constant of $\tilde{\omega}$. Consequently, by (4.19), (4.16) and the Lebesgue dominated convergence theorem, for every $\varphi \in \mathcal{D}_\omega$ one has

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}} \widehat{\varphi}(-z)R(iz)\,dz &= \sum_{k=0}^{\infty} \frac{1}{2\pi}\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}} \widehat{\varphi}(-z)U_k(z)\,dz = \sum_{k=0}^{\infty} S_k(\varphi)\\ &= \sum_{k=0}^{k(\varphi)-1} S_k(\varphi) = S(\varphi). \quad \square\end{aligned}$$

4.2 Proof of (A)^0

Suppose that $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$ where either $\omega(x) \equiv \log(1 + |x|)$ or $\omega(x) \equiv |x|^{1/s}$, $s = \text{const} > 1$. Let $\vartheta \in \mathcal{D}_\omega$ be a cut-off function. Take $\mathfrak{a} > 0$ and $\mathfrak{b} \ge 0$ for which (4.5) is satisfied. Statement $(\mathrm{A})^0$ will follow once it is shown that the pseudoresolvent R determined by (4.7), (4.8) and (4.11) belongs to $\mathcal{R}^0_{\mathfrak{a}\omega+\mathfrak{b}}(\mathcal{A})$. By (4.9), it is sufficient to show that there is $\varkappa \in \mathbb{R}^+$ such that

$$\sup_{\operatorname{Im} z \leq 0} \|S(e_{-iz}\vartheta)\|_{\mathcal{A}} \exp(\varkappa\omega(|z|)) < \infty. \tag{4.20}$$

To this end, take $\kappa \in \mathbb{N}$ for which (4.4) is satisfied. Let $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$, and for every $\varepsilon > 0$ and $z \in \overline{\mathbb{C}}_-$ put

$$f_\varepsilon(z) = \exp(-i\varepsilon z) S(e_{-iz}\vartheta) \cdot \begin{cases} (1+iz)^{-\kappa} & \text{if } \omega(x) \equiv \log(1+|x|), \\ \exp(-\kappa_s (iz)^{1/s}) & \text{if } \omega(x) \equiv |x|^{1/s},\ s = \text{const} > 1, \end{cases}$$

where $\kappa_s = \kappa(\cos\frac{\pi}{2s})^{-1}$, $(iz)^{1/s} = |z|^{1/s}\exp(\frac{i}{s}\operatorname{Arg}(iz))$ if $\operatorname{Im} z \leq 0$ and $z \neq 0$, $(i0)^{1/s} = 0$. For any $\varepsilon > 0$, f_ε is an X-valued function continuous on $\overline{\mathbb{C}}_-$ and holomorphic in $\mathbb{C}_-$. If $z \in \overline{\mathbb{C}}_-$, then

$$\begin{aligned} &\exp(\kappa\omega(-\operatorname{Re} z)) \\ &\quad \leq \begin{cases} (1+|\operatorname{Re} z|)^\kappa \leq 2^\kappa |1+iz|^\kappa & \text{if } \omega(x) \equiv \log(1+|x|), \\ \exp(\kappa|z|^{1/s}) \leq \exp(\kappa_s \operatorname{Re}((iz)^{1/s})) & \text{if } \omega(x) \equiv |x|^{1/s},\ s = \text{const} > 1. \end{cases} \end{aligned}$$

From these inequalities and from (4.4) it follows that for every $\varepsilon > 0$ the function f_ε is bounded in $\overline{\mathbb{C}}_-$. If $x \in \mathbb{R}$, then $\|f_\varepsilon(x)\|_X$ is independent of ε, and hence $M = \sup_{x\in\mathbb{R}} \|f_\varepsilon(x)\|$ is finite and independent of ε. For any $\varepsilon, \delta > 0$ one has

$$\sup_{z\in\overline{\mathbb{C}}_-} \|(1+i\delta z)^{-1} f_\varepsilon(z)\|_X \leq \sup_{z\in\overline{\mathbb{C}}_-} \|f_\varepsilon(z)\|_X < \infty$$

and

$$\lim_{z\in\overline{\mathbb{C}}_-,\, |z|\to\infty} (1+i\delta z)^{-1} f_\varepsilon(z) = 0,$$

and hence

$$\sup_{z\in\overline{\mathbb{C}}_-} \|(1+i\delta z)^{-1} f_\varepsilon(z)\|_X = \sup_{x\in\mathbb{R}} \|(1+i\delta x)^{-1} f_\varepsilon(x)\| \leq M,$$

by the maximum principle for holomorphic functions. Hence, for any $\varepsilon, \delta > 0$ and $z \in \overline{\mathbb{C}}_-$,

$$\begin{aligned} &\exp(\varepsilon \operatorname{Im} z)\|S(e_{-iz}\vartheta)\|_X \\ &\qquad \leq M|1+i\delta z| \cdot \begin{cases} |1+iz|^\kappa & \text{if } \omega(x) \equiv \log(1+|x|), \\ \exp(\kappa_s \operatorname{Re}(iz)^{1/s}) & \text{if } \omega(x) \equiv |x|^{1/s}, s = \text{const} > 1. \end{cases} \end{aligned}$$

Since $|1+iz|^\kappa \leq (1+|z|)^\kappa = \exp(\kappa\log(1+|z|))$ and $\exp(\kappa_s \operatorname{Re}(iz)^{1/s}) \leq \exp(\kappa_s|z|^{1/s})$, one concludes that for any $\varepsilon, \delta > 0$ and $z \in \overline{\mathbb{C}}_-$,

$$\exp(\varepsilon \operatorname{Im} z)\|S(e_{-iz}\varepsilon)\|_X \leq M|1+i\delta z| \exp(\varkappa\omega(|z|)),$$

where $\varkappa = \kappa$ or $\varkappa = \kappa_s$ according as $\omega(x) \equiv \log(1+|x|)$ or $\omega(x) \equiv |x|^{1/s}$. This inequality implies (4.20) by passing to the limit as $\varepsilon \downarrow 0$ and $\delta \downarrow 0$. $\square$

4.3 Proof of (B)

Suppose that $R \in \widetilde{\mathcal{R}}_\omega(\mathcal{A})$, so that the set

$$\mathfrak{C} = \{(\mathfrak{a}, \mathfrak{b}, \widetilde{\omega}) : \mathfrak{a}, \mathfrak{b} \geq 0,\ \widetilde{\omega} \in \widetilde{\Omega}_\omega,\ R \in \widetilde{\mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}}(\mathcal{A})\}$$

is non-empty. By means of estimations of $\|\widehat{\varphi}(-z)R(iz)\|_{\mathcal{A}}$ we will prove that

$$\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\omega}} \|\widehat{\varphi}(-z)R(iz)\|_{\mathcal{A}}\, |dz| < \infty \quad \text{ for } (\mathfrak{a}, \mathfrak{b}, \widetilde{\omega}) \in \mathfrak{C} \text{ and } \varphi \in \mathcal{D}_\omega, \tag{4.21}$$

$$\mathcal{D}_\omega \ni \varphi \mapsto \int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}}} \widehat{\varphi}(-z)R(iz)\, dz \in \mathcal{A} \quad \text{belongs to } \mathcal{D}'_\omega(\mathcal{A}), \tag{4.22}$$

for every fixed $(\mathfrak{a}, \mathfrak{b}, \widetilde{\omega}) \in \mathfrak{C}$,

$$\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}}} \widehat{\varphi}(-z)R(iz)\, dz \text{ is independent of } (\mathfrak{a}, \mathfrak{b}, \widetilde{\omega}) \in \mathfrak{C} \tag{4.23}$$

for every $\varphi \in \mathcal{D}_\omega$, and finally, for all $(\mathfrak{a}, \mathfrak{b}, \widetilde{\omega}) \in \mathfrak{C}$ and $\varphi \in \mathcal{D}_\omega$,

$$\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}}} \widehat{\varphi}(-z)R(iz)\, dz = 0 \quad \text{ whenever } \sup \operatorname{supp} \varphi < 0. \tag{4.24}$$

From (4.21)–(4.24) it follows that there is a unique distribution $S \in \mathcal{D}'_\omega(\mathcal{A})$ for which (2.16) holds, and that $\operatorname{supp} S \subset \mathbb{R}^+$. Thus, once (4.21)–(4.24) are proved, to complete the proof of (B) it will remain to show that

$$S \text{ is a distribution semigroup} \tag{4.25}$$

and

$$\text{the map (4.1) is invertible.} \tag{4.26}$$

Estimations of $\|\widehat{\varphi}(-z)R(iz)\|_{\mathcal{A}}$. By Definition 2.1 and Proposition 3.7, if $\mathfrak{a}, \mathfrak{b} \geq 0$ and $R \in \widetilde{\mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}}(\mathcal{A})$, then

$$\|\widehat{\varphi}(-z)R(iz)\|_{\mathcal{A}} \leq K_{n,\varepsilon} |||\varphi|||_{n,\omega} \exp((\kappa_\varepsilon - n)\omega(-\operatorname{Re} z) - (b+\varepsilon)\operatorname{Im} z) \tag{4.27}$$

for every $\varphi \in \mathcal{D}_\omega$ such that $\sup \operatorname{supp} \varphi \leq b$, $z \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$ (i.e. $\operatorname{Im} z \leq -\mathfrak{a}\omega(-\operatorname{Re} z) - \mathfrak{b}$), $\varepsilon > 0$ and $n \in \mathbb{N}$, with some $\kappa_\varepsilon \in \mathbb{R}$ depending only on ε, and some $K_{n,\varepsilon} \in]0, \infty[$ depending only on n and ε. From (4.27) it follows that if

$$\begin{gathered} \mathfrak{a}, \mathfrak{b} \geq 0,\ R \in \widetilde{\mathcal{R}_{\mathfrak{a}\omega+\mathfrak{b}}}(\mathcal{A}),\ \mathfrak{c} \in [\mathfrak{a}, \infty[,\ \mathfrak{d} \in [\max(0, \mathfrak{b}), \infty[, \\ \varepsilon > 0,\ b \in \mathbb{R},\ m, n \in \mathbb{N},\ n \geq m + \kappa_\varepsilon + \mathfrak{c}(b+\varepsilon), \end{gathered} \tag{4.28}$$

and either

$$\begin{gathered} b + \varepsilon \geq 0 \quad \text{and} \quad z \in (-i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}) \setminus (-i\Lambda_{\mathfrak{c}\omega+\mathfrak{d}}), \quad \text{i.e.} \\ -\mathfrak{c}\omega(-\operatorname{Re} z) - \mathfrak{d} < \operatorname{Im} z \leq -\mathfrak{a}\omega(-\operatorname{Re} z) - \mathfrak{b}, \end{gathered} \tag{4.29}$$

or

$$b + \varepsilon < 0 \quad \text{and} \quad z \in -i\Lambda_{\mathfrak{c}\omega+\mathfrak{d}}, \quad \text{i.e.} \quad \operatorname{Im} z \le -\mathfrak{c}\omega(-\operatorname{Re} z) - \mathfrak{d}, \tag{4.30}$$

then

$$\|\widehat{\varphi}(-z)R(iz)\|_{\mathcal{A}} \le K_{n,\varepsilon} \exp(\mathfrak{d}(b+\varepsilon)) \exp(-m\omega(-\operatorname{Re} z)) \tag{4.31}$$

for every $\varphi \in \mathcal{D}_\omega$ such that $\sup\operatorname{supp}\varphi \le b$. By (γ), for sufficiently large $m \in \mathbb{N}$ one has

$$\int_{-\infty}^{\infty} \exp(-m\omega(-x))\, dx = M_m < \infty. \tag{4.32}$$

From (4.28)–(4.31) it follows that if $m \in \mathbb{N}$ satisfies (4.32), $(\mathfrak{a}, \mathfrak{b}, \widetilde{\omega}) \in \mathfrak{C}$, L is the Lipschitz constant of $\widetilde{\omega}$, $\varepsilon > 0$ and $b \in \mathbb{R}$, then

(i) whenever $b > 0$ and $\mathfrak{c} \in [\mathfrak{a}, \infty[$, $\mathfrak{d} \in [\mathfrak{b}, \infty[$ are so large that $\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}} \subset (-i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}) \setminus (-i\Lambda_{\mathfrak{c}\omega+\mathfrak{d}})$, then there is $n \in \mathbb{N}$ such that

$$\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\widetilde{\omega}}} \|\widehat{\varphi}(-z)R(iz)\|_{\mathcal{A}}\, |dz| \le M_m (1 + \mathfrak{a}^2 L^2)^{1/2} K_{n,\varepsilon} \exp(\mathfrak{d}(b+\varepsilon)) |||\varphi|||_{n\omega}$$

for every $\varphi \in \mathcal{D}_\omega$ such that $\sup\operatorname{supp}\varphi \le b$,

(ii) whenever $b + \varepsilon < 0$, then there is $n \in \mathbb{N}$ such that

$$\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{d},\widetilde{\omega}}} \|\widehat{\varphi}(-z)R(iz)\|_{\mathcal{A}}\, |dz| \le M_m (1 + \mathfrak{a}^2 L^2)^{1/2} K_{n,\varepsilon} \exp(\mathfrak{d}(b+\varepsilon)) |||\varphi|||_{n\omega}$$

for every $\varphi \in \mathcal{D}_\omega$ such that $\sup\operatorname{supp}\varphi \le b$ and every $\mathfrak{d} \in [\mathfrak{b}, \infty[$.

Proof of (4.21) *and* (4.22). Both are consequences of (i).

Proof of (4.23). Suppose that $(\mathfrak{a}_k, \mathfrak{b}_k, \widetilde{\omega}_k) \in \mathfrak{C}$ for $k = 1, 2$ and $\varphi \in \mathcal{D}_\omega$. Let $\mathfrak{a}_0 = \max(\mathfrak{a}_1, \mathfrak{a}_2)$, $\mathfrak{b}_0 = \max(\mathfrak{b}_1, \mathfrak{b}_2)$ and $\widetilde{\omega}_0(x) = \max(\widetilde{\omega}_1(x), \widetilde{\omega}_2(x))$ for every $x \in \mathbb{R}$. Then $(\mathfrak{a}_0, \mathfrak{b}_0, \widetilde{\omega}_0) \in \mathfrak{C}$. We will prove that

$$\int_{\mathcal{C}_{\mathfrak{a}_k,\mathfrak{b}_k,\widetilde{\omega}_k}} \widehat{\varphi}(-z)R(iz)\, dz = \int_{\mathcal{C}_{\mathfrak{a}_0,\mathfrak{b}_0,\widetilde{\omega}_0}} \widehat{\varphi}(-z)R(iz)\, dz \qquad \text{for } k = 1, 2. \tag{4.33}$$

To this end fix k and choose $\mathfrak{c} > \mathfrak{a}_0$ and $\mathfrak{d} \ge \mathfrak{b}_0$ such that $\mathcal{C}_{\mathfrak{a}_0,\mathfrak{b}_0,\widetilde{\omega}_0} \subset (-i\Lambda_{\mathfrak{a}_0\omega+\mathfrak{b}_0}) \setminus (-i\Lambda_{\mathfrak{c}\omega+\mathfrak{d}})$. Then also $\mathcal{C}_{\mathfrak{a}_k,\mathfrak{b}_k,\widetilde{\omega}_k} \subset (-i\Lambda_{\mathfrak{a}_k\omega+\mathfrak{b}_k}) \setminus (-i\Lambda_{\mathfrak{c}\omega+\mathfrak{d}})$. Let

$$I(x) = \int_{\mathfrak{a}_k\widetilde{\omega}_k(-x)+\mathfrak{b}_k}^{\mathfrak{a}_0\widetilde{\omega}_0(-x)+\mathfrak{b}_0} \widehat{\varphi}(-(x+iy))R(i(x+iy))\, dy, \qquad x \in \mathbb{R}.$$

In the above the vertical segment of integration is contained in $(-i\Lambda_{\mathfrak{a}_k\omega+\mathfrak{b}_k}) \setminus (-i\Lambda_{\mathfrak{c}\omega+\mathfrak{d}})$ and hence its length is no greater than $\mathfrak{c}\omega(-x) + \mathfrak{d} - \mathfrak{b}_k$. Therefore applying (4.29) and (4.31) to $\mathfrak{a} = \mathfrak{a}_k$, $\mathfrak{b} = \mathfrak{b}_k$ one concludes that for any fixed $\varepsilon > 0$ and $b \ge 0$ with $\sup\operatorname{supp}\varphi \le b$, there are $m, n \in \mathbb{N}$ such that

$$\|I(x)\|_{\mathcal{A}} \le K_{n,\varepsilon} |||\varphi|||_{n\omega} \exp(\mathfrak{d}(b+\varepsilon))(\mathfrak{c}\omega(-x) + \mathfrak{d} - \mathfrak{b}_k) \exp(-m\omega(-x))$$

for every $x \in \mathbb{R}$. Since, by (γ), one has $\lim_{|x|\to\infty} \omega(-x) = \infty$, it follows that

$$\lim_{|x|\to\infty} \|I(x)\|_{\mathcal{A}} = 0. \tag{4.34}$$

Let

$$I_l(r) = \int_{\mathcal{C}_{\mathfrak{a}_l,\mathfrak{b}_l,\tilde{\omega}_l} \cap \{|\operatorname{Re} z|<r\}} \widehat{\varphi}(-z)R(iz)\,dz, \qquad r>0,\ l=k,0.$$

The equality (4.33) may be written in the form $I_k(\infty) = I_0(\infty)$. By Cauchy's integral theorem, for every $r \in]0,\infty[$ one has $I_k(r) - I_0(r) + I(r) - I(-r) = 0$. By (4.21), $\lim_{r\to\infty} I_l(r) = I_l(\infty)$ for $l = k, 0$. Hence, by (4.34),

$$I_k(\infty) - I_0(\infty) = \lim_{r\to\infty} [(I_k(\infty) - I_k(r)) - (I_0(\infty) - I_0(r)) + I(-r) - I(r)] = 0.$$

Proof of (4.24). Suppose that $(\mathfrak{a}, \mathfrak{b}, \tilde{\omega}) \in \mathfrak{C}$. Then $(\mathfrak{a}, \mathfrak{d}, \tilde{\omega}) \in \mathfrak{C}$ for every $\mathfrak{d} \in [\max(0, \mathfrak{b}), \infty[$. Suppose that $\varphi \in \mathcal{D}_\omega$ and $\sup \operatorname{supp} \varphi = b < 0$. Take $\varepsilon > 0$ such that $b + \varepsilon < 0$. Then $\lim_{\mathfrak{d}\to\infty} \exp(\mathfrak{d}(b+\varepsilon)) = 0$ and hence, by (4.23) and (ii),

$$\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}} \widehat{\varphi}(-z)R(iz)\,dz = \lim_{\mathfrak{d}\to\infty} \int_{\mathcal{C}_{\mathfrak{a},\mathfrak{d},\tilde{\omega}}} \widehat{\varphi}(-z)R(iz)\,dz = 0.$$

Proof of (4.25). By Proposition 2.1 and by (4.24), assertion (4.25) will follow once it is shown that

$$S(D\varphi)S(\psi) + \varphi(0)S(\psi) = S(\varphi)S(D\psi) + \psi(0)S(\varphi) \quad \text{for every } \varphi, \psi \in \mathcal{D}_\omega.$$

To prove this equality, let $(\mathfrak{a}, \mathfrak{b}, \tilde{\omega}) \in \mathfrak{C}$ and $\varphi, \psi \in \mathcal{D}_\omega$. Then $\widehat{D\varphi}(-z) = -iz\widehat{\varphi}(-z)$ and $\varphi(0) = \frac{1}{2\pi} \int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}} \widehat{\varphi}(-z)\,dz$, by Cauchy's integral theorem and the Fourier inversion theorem. Consequently, by (4.21), (2.16) and Fubini's theorem,

$$\begin{aligned} &S(D\varphi)S(\psi) + \varphi(0)S(\psi) \\ &\qquad = \frac{1}{4\pi^2} \iint_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}} \times \mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}} \widehat{\varphi}(-z_1)\widehat{\psi}(-z_2)(1 - iz_1R(iz_1))R(iz_2)(dz_1 \times dz_2). \end{aligned}$$

Similarly

$$\begin{aligned} &S(\varphi)S(D\psi) + \psi(0)S(\varphi) \\ &\qquad = \frac{1}{4\pi^2} \iint_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}} \times \mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}} \widehat{\varphi}(-z_1)\widehat{\psi}(-z_2)(1 - iz_2R(iz_2))R(iz_1)(dz_1 \times dz_2). \end{aligned}$$

These double integrals are equal in view of the Hilbert equality.

Proof of (4.26). Suppose that $R \in \widetilde{\mathcal{R}}_\omega(\mathcal{A})$, $S \in \mathcal{D}'_\omega\mathcal{S}(\mathcal{A})$, $(\mathfrak{a}, \mathfrak{b}, \tilde{\omega}) \in \mathfrak{C}$ and

$$S(\varphi) = \frac{1}{2\pi} \int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}} \widehat{\varphi}(-z)R(iz)\,dz \quad \text{for every } \varphi \in \mathcal{D}_\omega.$$

Then for every $\varphi \in \mathcal{D}_\omega$ and $z \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$ one has

$$\begin{aligned} S((D+iz)\varphi)R(iz) &= \frac{1}{2\pi}\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}} \widehat{\varphi}(-w)(iz-iw)R(iw)R(iz)\,dw \\ &= \frac{1}{2\pi}\int_{\mathcal{C}_{\mathfrak{a},\mathfrak{b},\tilde{\omega}}} \widehat{\varphi}(-w)[R(iw)-R(iz)]\,dw \\ &= S(\varphi)-\varphi(0)R(iz). \end{aligned} \tag{4.35}$$

Let $\vartheta \in \mathcal{D}_\omega$ be a cut-off function. Applying (4.35) to $\varphi = e_{-iz}\vartheta$ one concludes that

$$S(e_{-iz}D\vartheta)R(iz) = S(e_{-iz}\vartheta) - R(iz) \quad \text{for every } z \in -i\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}.$$

By (4.5) there are $\mathfrak{c} \geq \mathfrak{a}$ and $\mathfrak{d} \geq \mathfrak{b}$ such that $\|S(e_{-iz}D\vartheta)\|_{\mathcal{A}} \leq \frac{1}{2}$ for every $z \in -i\Lambda_{\mathfrak{c}\omega+\mathfrak{d}}$. Hence

$$R(iz) = S(e_{-iz}\vartheta) + \sum_{k=1}^{\infty}(-S(e_{-iz}D\vartheta))^k S(e_{-iz}\vartheta) \quad \text{for every } z \in -i\Lambda_{\mathfrak{c}\omega+\mathfrak{d}}.$$

This implies that the maximal $\mathcal{A}$-valued pseudoresolvent which extends R is uniquely determined by the distribution semigroup S, proving (4.26).

5 Proof of Corollary 1.2

By Theorem 2 of R. Beals' paper [Be], for every non-negative even continuous function ω on $\mathbb{R}$ such that $\omega|_{\mathbb{R}^+}$ is concave and

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2}\,dx = \infty \tag{5.1}$$

there exist a complex Hilbert space X and a closed linear operator A from X into X with domain $D(A)$ dense in X and with resolvent set $\varrho(A)$ such that

$$\begin{aligned} &\Lambda_\omega := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \omega(\operatorname{Im}\lambda)\} \subset \varrho(A) \subsetneq \mathbb{C}, \\ &\sup_{\lambda\in\Lambda_\omega} \|(\lambda - A)^{-1}\|_{L(X)}(1+\operatorname{Re}\lambda) < \infty \end{aligned} \tag{5.2}$$

and

for every $T \in \,]0,\infty[$ the only X-valued function u strongly differentiable on $[0,T[$ such that $u(t) \in D(A)$ and $\frac{du(t)}{dt} = Au(t)$ for every $t \in \,]0,T[$ is the identically zero function on $[0,T[$. (5.3)

Suppose that X and A satisfy (5.2) and (5.3), take any λ_0 in the spectrum $\sigma(A)$ of A, and put $A_0 = A - \lambda_0 \mathbb{1}_X$. Then $0 \in \sigma(A_0)$, $\varrho(A_0) = \varrho(A) - \lambda_0$, and if $\mathfrak{b} \geq \omega(\operatorname{Im}\lambda_0) - \operatorname{Re}\lambda_0$, then $\Lambda_{\omega+\mathfrak{b}} \subset \varrho(A_0)$, because by subadditivity of ω,

$$\begin{aligned}\lambda \in \Lambda_{\omega+\mathfrak{b}} &\Rightarrow \operatorname{Re}\lambda \geq \omega(\operatorname{Im}\lambda) + \omega(\operatorname{Im}\lambda_0) - \operatorname{Re}\lambda_0 \\ &\Rightarrow \operatorname{Re}(\lambda+\lambda_0) \geq \omega(\operatorname{Im}(\lambda+\lambda_0)) \\ &\Rightarrow \lambda+\lambda_0 \in \Lambda_\omega \subset \varrho(A) \Rightarrow \lambda \in \varrho(A) - \lambda_0 = \varrho(A_0).\end{aligned}$$

The condition (5.3) remains valid when A is replaced by A_0. Therefore one can modify Theorem 2 of [Be] replacing (5.2) by

$$\begin{aligned}&0 \in \sigma(A) \text{ and there is } \mathfrak{b} \in]0,\infty[\text{ such that } \Lambda_{\omega+\mathfrak{b}} \subset \varrho(A) \\ &\text{and } \sup_{\lambda\in\Lambda_{\omega+\mathfrak{b}}} \|(\lambda - A)^{-1}\|_{L(X)} \operatorname{Re}\lambda < \infty. \end{aligned} \qquad (5.2)_0$$

Take now any non-negative continuous even function ω on $\mathbb{R}$ such that $\omega|_{\mathbb{R}^+}$ is concave and (5.1) is satisfied, fix a complex Hilbert space X and a densely defined closed linear operator A from X into X satisfying $(5.2)_0$ and (5.3), and take any l.c.v.s. $\mathbb{D}$ contained in $C_c^\infty(\mathbb{R})$ and satisfying (2.1)–(2.4). Corollary 1.2 will follow once it is proved that

$$\text{there is no distribution semigroup } S \in \mathbb{D}'\mathcal{S}(L(X)) \text{ with generator } A. \qquad (5.4)$$

To this end we will proceed *ad absurdum*. Suppose that A is the generator of a distribution semigroup $S \in \mathbb{D}'\mathcal{S}(L(X))$. Then

$$D(A) = \left\{ \sum_{k=1}^n S(\varphi_k)x_k : n \in \mathbb{N},\ \varphi_k \in \mathbb{D} \text{ and } x_k \in X \text{ for } k = 1,\dots,n \right\} \qquad (5.5)$$

and

$$AS(\varphi) = -S(D\varphi) - \varphi(0)\mathbb{1}_X \quad \text{for every } \varphi \in \mathbb{D}. \qquad (5.6)$$

These equalities follow directly from the definition of the generator of a $\mathbb{D}$-distribution semigroup, identical with one discussed in [K] in the case when $\mathbb{D}$ is equal to the L. Schwartz space $\mathcal{D} = C_c^\infty(\mathbb{R})$. Furthermore,

$$S(\varphi)A = -S(D\varphi)|_{D(A)} - \varphi(0)\mathbb{1}_{D(A)} \quad \text{for every } \varphi \in \mathbb{D}. \qquad (5.7)$$

This follows from Proposition 3 of [K], formulated there for $\mathbb{D} = \mathcal{D}$, but valid for general $\mathbb{D}$. One has

$$\operatorname{supp} S = \mathbb{R}^+. \qquad (5.8)$$

Indeed, $\operatorname{supp} S \subset \mathbb{R}^+$ by the definition of a distribution semigroup, and if (5.8) were not true, then $[a,b] \cap \operatorname{supp} S = \emptyset$ for some $0 < a < b < \infty$. By (2.3) there is a $\varphi \in \mathbb{D}$ such that $\varphi = 1$ on $[-\frac{1}{2}, a]$ and $\operatorname{supp}\varphi \subset [-1,b]$. Then $\operatorname{supp} D\varphi \subset [-1,-\frac{1}{2}] \cup [a,b]$ so that $\operatorname{supp} D\varphi \cap \operatorname{supp} S = \emptyset$ and hence $S(D\varphi) = 0$. Consequently, by (5.6) and (5.7), $(-A)^{-1} = S(\varphi) \in L(X)$, which is impossible, because $0 \in \sigma(A)$.

By (5.8) there is $\varphi \in \mathbb{D}$ such that $\operatorname{supp}\varphi \subset \mathbb{R}^+$ and $S(\varphi) \neq 0$. Fix such a φ and take $x \in X$ such that

$$S(\varphi)x \neq 0. \qquad (5.9)$$

For every $t \in \mathbb{R}$ put

$$u(t) = S(\varphi_{-t})x,$$

where φ_{-t} is the translate of φ defined by $\varphi_{-t}(s) = \varphi(s-t)$. From (2.1), and from (2.2) applied to $t \mapsto \varphi_{-t}$ and $t \mapsto (-1)^n (D^n\varphi)_{-t}$, $n = 1, 2, \ldots$, it follows that $t \mapsto \varphi_{-t}$ is a $\mathbb{D}$-valued function infinitely differentiable on $\mathbb{R}$ in the topology of $\mathbb{D}$. Hence u is an X-valued function infinitely differentiable on $\mathbb{R}$. By (5.5), u takes values in $D(A)$. Furthermore, by (5.6),

$$\frac{du(t)}{dt} = -S(D(\varphi_{-t}))x = AS(\varphi_{-t})x + \varphi_{-t}(0)x = Au(t) + \varphi(-t)x = Au(t)$$

for every $t \in \mathbb{R}^+$. By (5.3) it follows that $u(t) = 0$ for every $t \in \mathbb{R}^+$. On the other hand, $u(0) = S(\varphi)x \neq 0$, by (5.9). This contradiction proves (5.4).

6 Distribution semigroups and degenerate differential equations in Banach spaces

Degenerate differential equations in Banach spaces, considered e.g. in [C-S], [Fe] and [F-Y], may be illustrated by examples such as the (algebraic-differential) system of Kirchhoff equations of an electrical RLC network or the Stokes system of linear PDE related to hydrodynamics. "Degeneracy" of these systems is related to absence of differentiation with respect to time in some part of the equations.

A connection between degenerate differential equations in Banach spaces and distribution semigroups follows from the results of J. Chazarain. To stay within the framework of his paper [C] we use only the distributions of L. Schwartz or those of M. Gevrey. Hence we assume that either $\omega(x) \equiv \log(1+|x|)$ or $\omega(x) \equiv |x|^{1/s}$, $s = \text{const} > 1$.

Let X, Y be Banach spaces, and L, M linear operators belonging to $L(Y, X)$. A distribution $\mathcal{E} \in \mathcal{D}'_\omega(L(X,Y))$ is called a *fundamental solution* for the differential operator $M\frac{d}{dt} - L$ if $\mathcal{P} * \mathcal{E} = \mathbb{I}_X \otimes \delta$ and $\mathcal{E} * \mathcal{P} = \mathbb{I}_Y \otimes \delta$, where $\mathcal{P} = M \otimes D\delta - L \otimes \delta \in \mathcal{D}'(L(Y,X))$. From Theorems 1.6 and 4.4 of [C] it follows that the following two assertions are equivalent:

(i) the operator $M\frac{d}{dt} - L$ has a unique fundamental solution $\mathcal{E} \in \mathcal{D}'_\omega(L(X,Y))$ with support in $\mathbb{R}^+$,

(ii) there are $\mathfrak{a}, \mathfrak{b} \geq 0$ and $\kappa \in \mathbb{R}$ such that for every $\lambda \in \Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$ the operator $\mathcal{P}(\lambda) = \lambda M - L$ is an isomorphism of Y onto X and

$$\sup_{\lambda \in \Lambda_{\mathfrak{a}\omega+\mathfrak{b}}} \|\mathcal{P}(\lambda)^{-1}\|_{L(X,Y)} \exp(-\kappa\omega(|\lambda|)) < \infty.$$

It is easy to prove that if (ii) holds, then the formulas

(iii) $R(\lambda) = M(\lambda M - L)^{-1}, \quad \widetilde{R}(\lambda) = (\lambda M - L)^{-1}M, \quad \lambda \in \Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$,

determine pseudoresolvents $R \in \mathcal{R}^0_{\mathfrak{a}\omega+\mathfrak{b}}(L(X))$ and $\widetilde{R} \in \mathcal{R}^0_{\mathfrak{a}\omega+\mathfrak{b}}(L(Y))$. By Theorem 2.1 there exist distribution semigroups $S \in \mathcal{D}'_\omega\mathcal{S}(L(X))$ and $\widetilde{S} \in \mathcal{D}'_\omega\mathcal{S}(L(Y))$ which are the inverse Laplace transforms of R and $\widetilde{R}$. Since, by

Chazarain's results, $\mathcal{E}$ is the inverse Laplace transform of the $L(X,Y)$-valued function $\lambda \mapsto \mathcal{P}(\lambda)^{-1}$, one has

(iv) $S = M\mathcal{E}$ and $\widetilde{S} = \mathcal{E}M$.

The operator $M \in L(Y,X)$ need not be invertible, and if it is not, then the abstract Cauchy problem for the operator $M\frac{d}{dt} - L$ is "degenerate". Let $\mathfrak{I}(M) =$ the range of M, $\mathcal{N}(M) =$ the null space of M. The generator of R and S is the operator

(v) $LM^{-1} : \mathfrak{I} \to X/\mathcal{N}$ where $\mathfrak{I} = \mathfrak{I}(M)$ and $\mathcal{N} = L\mathcal{N}(M)$.

The generator of $\widetilde{R}$ and $\widetilde{S}$ is the operator

(vi) $M^{-1}L : \widetilde{\mathfrak{I}} \to Y/\widetilde{\mathcal{N}}$ where $\widetilde{\mathfrak{I}} = L^{-1}\mathfrak{I}(M)$ and $\widetilde{\mathcal{N}} = \mathcal{N}(M)$.

Remarks. I. Theorem 1.6 of [C], concerning $\omega(x) \equiv \log(1 + |x|)$, was generalized by H. O. Fattorini in Theorem 8.4.8 of his book [F] to distributions $\mathcal{P} \in \mathcal{S}'(L(Y,X))$ with $\operatorname{supp} \mathcal{P} \subset \mathbb{R}^+$.

II. Pseudoresolvents of both kinds in (iii) and the operators (v) and (vi) appear in the paper [Fe] of V. Fedorov. In the case of reflexive spaces X and Y the distribution semigroups S and $\widetilde{S}$ are represented in [Fe] by strongly continuous semigroups of operators $(S_t)_{t\geq 0} \subset L(X)$ and $(\widetilde{S}_t)_{t\geq 0} \subset L(Y)$ such that S_0 is the projector of X onto $\overline{\mathfrak{I}(M)}$ along $L\mathcal{N}(M)$, and $\widetilde{S}_0$ is the projector of Y onto $\overline{L^{-1}\mathfrak{I}(M)}$ along $\mathcal{N}(M)$.

III. In Chapter V of the book of A. Favini and A. Yagi [F-Y] the degenerate Cauchy problem for the operator $M\frac{d}{dt} - L$ is considered under the additional assumptions that Y is continuously embedded in X and L is an isomorphism of Y onto X.

IV. Directly from Theorem 4.4 of [C] it follows that if $\omega(x) \equiv |x|^{1/s}$, $s =$ const > 1, then (i) is equivalent to the condition

(ii)′ there are $\mathfrak{a}, \mathfrak{b} \geq 0$ such that for every $\lambda \in \Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$ the operator $\mathcal{P}(\lambda) = \lambda M - L$ is an isomorphism of Y onto X and for every $\varepsilon > 0$ there is $\kappa \in \mathbb{R}$ such that

$$\sup_{\lambda \in \Lambda_{\mathfrak{a}\omega+\mathfrak{b}}} \|\mathcal{P}(\lambda)^{-1}\|_{L(X,Y)} \exp(-\varepsilon \operatorname{Re}\lambda - \kappa\omega(|\lambda|)) < \infty.$$

Evidently (ii)⇒(ii)′. By Theorem 4.4 of [C], (ii)′⇒(i). If (i) holds and ϑ is a cut-off function (defined in Section 4.1), then for every λ in some $\Lambda_{\mathfrak{a}\omega+\mathfrak{b}}$ one has $\|M\mathcal{E}(e_{-\lambda}D\vartheta)\|_{L(X)} \leq \frac{1}{2}$, $\mathcal{P}(\lambda)^{-1} = \mathcal{E}(e_{-\lambda}\vartheta)[I + M\mathcal{E}(e_{-\lambda}D\vartheta)]^{-1}$, and $\|\mathcal{P}(\lambda)^{-1}\|_{L(X,Y)} \leq 2\|\mathcal{E}(e_{-\lambda}\vartheta)\|_{L(X,Y)}$. This implies (ii) by an argument similar to one used in Section 4.2.

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