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Invariant measures for forward rate HJM model with Lévy noise

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Invariant measures for forward rate HJM model with Lévy noise

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Abstract

We give a sufficient condition for the existence of an invariant measure for Heath-Jarrow-Morton model of forward rate function driven by a Lévy process. We also prove that there exists an invariant measure for the model, if noise is small enough.

1 Introduction

Stochastic process has mean reversion property if it moves towards its average value. Since mean reversion may not reveal itself over short horizon, it is worthwhile to examine long-time behaviour. Whenever process converges in law, it also moves towards a long-run equilibrium level. It is believed that interest rates have mean reversion property as practitioners use to say: rates drop when they are high and rise when they are low.

We consider the existence of an invariant measure for Heath-Jarrow-Morton model of forward rate function in Musiela parametrization. If $P(t,\theta)$ denotes price at time t of bond paying 1 at moment $\theta \geq t$, then forward rate function f_t is given by

$$f_t(x) = -\frac{\partial}{\partial x} \ln P(t, t+x).$$

Heath, Jarrow and Morton [6] assumed that for fixed $\theta > 0$, $(f_t(\theta - t))_{0 \leq t \leq \theta}$ is an Itô process with d-dimensional Wiener noise. We work with parametrization proposed by Musiela [8]. Forward rate HJM model in Musiela parametrization with Lévy noise on separable Hilbert space H of real functions defined on $[0, +\infty)$ is given by equation

$$f_t = S(t)f_0 + \int_0^t S(t-s)\alpha(f_s)ds + \int_0^t S(t-s)\sigma(f_s)dZ(s),$$

where $S(t)_{t\geq 0}$ is the semigroup of shift operators, $Z(t)_{t\geq 0}$ is a Lévy process. We consider one-dimensional noise, so the coefficient σ (the volatility) is a mapping from H into H. We assume that σ is bounded and Lipschitz mapping. We make no assumption about

$\alpha : H \to H$, because no-arbitrage implies that α is given, once we have σ. More precisely, it was proved in Jakubowski and Zabczyk [7] that

$$\alpha(f)(x) = J'\left(\int_0^x \sigma(f)(s)ds\right)\sigma(f)(x),$$

where function $J : \mathbb{R} \to (-\infty, +\infty]$, which we will call *Laplace exponent* of Lévy process, is connected to $Z(t)_{t\geq 0}$ by

$$\mathbb{E}\, e^{-zZ(1)} = e^{J(z)}.$$

The main result of this paper is Theorem 4.1, which gives a sufficient condition for the existence of an invariant measure for HJM model on extended weighted L^2 space. We will also present condition (4.6) - more restrictive, but simpler - that also implies the existence of an invariant measure. Proposition 5.1 ensures the existence of an invariant measure for the model, if noise is small enough.

In conditions for the existence of an invariant measure appear expressions of the form $\sup\left|J^{(k)}(z)\right|$, where $J^{(k)}$ denotes derivative of J of order k. In Proposition 2.1 we prove that

$$\sup_{z\in\mathcal{A}}\left|J^{(k)}(z)\right| = \max\left\{\left|J^{(k)}(\inf\mathcal{A})\right|, \left|J^{(k)}(\sup\mathcal{A})\right|\right\},$$

where $\mathcal{A}$ is a subset of the domain of function J.

Invariant measures for HJM model with Hilbert space valued Wiener noise are considered by Tehranchi in [11]. He gives a sufficient condition for the existence of an invariant measure for HJM model on weighted Sobolev space H_w, proposed as appropriate state space in Filipovic [3]. In Tehranchi [11] the condition is derived from a known theorem on invariant measures for stochastic evolution equations with Wiener noise (Da Prato and Zabczyk [1], Theorem 6.3.2).

We derive our condition from general theorem on invariant measures for stochastic evolution equations with Lévy noise (Rusinek [10], Theorem 4.1).

In Filipovic and Tappe [4] it is shown that HJM function $\mathcal{T}$

$$\mathcal{T}(f)(x) = J'\left(\int_0^x f(s)ds\right)f(x),$$

is locally Lipschitz mapping on $H_w^0 = H_w \cap \{f : f(\infty) = 0\}$.
We consider HJM model on a different space. Computing Lipschitz constant of α, we prove that HJM function $\mathcal{T}$ is locally Lipschitz mapping on weighted L^2 space.
In Filipovic and Tappe [4] the existence of solutions is considered, but not the existence of invariant measures. For the existence of solutions it suffices to know that α is Lipschitz mapping. For the existence of invariant measures it is worthwhile to examine its Lipschitz constant. It makes difference if it is large or small. In Proposition 2.1 we obtain results on expressions appearing in Lipschitz constant of α.

In Section 2 results on Laplace exponent are stated. Section 3 presents forward rate HJM model in Musiela parametrization. In Section 4 we define space chosen as a state space and formulate the main theorem. In Section 5 we give a few examples. Section 6 contains the proof of the main theorem and Section 7 is devoted to the study of Laplace exponent.

2 Properties of Laplace exponent

Let $Z(t)_{t\geq 0}$ be a one-dimensional Lévy process, i.e. a process with independent and stationary increments taking values in $\mathbb{R}$. Its Laplace exponent $J : \mathbb{R} \to (-\infty, +\infty]$ is given by

$$\mathbb{E}\, e^{-zZ(1)} = e^{J(z)}. \tag{2.1}$$

Let $\mathcal{D}_J = \{z \in \mathbb{R} : J(z) < +\infty\}$. By Lemma 8.1 and Lemma 8.2 from Appendix for every k derivative of J of order k, denoted by $J^{(k)}$, is well-defined on interior of $\mathcal{D}_J$. The following result will be proved in Section 7.

Proposition 2.1. *Suppose that $\mathcal{A} \subset \mathcal{D}_J$. Then $\sup_{z\in\mathcal{A}} \left|J^{(k)}(z)\right| < +\infty$ if and only if $J^{(k)}(\inf \mathcal{A})$, $J^{(k)}(\sup \mathcal{A})$ are well-defined. For every $k \geq 1$*

$$\sup_{z\in\mathcal{A}} \left|J^{(k)}(z)\right| = \max\left\{\left|J^{(k)}(\inf \mathcal{A})\right|, \left|J^{(k)}(\sup \mathcal{A})\right|\right\}. \tag{2.2}$$

If Z has only negative jumps, then for $k \geq 2$

$$\sup_{z\in\mathcal{A}} \left|J^{(k)}(z)\right| = J^{(k)}(\sup \mathcal{A}). \tag{2.3}$$

If Z has only positive jumps, then for $k \geq 2$

$$\sup_{z\in\mathcal{A}} \left|J^{(k)}(z)\right| = \left|J^{(k)}(\inf \mathcal{A})\right|. \tag{2.4}$$

We abbreviate $J^{(1)}$ to J' and $J^{(2)}$ to J''. Since J'' is always positive, as a corollary from Proposition 2.1 we obtain the following result.

Proposition 2.2. *Suppose that $0 \in \mathcal{A} \subset \mathcal{D}_J$ and $J'(0) = 0$. Then*

$$\sup_{z\in\mathcal{A}} |J'(z)| = \max\{-J'(\inf \mathcal{A}), J'(\sup \mathcal{A})\}, \tag{2.5}$$

$$\sup_{z\in\mathcal{A}} |J''(z)| = \max\{J''(\inf \mathcal{A}), J''(\sup \mathcal{A})\}. \tag{2.6}$$

If Z has only negative jumps, then

$$\sup_{z\in\mathcal{A}} |J''(z)| = J''(\sup \mathcal{A}). \tag{2.7}$$

If Z has only positive jumps, then

$$\sup_{z\in\mathcal{A}} |J''(z)| = J''(\inf \mathcal{A}). \tag{2.8}$$

3 HJM model

Let H be a separable Hilbert space of real functions defined on $[0, +\infty)$. **Forward rate HJM model** on H driven by a Lévy process Z is given by equation

$$f_t = S(t)f_0 + \int_0^t S(t-s)\alpha(f_s)ds + \int_0^t S(t-s)\sigma(f_s)dZ(s), \tag{3.1}$$

where $f_0 \in H$ and

- $S(t)_{t\geq 0}$ is the semigroup of shift operators, i.e. $(S(t)f)(x) = f(x+t)$.
- σ is a mapping from H into H such that J' is well-defined on set

$$\mathcal{O} = \left\{ \int_0^x \sigma(f)(s)ds : x \geq 0, f \in H \right\}.$$

- $\alpha : H \to H$ is given by

$$\alpha(f)(x) = J'\left(\int_0^x \sigma(f)(s)ds \right) \sigma(f)(x).$$

Let $Z(t)_{t\geq 0}$, $Z_m(t)_{t\geq 0}$ be two Lévy processes with Laplace exponents J, J_m given by

$$\mathbb{E}\, e^{-zZ(1)} = e^{J(z)}, \quad \mathbb{E}\, e^{-zZ_m(1)} = e^{J_m(z)},$$

such that $Z_m(t) = Z(t) + mt$. Then $\mathbb{E}\, Z(1) = 0$ if and only if $\mathbb{E}\, Z_m(1) = m$. For mapping $\sigma : H \to H$ coefficients α, α_m are given by

$$\alpha(f)(x) = J'\left(\int_0^x \sigma(f)(s)ds \right) \sigma(f)(x),$$
$$\alpha_m(f)(x) = J'_m\left(\int_0^x \sigma(f)(s)ds \right) \sigma(f)(x).$$

The following proposition shows that without any loss of generality one can restrict considerations to HJM equations (3.1) driven by Lévy processes with mean zero.

Proposition 3.1. *Process $(f_t)_{t\geq 0}$ satisfies equation*

$$f_t = S(t)f_0 + \int_0^t S(t-s)\alpha_m(f_s)ds + \int_0^t S(t-s)\sigma(f_s)dZ_m(s),$$

if and only if it satisfies equation

$$f_t = S(t)f_0 + \int_0^t S(t-s)\alpha(f_s)ds + \int_0^t S(t-s)\sigma(f_s)dZ(s).$$

Proof of Proposition 3.1. Since $-zZ_m(1) = -z(Z(1)+m)$, we have

$$J_m(z) = J(z) - mz,$$

from which

$$\alpha_m(f) = \alpha(f) - m\sigma(f).$$

Therefore

$$\begin{aligned} f_t &= S(t)f_0 + \int_0^t S(t-s)\alpha_m(f_s)ds + \int_0^t S(t-s)\sigma(f_s)dZ_m(s) \\ &= S(t)f_0 + \int_0^t S(t-s)\left(\alpha(f_s) - m\sigma(f_s)\right)ds + \int_0^t S(t-s)\sigma(f_s)\left(dZ(s) + mds\right) \\ &= S(t)f_0 + \int_0^t S(t-s)\alpha(f_s)ds + \int_0^t S(t-s)\sigma(f_s)dZ(s). \end{aligned}$$

□

From now on we assume that $\mathbb{E}\, Z(1) = 0$ and denote

$$\overline{\lambda} = \mathrm{Var}\, Z(1).$$

Set $\mathcal{O}$ is an interval containing 0. From (2.1)

$$J'(0) = -\mathbb{E}\, Z(1) = 0,$$

therefore we can apply Proposition 2.2 to expressions $\sup_{z\in\mathcal{O}} |J'(z)|$, $\sup_{z\in\mathcal{O}} |J''(z)|$. Those expressions appear in conditions for the existence of an invariant measure for HJM model. From now on we make the assumption:

$$J''(\inf \mathcal{O}) < +\infty, \quad J''(\sup \mathcal{O}) < +\infty.$$

Example Suppose that function $\sigma(f) \in H$ is positive for every $f \in H$ and Z has only positive jumps. We claim that

$$\sup_{z\in\mathcal{O}} |J''(z)| = \overline{\lambda}. \tag{3.2}$$

Indeed, to obtain (3.2) from (2.8), see that $\inf \mathcal{O} = 0$ and $J''(0) = \overline{\lambda}$.

4 The main theorem

We will denote by L^1 the space of all functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|f\|_{L^1} = \int_0^{+\infty} |f(x)|dx < +\infty.$$

For $\gamma > 0$, let L^2_γ denote the space of all functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $\int_0^{+\infty} |f(x)|^2 e^{\gamma x} dx < +\infty$, with inner product

$$\langle f, g \rangle_{L^2_\gamma} = \int_0^{+\infty} f(x) g(x) e^{\gamma x} dx.$$

By Hölder inequality

$$\int_0^{+\infty} |f(x)| dx \le \left(\int_0^{+\infty} |f(x)|^2 e^{\gamma x} dx \right)^{\frac{1}{2}} \left(\int_0^{+\infty} e^{-\gamma x} dx \right)^{\frac{1}{2}},$$

so $L^2_\gamma \subset L^1$ and

$$\|f\|_{L^1} \le \gamma^{-\frac{1}{2}} \|f\|_{L^2_\gamma}. \tag{4.1}$$

We define $\hat{L^2_\gamma}$ to be the space of all functions $f : \mathbb{R}_+ \to \mathbb{R}$ for which there exists $\varphi(f) \in \mathbb{R}$ such that $f - \varphi(f)\mathbf{1} \in L^2_\gamma$, with inner product

$$\langle f, g \rangle_{\hat{L^2_\gamma}} = \varphi(f)\varphi(g) + \langle f - \varphi(f)\mathbf{1}, g - \varphi(g)\mathbf{1} \rangle_{L^2_\gamma}.$$

The following theorem will be proved in Section 6.

Theorem 4.1. *Assume that $\sigma : \hat{L^2_\gamma} \to L^2_\gamma \subset \hat{L^2_\gamma}$ and there exist $K_\gamma, M_\gamma > 0$ such that for every $f, g \in \hat{L^2_\gamma}$*

$$\|\sigma(f) - \sigma(g)\|_{\hat{L^2_\gamma}} \le K_\gamma \|f - g\|_{\hat{L^2_\gamma}}, \tag{4.2}$$

$$\|\sigma(f)\|_{\hat{L^2_\gamma}} \le M_\gamma. \tag{4.3}$$

Let

$$\mathcal{O}_\gamma = \left\{ \int_0^x \sigma(f)(s) ds : x \ge 0, f \in \hat{L^2_\gamma} \right\}. \tag{4.4}$$

If

$$-\gamma^{\frac{3}{2}} + \left(\overline{\lambda} {K_\gamma}^2 + 2K_\gamma \sup_{z \in \mathcal{O}_\gamma} |J'(z)| \right) \gamma^{\frac{1}{2}} + 2K_\gamma M_\gamma \sup_{z \in \mathcal{O}_\gamma} |J''(z)| < 0, \tag{4.5}$$

then there exists an invariant measure for HJM model (3.1) *on $\hat{L^2_\gamma}$.*

For $\sigma : \hat{L^2_\gamma} \to L^2_\gamma \subset \hat{L^2_\gamma}$ we will always define K_γ, M_γ and $\mathcal{O}_\gamma$ by (4.2),(4.3) and (4.4). We also define

$$r_\gamma = \sup_{f \in \hat{L^2_\gamma}} \|\sigma(f)\|_{L^1}.$$

From (4.1)

$$\mathcal{O}_\gamma \subset [-r_\gamma, r_\gamma] \subset \left[-\gamma^{-\frac{1}{2}} M_\gamma, \gamma^{-\frac{1}{2}} M_\gamma \right].$$

Remark The following easier to check condition

$$\sup_{z\in\mathcal{O}_\gamma} |J''(z)| < \gamma^{\frac{3}{2}} \left({K_\gamma}^2 \gamma^{\frac{1}{2}} + 4K_\gamma M_\gamma \right)^{-1}, \tag{4.6}$$

also implies the existence of an invariant measure. In fact, it is stronger then condition (4.5).
Indeed, $\mathcal{O}_\gamma$ is a convex set containing 0, so for $z \in \mathcal{O}_\gamma$, by Lagrange theorem, $J'(z) = J'(z) - J'(0) = J''(\xi)z$, where $\xi \in \mathcal{O}_\gamma$. We get

$$\sup_{z\in\mathcal{O}_\gamma} |J'(z)|\gamma^{\frac{1}{2}} \leq \sup_{z\in\mathcal{O}_\gamma} |J''(z)|M_\gamma, \tag{4.7}$$

since for $z \in \mathcal{O}_\gamma$, we have $|z| \leq \gamma^{-\frac{1}{2}} M_\gamma$. Assume that (4.6) is fulfilled. Then

$$\begin{aligned}
\gamma^{\frac{3}{2}} &> \sup_{z\in\mathcal{O}_\gamma} |J''(z)| {K_\gamma}^2 \gamma^{\frac{1}{2}} + 2 \sup_{z\in\mathcal{O}_\gamma} |J''(z)| K_\gamma M_\gamma + 2 \sup_{z\in\mathcal{O}_\gamma} |J''(z)| K_\gamma M_\gamma \\
&\geq \overline{\lambda} {K_\gamma}^2 \gamma^{\frac{1}{2}} + 2 \sup_{z\in\mathcal{O}_\gamma} |J''(z)| K_\gamma M_\gamma + 2 \sup_{z\in\mathcal{O}_\gamma} |J''(z)| K_\gamma M_\gamma,
\end{aligned}$$

since $\overline{\lambda} = J''(0)$. Now, from (4.7), we get

$$\gamma^{\frac{3}{2}} > \overline{\lambda} {K_\gamma}^2 \gamma^{\frac{1}{2}} + 2K_\gamma \sup_{z\in\mathcal{O}_\gamma} |J'(z)|\gamma^{\frac{1}{2}} + 2 \sup_{z\in\mathcal{O}_\gamma} |J''(z)| K_\gamma M_\gamma.$$

5 Examples

Consider HJM model with volatility $\tilde{\sigma}$ such that condition (4.6) does not hold. If, for small $\varepsilon > 0$, we consider HJM model with volatility $\varepsilon\tilde{\sigma}$ instead of $\tilde{\sigma}$, then condition (4.6) will hold. The following proposition states that there always exists an invariant measure for HJM model, if noise is small enough.

Proposition 5.1. *Assume that $\tilde{\sigma} : \hat{L}^2_\gamma \to L^2_\gamma \subset \hat{L}^2_\gamma$ is Lipschitz and bounded mapping and for some $r^* > 0$*

($*$) *J', J'' are well-defined on $[-r^*, r^*]$.*

Then for ε sufficiently small there exists an invariant measure for HJM model (3.1) *on $\hat{L}^2_\gamma$ with σ given by*

$$\sigma(f)(x) = \varepsilon\tilde{\sigma}(f)(x).$$

Proof of Proposition 5.1. There exist $\tilde{K}_\gamma, \tilde{M}_\gamma > 0$ such that for every $f, g \in \hat{L}^2_\gamma$

$$\begin{aligned}
\|\tilde{\sigma}(f) - \tilde{\sigma}(g)\|_{\hat{L}^2_\gamma} &\leq \tilde{K}_\gamma \|f - g\|_{\hat{L}^2_\gamma}, \\
\|\tilde{\sigma}(f)\|_{\hat{L}^2_\gamma} &\leq \tilde{M}_\gamma.
\end{aligned}$$

Then for some $\tilde{r}_\gamma > 0$, we have $\|\tilde{\sigma}(f)\|_{L^1} \leq \tilde{r}_\gamma$ for every $f \in \hat{L}^2_\gamma$. Fix $\varepsilon > 0$ such that

$$\varepsilon < \frac{r^*}{\tilde{r}_\gamma}, \tag{5.1}$$

$$\varepsilon^2 < \gamma^{\frac{3}{2}} \left(\max_{|z|=r^*} J''(z) \right)^{-1} \left(\tilde{K}_\gamma^2 \gamma^{\frac{1}{2}} + 4\tilde{K}_\gamma \tilde{M}_\gamma \right)^{-1}. \tag{5.2}$$

Let $\sigma = \varepsilon\tilde{\sigma}$. Then $M_\gamma = \varepsilon\tilde{M}_\gamma$, $K_\gamma = \varepsilon\tilde{K}_\gamma$. It follows that J' is well-defined on $\mathcal{O}_\gamma$, since

$$\mathcal{O}_\gamma \subset [-\varepsilon\tilde{r}_\gamma, \varepsilon\tilde{r}_\gamma] \subset [-r^*, r^*],$$

by (5.1). And from (2.6)

$$\sup_{z\in\mathcal{O}_\gamma} |J''(z)| \leq \sup_{|z|\leq r^*} |J''(z)| = \max_{|z|=r^*} J''(z).$$

Inequality (5.2) now leads to

$$\begin{aligned} \sup_{z\in\mathcal{O}_\gamma} |J''(z)| &< \gamma^{\frac{3}{2}} \left(\varepsilon^2 \tilde{K}_\gamma^2 \gamma^{\frac{1}{2}} + 4\varepsilon\tilde{K}_\gamma \varepsilon\tilde{M}_\gamma \right)^{-1} \\ &= \gamma^{\frac{3}{2}} \left({K_\gamma}^2 \gamma^{\frac{1}{2}} + 4K_\gamma M_\gamma \right)^{-1}. \end{aligned}$$

□

Remark Condition $(*)$ is fulfilled if and only if $J''(-r^*), J''(r^*) < +\infty$.
If $\sigma(f)$ is positive for every $f \in \hat{L}^2_\gamma$, then the proposition still holds if we replace $(*)$ by assumption: $J''(r^*) < +\infty$.

In the following proposition we compute constants K_γ, M_γ, r_γ for volatility of the form

$$\sigma(f)(x) = v\left(f(x)\right)\psi(x), \tag{5.3}$$

for some real functions v, ψ. The proof is left to Appendix.

Proposition 5.2. *Consider volatility given by* (5.3)*, where* $\|\psi\|_{L^\infty} < +\infty$*, and* v *is a real function such that for some* $K, M > 0$

$$\begin{aligned} |v(x) - v(y)| &\leq K|x-y|, \\ |v(x)| &\leq M, \end{aligned}$$

for every $x, y \in \mathbb{R}$*. If in addition for some* $\gamma > 0$ $\psi \in L^2_\gamma$*, then*

$$\begin{aligned} K_\gamma &= \sqrt{2}K \max\left\{ \|\psi\|_{L^2_\gamma}, \|\psi\|_{L^\infty} \right\}, \\ M_\gamma &= M\|\psi\|_{L^2_\gamma}, \quad r_\gamma = M\|\psi\|_{L^1}. \end{aligned}$$

As an application of Proposition 5.2 we have the following result.

Proposition 5.3. *Assume that $\overline{\lambda} = 1$ and Z has only positive jumps. Let $v : \mathbb{R} \to \mathbb{R}$ satisfy*

$$|v(x) - v(y)| \leq |x - y|,$$
$$0 \leq v(x) \leq 1,$$

for every $x, y \in \mathbb{R}$. If

$$\gamma > 1 + \sqrt{1 + 4\sqrt{2}} \approx 3,5801,$$

then there exists an invariant measure for HJM model (3.1) *on $\hat{L}^2_\gamma$ with σ given by*

$$\sigma(f)(x) = v\left(f(x)\right) e^{-\gamma x}.$$

Proof of Proposition 5.3. Applying Proposition 5.2 with $K = M = 1$ and $\psi(x) = e^{-\gamma x}$, we compute

$$K_\gamma = \sqrt{2}, \quad M_\gamma = \gamma^{-\frac{1}{2}}.$$

Thus condition (4.6) takes the following form:

$$\sup_{z \in \mathcal{O}_\gamma} |J''(z)| \left(2\gamma^{\frac{1}{2}} + 4\sqrt{2}\gamma^{-\frac{1}{2}}\right) < \gamma^{\frac{3}{2}}.$$

Multiplying by $\gamma^{\frac{1}{2}}$, we get

$$\sup_{z \in \mathcal{O}_\gamma} |J''(z)| \left(2\gamma + 4\sqrt{2}\right) < \gamma^2.$$

But $\sup_{z \in \mathcal{O}_\gamma} |J''(z)| = 1$, which follows from (3.2). Hence

$$-\gamma^2 + 2\gamma + 4\sqrt{2} < 0.$$

□

6 Proof of Theorem 4.1

Let H be a separable Hilbert space of real functions defined on $[0, +\infty)$. It is worthwhile to rewrite HJM equation (3.1) on H as

$$df_t = \left(\frac{\partial}{\partial x} f_t + \alpha(f_t)\right) dt + \sigma(f_t) dZ(t). \tag{6.1}$$

Let η be a random variable with distribution $\mathcal{L}(\eta)$. A probability measure μ on H is an invariant measure for equation (6.1), if for all $t \geq 0$, we have $\mathcal{L}\left(f_t\right) = \mu$, where $(f_t)_{t \geq 0}$ is a solution of equation (6.1) with a random initial condition η such that $\mathcal{L}(\eta) = \mu$.

We recall Theorem 4.1 from Rusinek [10], which gives a sufficient condition for the existence of an invariant measure for equation of the form

$$dX(t) = \left(AX + F(X)\right) dt + B(X) dZ(t). \tag{6.2}$$

The theorem will be formulated in case Z takes values in $\mathbb{R}$ and $\mathbb{E}\, Z(1) = 0$.

Theorem 6.1. *Let $(H, \|\cdot\|)$ be a separable Hilbert space. Assume that*

- *Stroungly continuous semigroup $S(t)_{t\geq 0}$ is generated by A - linear operator on H with dense domain - which in general may be unbounded.*
- *F, B are bounded mappings from H into H.*
- *Z is a one-dimensional Lévy process such that $\operatorname{Var} Z(1) < +\infty$ and $\mathbb{E}\, Z(1) = 0$.*

If

$$\begin{aligned} \|S(t)x\| &\leq e^{-\beta t} \|x\|, \\ \|F(x) - F(y)\| &\leq L_F \|x - y\|, \\ \|B(x) - B(y)\| &\leq L_B \|x - y\|, \end{aligned}$$

and

$$-2\beta + 2L_F + \operatorname{Var} Z(1) {L_B}^2 < 0, \tag{6.3}$$

then there exists an invariant measure for equation (6.2).

Remark Let $(f_t^\eta)_{t\geq 0}$ denote the solution of HJM equation (3.1) on $\hat{L}^2_\gamma$ with initial condition $\eta \in \hat{L}^2_\gamma$. Theorem 4.1 from Rusinek [10] ensures the uniqueness of an invariant measure. Hence in the proof of Theorem 4.1 we obtain that if $\kappa, \eta \in \hat{L}^2_\gamma$ are such that

$$\kappa - c\mathbf{1}, \eta - c\mathbf{1} \in L^2_\gamma,$$

for the same $c \in \mathbb{R}$, then $\mathcal{L}(f_t^\kappa), \mathcal{L}(f_t^\eta)$ converge to the same invariant measure μ_c, depending only on c.

Dependence between the coefficients α and σ in HJM equation (3.1) is a consequence of no-arbitrage and it was proved in Jakubowski and Zabczyk [7] that $\alpha(f) = \mathcal{T}(\sigma(f))$, where $\mathcal{T}$ is the so-called HJM function given by

$$\mathcal{T}(f)(x) = J'\left(\int\limits_0^x f(s)ds\right) f(x).$$

In order to prove Theorem 4.1, we show that $\alpha : \hat{L}^2_\gamma \to L^2_\gamma \subset \hat{L}^2_\gamma$ is Lipschitz mapping, whenever σ is Lipschitz and bounded, computing Lipschitz constant of α in the following lemma.

Lemma 6.2. *Assume that $\sigma : \hat{L}^2_\gamma \to L^2_\gamma \subset \hat{L}^2_\gamma$. Let $\alpha(f) = \mathcal{T}(\sigma(f))$. For every $F, G \in \hat{L}^2_\gamma$, we have*

$$\|\alpha(F) - \alpha(G)\|_{L^2_\gamma} \leq \left(\sup_{z\in\mathcal{O}_\gamma} |J''(z)|\gamma^{-\frac{1}{2}} M_\gamma + \sup_{z\in\mathcal{O}_\gamma} |J'(z)|\right) K_\gamma \|F - G\|_{\hat{L}^2_\gamma},$$

where $K_\gamma, M_\gamma, \mathcal{O}_\gamma$ are given by (4.2), (4.3), (4.4).

Proof of Lemma 6.2. Let $F, G \in \hat{L}^2_\gamma$ and write $f = \sigma(F)$, $g = \sigma(G)$. Then $f, g \in L^2_\gamma$ and for every $x \geq 0$

$$\int_0^x f(s)ds, \int_0^x g(s)ds \in \mathcal{O}_\gamma.$$

Throughout the proof, V_1 stands for $\sup\limits_{z\in\mathcal{O}_\gamma} |J'(z)|$, V_2 stands for $\sup\limits_{z\in\mathcal{O}_\gamma} |J''(z)|$. With this notation,

$$\left| J'\left(\int_0^x f(s)ds\right)\right| \leq V_1. \tag{6.4}$$

By Lagrange theorem, $J'\left(\int_0^x f(s)ds\right) - J'\left(\int_0^x g(s)ds\right) = J''(\xi)\int_0^x (f-g)(s)ds$, where $\xi \in \mathcal{O}_\gamma$. Hence

$$\left| J'\left(\int_0^x f(s)ds\right) - J'\left(\int_0^x g(s)ds\right)\right| \leq V_2 \left\|f-g\right\|_{L^1}.$$

Thus

$$\left| J'\left(\int_0^x f(s)ds\right) - J'\left(\int_0^x g(s)ds\right)\right| \leq V_2\gamma^{-\frac{1}{2}} \left\|f-g\right\|_{L^2_\gamma}, \tag{6.5}$$

by (4.1). With the notation $h = \mathcal{T}(f) - \mathcal{T}(g)$, we have

$$\|\mathcal{T}(f) - \mathcal{T}(g)\|^2_{L^2_\gamma} = \langle \mathcal{T}(f) - \mathcal{T}(g), h\rangle_{L^2_\gamma} = I_1 + I_2,$$

where

$$I_1 = \int_0^{+\infty} J'\left(\int_0^x f(s)ds\right)(f(x) - g(x))\,h(x)e^{\gamma x}dx,$$
$$I_2 = \int_0^{+\infty} \left(J'\left(\int_0^x f(s)ds\right) - J'\left(\int_0^x g(s)ds\right)\right) g(x)h(x)e^{\gamma x}dx.$$

From (6.4), (6.5)

$$|I_1| \leq V_1 \int_0^{+\infty} |f(x) - g(x)|\,|h(x)|\,e^{\gamma x}dx,$$

$$|I_2| \leq V_2\gamma^{-\frac{1}{2}} \left\|f-g\right\|_{L^2_\gamma} \int_0^{+\infty} |g(x)|\,|h(x)|\,e^{\gamma x}dx.$$

By Hölder inequality,

$$|I_1| \le V_1 \|f-g\|_{L^2_\gamma} \|h\|_{L^2_\gamma},$$
$$|I_2| \le V_2 \gamma^{-\frac{1}{2}} \|f-g\|_{L^2_\gamma} \|g\|_{L^2_\gamma} \|h\|_{L^2_\gamma},$$

so

$$\begin{aligned}\langle \mathcal{T}(f) - \mathcal{T}(g), h\rangle_{L^2_\gamma} &\le |I_1| + |I_2| \\ &\le \left(V_2\gamma^{-\frac{1}{2}} \|g\|_{L^2_\gamma} + V_1\right) \|f-g\|_{L^2_\gamma} \|h\|_{L^2_\gamma}.\end{aligned}$$

Therefore

$$\begin{aligned}\|\alpha(F) - \alpha(G)\|_{L^2_\gamma} = \|\mathcal{T}(f) - \mathcal{T}(g)\|_{L^2_\gamma} &\le \left(V_2\gamma^{-\frac{1}{2}} \|g\|_{L^2_\gamma} + V_1\right) \|f-g\|_{L^2_\gamma} \\ &\le \left(V_2\gamma^{-\frac{1}{2}} M_\gamma + V_1\right) K_\gamma \|F-G\|_{\hat{L^2_\gamma}}.\end{aligned}$$

□

Proof of Theorem 4.1. For $\eta \in \hat{L^2_\gamma}$ let $c = \varphi(\eta)$. Then

$$\eta^c = \eta - c\mathbf{1} \in L^2_\gamma.$$

Let $\sigma^c, \alpha^c : L^2_\gamma \to L^2_\gamma$ be defined by $\sigma^c(f) = \sigma(f + c\mathbf{1})$, $\alpha^c(f) = \alpha(f + c\mathbf{1})$. Consider equation on L^2_γ

$$g_t = S(t)\eta^c + \int_0^t S(t-s)\alpha^c(g_s)ds + \int_0^t S(t-s)\sigma^c(g_s)dZ(s). \tag{6.6}$$

The idea of defining σ^c, α^c and working on subspace comes from Tehranchi [11]. Let us apply condition (6.3) from Theorem 6.1. In our case $\beta = \frac{1}{2}\gamma$. From Lemma 6.2

$$L_F = \left(\sup_{z\in\mathcal{O}_\gamma} |J''(z)|\gamma^{-\frac{1}{2}} M_\gamma + \sup_{z\in\mathcal{O}_\gamma} |J'(z)|\right) K_\gamma.$$

And $L_B = K_\gamma$, so condition (6.3) will take the following form:

$$-\gamma + 2\left(\sup_{z\in\mathcal{O}_\gamma} |J''(z)|\gamma^{-\frac{1}{2}} M_\gamma + \sup_{z\in\mathcal{O}_\gamma} |J'(z)|\right) K_\gamma + \overline{\lambda} {K_\gamma}^2 < 0.$$

After multiplying by $\gamma^{\frac{1}{2}}$, you get exactly condition (4.5). It follows that $\mathcal{L}(g_t)$ converges to $\mathcal{L}(\mathcal{X}_c)$ for random variable $\mathcal{X}_c$ such that $\mathbb{E}\,{\mathcal{X}_c}^2 < +\infty$. If $(g_t)_{t\ge 0}$ is a solution to (6.6), then $(f_t)_{t\ge 0}$ given by $f_t = g_t + c\mathbf{1}$ is a solution to

$$f_t = S(t)\eta + \int_0^t S(t-s)\alpha(f_s)ds + \int_0^t S(t-s)\sigma(f_s)dZ(s).$$

So $\mathcal{L}(f_t)$ converges to $\mathcal{L}(\mathcal{X}_c + c\mathbf{1})$, hence $\mu_c = \mathcal{L}(\mathcal{X}_c + c\mathbf{1})$ is an invariant measure for HJM model space $\hat{L^2_\gamma}$. □

7 Proof of Proposition 2.1

Let $Z(t)_{t\geq 0}$ be a Lévy process taking values in $\mathbb{R}$. Associated with Z is the so-called Lévy measure of Z, denoted ν, measure on $\mathbb{R}$, given by

$$\nu\left(\Gamma\right)=\mathbb{E}\left(\sum_{0<s\leq 1}\mathbf{1}_{\Gamma}\left(Z(s)-Z\left(s^{-}\right)\right)\right),$$

where Γ is a Borel subset of $\mathbb{R}$ such that $\overline{\Gamma}\subset\mathbb{R}\setminus\{0\}$. And $\nu\left(\{0\}\right)=0$. It is well-known that ν satisfies the following conditions

$$\int\limits_{|y|\leq 1} y^2\nu(dy)<+\infty, \tag{7.1}$$

$$\int\limits_{|y|>1} \nu(dy)<+\infty. \tag{7.2}$$

Process $Z(t)$ can be represented as

$$Z(t)=at+\sqrt{q}W(t)+\xi(t),$$

where $a\in\mathbb{R}$, $q>0$, W is a standard one-dimensional Wiener process and $\xi(t)$ is a jump process. We can present function J given by (2.1) in terms of a,q and function J_0 connected to measure ν. We have

$$J(z)=-az+\frac{1}{2}qz^2+J_0(z),$$

where function J_0 is given by

$$J_0(z)=\int\limits_{|y|\leq 1}(e^{-zy}-1+zy)\nu(dy)+\int\limits_{|y|>1}(e^{-zy}-1)\nu(dy).$$

$J(z)$ is well-defined if and only if $\int\limits_{|y|>1} e^{-zy}\nu(dy)<+\infty$ by assumptions (7.1), (7.2) and Lemma 8.1 from Appendix.

Under assumption (7.2) integral $\int\limits_{-\infty}^{-1} e^{-zy}\nu(dy)$ is well-defined for every $z<0$ and integral $\int\limits_{1}^{+\infty} e^{-zy}\nu(dy)$ is well-defined for every $z>0$. If $\int\limits_{-\infty}^{-1} e^{-ry}\nu(dy)<+\infty$ for some $r>0$ and $z<r$, then also $\int\limits_{-\infty}^{-1} e^{-zy}\nu(dy)<+\infty$. If $\int\limits_{1}^{+\infty} e^{-ry}\nu(dy)<+\infty$ for some $r<0$ and $z>r$, then also $\int\limits_{1}^{+\infty} e^{-zy}\nu(dy)<+\infty$. Define

$$r_\nu^-=\inf\left\{r\leq 0:\int\limits_{1}^{+\infty} e^{-ry}\nu(dy)<+\infty\right\}, \tag{7.3}$$

$$r_\nu^+=\sup\left\{r\geq 0:\int\limits_{-\infty}^{-1} e^{-ry}\nu(dy)<+\infty\right\}. \tag{7.4}$$

Then interior of $\mathcal{D}_J = \{z \in \mathbb{R} : J(z) < +\infty\}$ is given by

$$\mathrm{int}\mathcal{D}_J = \left(r_\nu^-, r_\nu^+\right). \tag{7.5}$$

By Lemma 8.1 and Lemma 8.2 from Appendix function J as well as derivative of J of any order are well-defined on set $\mathrm{int}\mathcal{D}_J$. We have

$$J'(z) = -a + qz + \int\limits_{|y|\le 1} \left(-ye^{-zy} + y\right)\nu(dy) + \int\limits_{|y|>1} -ye^{-zy}\nu(dy),$$

$$J''(z) = q + \int\limits_{\mathbb{R}} y^2 e^{-zy}\nu(dy).$$

And for $k \geq 3$

$$J^{(k)}(z) = \int\limits_{\mathbb{R}} (-y)^k e^{-zy}\nu(dy).$$

Proof of Proposition 2.1. Throughout the proof, $c = \inf \mathcal{A}$, $d = \sup \mathcal{A}$. The proof will be divided into 2 steps.
I. Assume that $\mathcal{A} = [c, d] \subset \mathrm{int}\mathcal{D}_J$. Consider $F : [c, d] \to \mathbb{R}$ such that F' and F'' exist on $[c, d]$. The proof is based on the following observations.

(i) If $F, F'' \geq 0$ on $[c, d]$, then $\sup\limits_{x\in[c,d]} |F(x)| = \max\{|F(c)|, |F(d)|\}$.

(ii) If $F' \geq 0$ on $[c, d]$, then $\sup\limits_{x\in[c,d]} |F(x)| = \max\{|F(c)|, |F(d)|\}$.

(iii) If $F, F' \geq 0$ on $[c, d]$, then $\sup\limits_{x\in[c,d]} |F(x)| = F(d)$.

(iv) If $F \geq 0$, $F' \leq 0$ on $[c, d]$, then $\sup\limits_{x\in[c,d]} |F(x)| = F(c)$.

(v) If $F \leq 0$, $F' \geq 0$ on $[c, d]$, then $\sup\limits_{x\in[c,d]} |F(x)| = -F(c)$.

Applying (i) to $J^{(k)}$ for even k and (ii) to $J^{(k)}$ for odd k, we obtain (2.2).
If Z has only negative jumps, then for $k \geq 3$

$$J^{(k)}(z) = \int\limits_{-\infty}^{-1} (-y)^k e^{-zy}\nu(dy) \geq 0,$$

so applying (iii) to $J^{(k)}$, we obtain (2.3).
If Z has only positive jumps, then for $k \geq 3$

$$J^{(k)}(z) = (-1)^k \int\limits_{1}^{+\infty} y^k e^{-zy}\nu(dy).$$

Thus function $J^{(k)}$ is positive for even k and negative for odd k. Applying (iv) to $J^{(k)}$ for even k and (v) to $J^{(k)}$ for odd k, we obtain (2.4).

II. Now let $\mathcal{A} \subset \mathcal{D}_J$ be any. To avoid technicalities, we shall prove that if $\sup_{z\in\mathcal{A}} |J'(z)| = M < +\infty$, then $J'(d)$ is well-defined. Let $\{d_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, $d_n \nearrow d$. Then sequence $\{X_n\}_{n\in\mathbb{N}}$ given by $X_n = J'(d_n)$ is nondecreasing and bounded, since $|X_n| \le M$, so there exists $X = \lim_{n\to\infty} X_n$, from which

$$|J'(d)| = |X| \le M = \sup_{z\in\mathcal{A}} |J'(z)| \,.$$

In the same manner we obtain $|J'(c)| \le \sup_{z\in\mathcal{A}} |J'(z)|$. It follows that

$$\begin{aligned}\max\{|J'(c)|\,, |J'(d)|\} \le \sup_{z\in\mathcal{A}} |J'(z)| &\le \sup_{z\in[c,d]} |J'(z)| \\ &= \max\{|J'(c)|\,, |J'(d)|\}\,.\end{aligned}$$

□

Remark If $\mathbb{E}\, Z(1) = 0$, then $J'(0) = 0$, J' is negative on $(-\infty, 0)$ and J' is positive on $(0, +\infty)$. Therefore, since $J(0) = 0$, we have $J \ge 0$, so (2.2) holds for J instead of $J^{(k)}$ as well. If $\mathbb{E}\, Z(1) = 0$, then

$$J(z) = \frac{1}{2} q z^2 + \int\limits_{\mathbb{R}} \left(e^{-zy} - 1 + zy\right) \nu(dy).$$

8 Appendix

Let ψ_X denote moment-generating function of a random variable X. Then function J is simply $J(z) = \ln \psi_{Z(1)}(-z)$. Moment-generating function together with characteristic function φ_X (and φ_X is connected to ψ_X by $\varphi_X(t) = \psi_X(it)$) are basic concepts of probability theory, but we add to the paper two well-known statements about function J also with proofs, which use concept of Lévy measure of a Lévy process.

Lemma 8.1. *Under assumption* (7.1) *for every* $z \in \mathbb{R}$ *integrals*

$$\int\limits_{|y|\le 1} \left(e^{-zy} - 1 + zy\right) \nu(dy), \qquad \int\limits_{|y|\le 1} y\left(e^{-zy} - 1\right) \nu(dy),$$

$$\int\limits_{|y|\le 1} y^k e^{-zy} \nu(dy), k \ge 2,$$

are well-defined.

Proof of Lemma 8.1. For every $x \in \mathbb{R}$ and $n \in \mathbb{N}$ there exists $\theta \in [0, 1]$ such that

$$e^x = \sum_{k=0}^{n-1} \frac{x^k}{k!} + e^{\theta x} \frac{x^n}{n!}.$$

So

$$\left| e^x - \sum_{k=0}^{n-1} \frac{x^k}{k!} \right| = \left| e^{\theta x} \frac{x^n}{n!} \right| \leq e^{|x|} \frac{|x|^n}{n!}.$$

For $n = 1$ we get $|e^x - 1| \leq e^{|x|}|x|$, for $n = 2$ we get $|e^x - 1 - x| \leq \frac{1}{2}e^{|x|}|x|^2$, thus

$$\int\limits_{|y|\leq 1} \left|e^{-zy} - 1 + zy\right| \nu(dy) \leq \int\limits_{|y|\leq 1} \frac{1}{2} e^{|zy|}|zy|^2 \nu(dy) \leq \frac{1}{2} e^{|z|}|z|^2 \int\limits_{|y|\leq 1} y^2 \nu(dy),$$

$$\int\limits_{|y|\leq 1} \left|\left(e^{-zy} - 1\right) y\right| \nu(dy) \leq \int\limits_{|y|\leq 1} e^{|zy|} \cdot |zy| \cdot |y| \nu(dy) \leq e^{|z|}|z| \int\limits_{|y|\leq 1} y^2 \nu(dy).$$

And for $k \geq 2$

$$\int\limits_{|y|\leq 1} |y|^k e^{-zy} \nu(dy) \leq e^{|z|} \int\limits_{|y|\leq 1} y^2 \nu(dy).$$

$\square$

Lemma 8.2. *For $k \geq 0$ define $\mathcal{D}_J^k = \left\{ z \in \mathbb{R} : \int\limits_{|y|>1} |y|^k e^{-zy} \nu(dy) < +\infty \right\}$. For every $k \geq 0$, we have $(r_\nu^-, r_\nu^+) \subset \mathcal{D}_J^k$, where r_ν^-, r_ν^+ are given by* (7.3), (7.4).

Proof of Lemma 8.2. For $k = 0$ we have $(r_\nu^-, r_\nu^+) \subset \mathcal{D}_J^0$ by definition of r_ν^- and r_ν^+. We now proceed by induction. Assume that $(r_\nu^-, r_\nu^+) \subset \mathcal{D}_J^k$. Let $z \in (r_\nu^-, r_\nu^+)$. We shall prove that $z \in \mathcal{D}_J^{k+1}$. Fix $\varepsilon > 0$ such that $z + \varepsilon < r_\nu^+$ and $z - \varepsilon > r_\nu^-$. Then $z - \varepsilon, z + \varepsilon \in \mathcal{D}_J^k$. We have

$$\int\limits_{|y|>1} |y|^{k+1} e^{-zy} \nu(dy) = \int\limits_{|y|>1} |y|^k |y| e^{-\varepsilon|y|} e^{\varepsilon|y|} e^{-zy} \nu(dy).$$

Let $F(y) = ye^{-\varepsilon y}$, $y \geq 0$. Then $F'(y^*) = 0$ for $y^* = \varepsilon^{-1}$, $F(0) = 0$, $F\left(\varepsilon^{-1}\right) = (\varepsilon e)^{-1}$ and $F(+\infty) = 0$, so

$$\sup_{y\geq 0} ye^{\varepsilon y} = \frac{1}{\varepsilon e}.$$

It follows that

$$\begin{aligned}
\int\limits_{|y|>1} |y|^{k+1} e^{-zy} \nu(dy) &\leq \frac{1}{\varepsilon e} \int\limits_{|y|>1} |y|^k e^{\varepsilon|y|} e^{-zy} \nu(dy) \\
&= \frac{1}{\varepsilon e} \int\limits_{-\infty}^{-1} |y|^k e^{-(z+\varepsilon)y} \nu(dy) + \frac{1}{\varepsilon e} \int\limits_{1}^{+\infty} |y|^k e^{-(z-\varepsilon)y} \nu(dy).
\end{aligned}$$

Hence $\int\limits_{|y|>1} |y|^{k+1} e^{-zy} \nu(dy) < +\infty$, since $z - \varepsilon, z + \varepsilon \in \mathcal{D}_J^k$. $\square$

We end Appendix with the proof of Proposition 5.2.

Proof of Proposition 5.2. Let $f, g \in \hat{L^2_\gamma}$. We have

$$\|\sigma(f)\|_{L^1} = \int_0^{+\infty} |v(f(x))| \, |\psi(x)| \, dx \leq M \, \|\psi\|_{L^1} \,,$$

$$\|\sigma(f)\|^2_{L^2_\gamma} = \int_0^{+\infty} |v(f(x))|^2 \, |\psi(x)|^2 \, e^{\gamma x} dx \leq M^2 \, \|\psi\|^2_{L^2_\gamma} \,.$$

And

$$\begin{aligned}
\|\sigma(f) - \sigma(g)\|^2_{L^2_\gamma} &= \int_0^{+\infty} |v(f(x)) - v(g(x))|^2 \, |\psi(x)|^2 \, e^{\gamma x} dx \\
&\leq K^2 \int_0^{+\infty} |f(x) - g(x)|^2 \, |\psi(x)|^2 \, e^{\gamma x} dx.
\end{aligned}$$

Let $c = \varphi(f), d = \varphi(g) \in \mathbb{R}$ be such that $f - c\mathbf{1}, g - d\mathbf{1} \in L^2_\gamma$. Then

$$\begin{aligned}
\|\sigma(f) - \sigma(g)\|^2_{L^2_\gamma} \leq 2K^2 &\int_0^{+\infty} |f(x) - c - (g(x) - d)|^2 \, |\psi(x)|^2 \, e^{\gamma x} dx \\
&+ 2K^2 \int_0^{+\infty} |c - d|^2 \, |\psi(x)|^2 \, e^{\gamma x} dx \\
&\leq 2K^2 \, \|\psi\|^2_{L^\infty} \, \|f - c\mathbf{1} - (g - d\mathbf{1})\|^2_{L^2_\gamma} + 2K^2 \, \|\psi\|^2_{L^2_\gamma} \, |c - d|^2 \\
&\leq 2K^2 \max\left\{ \|\psi\|^2_{L^\infty} \,, \|\psi\|^2_{L^2_\gamma} \right\} \|f - g\|^2_{\hat{L^2_\gamma}} \,.
\end{aligned}$$

□

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