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# Exponential moments for HJM models with jumps

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# Exponential moments for HJM models with jumps *

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**Abstract**

General HJM models driven by a Lévy process are considered. Necessary moment conditions for the discounted bond prices to be local martingales are derived. It is proved, under the moment conditions, that the discounted bond prices are local martingales if and only if a generalized HJM condition holds.

## 0.1 Introduction

Let $P(t,\theta)$, $0 \le t \le \theta$, be the market price at time $t$ of a bond paying 1 at the maturity time $\theta$. Let $T$ be a finite time horizon, i.e. $\theta \le T$. The *forward rate curve* is a function $f(t,\theta)$ such that

$$P(t,\theta) = e^{-\int_t^\theta f(t,s)ds}. \tag{1}$$

Heath, Jarrow and Morton [13] proposed to model the forward curves as Itô processes

$$df(t,\theta) = \alpha(t,\theta)dt + \langle \sigma(t,\theta),\, dZ(t)\rangle, \qquad 0 \le t \le \theta, \tag{2}$$

with $Z$ the $d$-dimensional standard Wiener process, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \boldsymbol{P})$. According to the observed bond prices, the (random) function $f(t,\theta)$ should be regular in $\theta$ for fixed $t$ and chaotic in $t$ for fixed $\theta$. The latter property is implied by the presence of the process $Z$ in the representation and the former by the regular dependence of $\alpha(t,\theta)$ and $\sigma(t,\theta)$ on $\theta$ for fixed $t$.

One says that the *HJM postulate* is satisfied for (1) if the discounted bond price processes $\hat{P}(t,\theta) := P(t,\theta)e^{-\int_0^t f(t,s)ds}$, $t \le \theta$, for $\theta \in [0,T]$ are local martingales (see e.g. [7] or [15]). When $Z$ is a Wiener process, conditions under which the HJM postulate is satisfied for (2) are well known.

We consider a generalization of this model replacing the Wiener process by a general Lévy process $Z$ with values in a separable Hilbert space $U$. Of course, $Z$

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can be finite-dimensional if $U = \mathbb{R}^d$. The importance of treating models with an infinite number of factors was stressed in recent papers of Carmona and Tehranchi [4], Ekeland and Taflin [11], Cont [5] and Özkan and Schmidt [16]. Özkan and Schmidt considered a model of defaultable bonds driven by a general Lévy process.

Basic results on HJM models driven by discontinuous noise were obtained in the pioneering papers by Björk at al. [2], [3]. For more recent contributions see Eberlein at al. [8], [9], [10].

Let

$$b(u) = \int_{|y|>1} e^{-\langle u,\, y\rangle} \nu(dy), \quad B = \{u \in U : \; b(u) < \infty\},$$

where $\nu$ is the jump measure of the Lévy process $Z$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $U$.

In Theorem 1 we show, under natural assumptions, that if the HJM postulate holds, then for each $\theta \leq T$,

$$\int_t^\theta \sigma(t,v)\, dv \in B, \quad dt \otimes dP\text{-almost surely.} \tag{3}$$

Thus the HJM postulate imposes existence of some exponential moments of the jump measure $\nu$, related to the behavior of the volatility. Recall that for an arbitrary Lévy process $Z$,

$$u \in B \;\; \text{iff} \;\; \boldsymbol{E} e^{-\langle u, Z_t\rangle} < \infty \quad \text{for some } t > 0 \tag{4}$$

(see Sato [19] and for the infinite dimensional extension [17]). Thus our result can be regarded as a stochastic analog of (4) but it is not a consequence of (4) and requires a new proof.

Part (ii) of Theorem 1 states that if the necessary condition (3) is satisfied then the HJM postulate holds if and only if the drift coefficient satisfies a generalized HJM type condition formulated in terms of the logarithm of the moment generating function of the process $Z$. The HJM condition as given in this paper was first formulated in Eberlein and Özkan [9] for models with a finite number of factors, and later, independently and for an infinite number of factors, in the first version of the present paper [14]. Our derivation of the HJM condition is obtained under minimal assumptions on the model. In [2], [3], [8], [9] some a priori requirements were imposed on the moments of the jump measure $\nu$. In particular in [9] it is required that there exists a constant $M > 0$ such that

$$\int_{|y|>1} e^{-\langle c,y\rangle} \nu(dy) < \infty \quad \text{for all } c \in [-M, M]^d. \tag{5}$$

However, as follows from Theorem 1, if $\sigma$ is a positive process and $Z$ has only positive jumps, no a priori requirements on $\nu$ are necessary.

In section 4 the properties of the set $B$ are discussed. It turns out that many properties of $B$ which hold in finite dimensions are not true in infinite dimensions. In particular, in infinite dimensions the set $B$ could be the difference of an open set and a dense subset. We also investigate the interplay between condition (3) and the supports of the distributions of $\int_t^\theta \sigma(t,v)\, dv$.

Section 5 concerns existence of strong exponential moments

$$\int_{|y|>1} e^{\gamma |y|} \nu(dy) < \infty.$$

In section 6 some extensions of our results are obtained for models

$$df(t,\theta) = \alpha(t,\theta)dt + \sigma(t,\theta)\, dW(t) + \int_{|y|\le 1} \sigma_0(t,\theta,y)(\mu(dt,dy) - dt\nu(dy)) + \int_{|y|>1} \sigma_1(t,\theta,y)\mu(dt,dy)$$

with $\mu$ a Poissonian random measure with intensity $\nu(dx)\otimes dt$. Such models have already been studied (see e.g. Björk et al. [2], [3], Eberlein and Raible [8], Eberlein and Özkan [9], Eberlein et al. [10]). In Björk et al. [2] sufficient, but not necessary, conditions for the HJM postulate to hold are given (Proposition 5.3 and Assumption 5.1 in [2]).

The present paper is a rewritten version of our preprint [14]. One of the referees indicated that in a forthcoming paper [16] the HJM conditions are obtained under an integrability condition.

For basic information on Lévy processes we refer to the books by Bertoin [1], Sato [19] and Gihman and Skorohod [12].

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## 0.2 Forward rate function driven by a Lévy process

We assume that the basic probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ is complete. Let $Z$ be a Lévy process in a separable Hilbert space $U$, i.e. a càdlàg process with stationary independent increments and values in $U$. Let $\mathcal{F}_t^0 = \sigma(Z(s); s \le t)$ be the $\sigma$-fields generated by $Z(t), t \ge 0$, and $\mathcal{F}_t$ be the completion of $\mathcal{F}_t^0$ by all sets of $\boldsymbol{P}$ probability zero. It is known that this filtration is right continuous, so it satisfies the "usual conditions". We denote by $\mu$ the measure associated to the jumps of $Z$, i.e for $\Gamma \in \mathcal{B}(U)$ with $\overline{\Gamma} \subset U \setminus \{0\}$,

$$\mu([0,t],\Gamma) = \sum_{0<s\le t} \mathbf{1}_\Gamma(\Delta Z(s)).$$

A measure $\nu$ such that

$$\boldsymbol{E}(\mu([0,t],\Gamma)) = t\nu(\Gamma)$$

is called a *Lévy measure* of the process $Z$.

Throughout the paper we denote the inner product in $U$ by $\langle \cdot\, , \cdot\rangle$ and the norm in $U$ by $|\cdot|$.

The characteristic function of $Z(t)$ has the form (Lévy-Khintchine formula)

$$\boldsymbol{E} e^{i\langle\lambda, Z(t)\rangle} = e^{t\psi(\lambda)},$$

where

$$\psi(\lambda) = i\langle a,\lambda\rangle - \frac{1}{2}\langle Q\lambda,\lambda\rangle + \int_U (e^{i\langle\lambda,x\rangle} - 1 - i\langle\lambda,x\rangle \mathbf{1}_{\{|x|\le 1\}}(x))\nu(dx),$$

and $a \in U$, $Q$ is a symmetric non-negative nuclear operator on $U$, $\nu$ is the measure on $U$ with $\nu(\{0\}) = 0$ and

$$\int_U (|x|^2 \wedge 1)\, \nu(dx) < \infty. \tag{6}$$

Moreover $Z$ has a decomposition

$$Z(t) = at + W(t) + \int_0^t \int_{|y|\le 1} y(\mu(ds,dy) - ds\nu(dy)) \tag{7}$$
$$+ \int_0^t \int_{|y|>1} y\mu(ds,dy),$$

where $W$ is a Wiener process with values in $U$ and covariance operator $Q$. Under additional conditions, the distributions of $Z(t), t \ge 0$, have exponential moments and the Laplace transform exists, and

$$\boldsymbol{E}\, e^{-\langle u, Z(t)\rangle} = e^{tJ(u)},$$

where

$$J(u) = -\langle a, u\rangle + \frac{1}{2}\langle Qu, u\rangle + J_0(u), \tag{8}$$
$$J_0(u) = \int_U \Big[e^{-\langle u,y\rangle} - 1 + \langle u, y\rangle \mathbf{1}_{\{|y|\le 1\}}(y)\Big]\nu(dy), \quad u \in U. \tag{9}$$

Let $b$ be the Laplace transform of the measure $\nu$ restricted to the complement of the ball $\{y:\ |y|\le 1\}$:

$$b(u) = \int_{|y|>1} e^{-\langle u,\, y\rangle}\nu(dy),$$

and $B$ the set of those $u \in U$ for which the Laplace transform is finite:

$$B = \{u \in U :\ b(u) < \infty\}.$$

It follows from Fatou's lemma that $b$ is lower semicontinuous and $B$ is a countable union of closed sets.

As was stated in the Introduction, we consider a generalized Heath, Jarrow and Morton model (2) taking a Lévy process $Z$ in $U$ instead of Wiener process,

$$df(t,\theta) = \alpha(t,\theta)dt + \langle \sigma(t,\theta),\ dZ(t)\rangle, \qquad 0 \le t \le \theta \le T, \tag{10}$$

where $T$ is a finite horizon. Equivalently, for $t \le \theta$,

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(s,\theta)ds + \int_0^t \langle \sigma(s,\theta),\ dZ(s)\rangle. \tag{11}$$

For each $\theta$ the processes $\alpha(t,\theta)$, $\sigma(t,\theta)$, $t \le \theta$, are assumed to be predictable with respect to a given filtration $(\mathcal{F}_t)$ and such that integrals in (11) are well defined. The forward rate curve function $f(t,\theta)$ defined by (1) is usually interpreted as the anticipated short rate at time $\theta$ as seen by the market at time $t$.

Let $r(t)$, $t \ge 0$, be the short rate process. If at time 0 one puts 1 into the bank account then at time $t$ one has

$$B_t = e^{\int_0^t r(\sigma)d\sigma}.$$

It is convenient to assume that once a bond has matured its cash equivalent goes to the bank account. Thus $P(t,\theta)$, the market price at time $t$ of a bond paying 1 at the maturity time $\theta$, is defined also for $t \ge \theta$ by the formula

$$P(t,\theta) = e^{\int_\theta^t r(\sigma)d\sigma}. \tag{12}$$

For $\theta < t$ we put

$$\alpha(t,\theta) = \sigma(t,\theta) = 0, \tag{13}$$

so the forward rate $f$ is defined for $t, \theta \in [0,T]$. By (13) we deduce from (11) that for $t > \theta$,

$$f(t,\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)ds + \int_0^\theta \langle \sigma(s,\theta),\ dZ(s)\rangle.$$

Consequently, for each $\theta > 0$ the process $f(t,\theta)$, $t > \theta$, is constant in $t$ and could be identified with the short rate:

$$r(\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)ds + \int_0^\theta \langle \sigma(s,\theta),\ dZ(s)\rangle. \tag{14}$$

>From now on we assume (10) and (13) and that the short rate is given by (14). We will assume that for given $T$, the integrals in the definition of $f$ exist in the sense of the Hilbert space $H = L^2(0,T)$ with the scalar product $(\cdot,\cdot)_H$. We will regard the coefficients $\alpha$ and $\sigma$ in (10) as, respectively, $H$ and $L(U,H)$ valued, predictable processes:

$$\alpha(t)(\theta) = \alpha(t,\theta), \theta \in [0,T],\ \ \sigma(t)u(\theta) = \langle \sigma(t,\theta), u\rangle,\ u \in U,\ \theta \in [0,T].$$

Then (10) can be written as

$$df(t) = \alpha(t)dt + \sigma(t)dZ(t). \tag{15}$$

## 0.3 Arbitrage free models

Let us recall that the *HJM postulate* is the requirement that the discounted bond price processes $\hat{P}(\cdot,\theta)$, $\theta \in [0,T]$, given by

$$\hat{P}(t,\theta) = P(t,\theta)/B_t = e^{-\int_t^\theta f(t,s)ds} e^{-\int_0^t f(t,s)ds} = e^{-\int_0^\theta f(t,s)ds}$$

are local martingales.

We now intend to prove a theorem giving necessary and sufficient conditions for the HJM postulate to be satisfied. The following assumptions are used throughout this paper:

$(H1)$ The processes $\alpha$ and $\sigma$ are predictable and with probability one have bounded trajectories.

$(H2)$ For arbitrary $r > 0$ the function $b$ is bounded on $\{u : |u| \le r,\ b(u) < \infty\}$.

**Theorem 1** *Assume that (H1) holds.*

(i) *If the HJM postulate holds then, for arbitrary $\theta \le T$, $\boldsymbol{P}$-almost surely,*

$$\int_t^\theta \sigma(t,v)\,dv \in B \tag{16}$$

*for almost all $t \in [0,\theta]$.*

(ii) *Assume (H2) and that for all* $\theta \le T$, $\boldsymbol{P}$ *-almost surely* (16) *holds for almost all* $t \in [0,\theta]$. *Then the HJM postulate holds if and only if the following HJM condition holds:*

$$\int_t^\theta \alpha(t,v)\,dv = J\left(\int_t^\theta \sigma(t,v)\,dv\right) \tag{17}$$

*for almost all* $t \in [0,\theta]$.

*Proof* Fix $\theta \le T$. Set, for $t \in [0,T]$,

$$A(t,\theta) = (\mathbf{1}_{[0,\theta]},\alpha)_H = \int_t^\theta \alpha(t,\eta)\,d\eta, \quad \Sigma(t,\theta) = \sigma^*(t)\mathbf{1}_{[0,\theta]} = \int_t^\theta \sigma(t,\eta)\,d\eta,$$

where $\sigma^*(t)$ is the adjoint operator to $\sigma(t)$.
Since $\theta$ is fixed in the following calculations we omit $\theta$ and write $A(t)$, $\Sigma(t)$. Let $X(t) = (\mathbf{1}_{[0,\theta]}, f(t))_H$. Then

$$\begin{aligned} dX(t) &= A(t)dt + \langle \Sigma(t), dZ(t)\rangle \\ &= A(t)dt + \langle \Sigma(t), a\,dt + dW(t)\rangle \\ &\quad + \int_U \mathbf{1}_{\{|z|\le 1\}}(z)\langle \Sigma(s), z\rangle\big(\mu(dt,dz) - dt\,\nu(dz)\big) \\ &\quad + \int_U \mathbf{1}_{\{|z|>1\}}(z)\langle \Sigma(s), z\rangle \mu(ds,dz). \end{aligned}$$

To apply Itô's formula (see e.g. [6]) to the process $\psi(X(t))$ for a function $\psi \in C^2$, denote by $\mu_X$ the jump measure of the semimartingale $X$. We have
$\Delta X(t) = \langle \Sigma(t), \Delta Z(t)\rangle$, therefore

$$\mu_X([0,t],\Gamma) = \sum_{s\le t} \mathbf{1}_\Gamma(\langle \Sigma(s), \Delta Z(s)\rangle) = \int_0^t \int_U \mathbf{1}_\Gamma(\langle \Sigma(s), z\rangle)\,\mu(ds,dz)$$

for $\Gamma \in \mathcal{B}(\mathbb{R})$ with $0 \notin \overline{\Gamma}$, and, more generally, for a non-negative predictable field $\varphi(s,y)$, $s \ge 0$, $y \in \mathbb{R}$

$$\int_0^t \int_{\mathbb{R}} \varphi(s,y)\,\mu_X(ds,dy) = \int_0^t \int_U \varphi(s, \langle \Sigma(s), z\rangle)\,\mu(ds,dz).$$

Moreover, the quadratic variation process of $\int_0^t \langle \Sigma(s), dW(s)\rangle$ is

$$\int_0^t \langle Q\Sigma(s), \Sigma(s)\rangle\,ds, \quad t \ge 0.$$

Consequently, the Itô formula gives

$$\begin{aligned} \psi(X(t)) &= \psi(X(0)) + \int_0^t \psi'(X(s-))\,dX(s) \\ &\quad + \frac{1}{2}\int_0^t \psi''(X(s))\langle Q\Sigma(s), \Sigma(s)\rangle\,ds \\ &\quad + \sum_{s\le t}\big[\psi(X(s)) - \psi(X(s-)) - \psi'(X(s-))\Delta X(s)\big] \\ &= \psi(X(0)) + I_1(t) + I_2(t) + I_3(t), \end{aligned}$$

where

$$I_1(t) = M_1(t) + \int_0^t \psi'(X(s-))\big[A(t) + \langle \Sigma(s), a\rangle\big]\, ds$$
$$+ \int_0^t \int_U \mathbf{1}_{\{|z|>1\}}(z)\psi'(X(s-))\langle \Sigma(s), z\rangle\, \mu(ds, dz)$$

and $M_1(t)$ is a local martingale as a sum of a Wiener integral and a stochastic integral with respect to the compensated jump measure $\mu(ds, dy) - ds\, \nu(dy)$,

$$I_2(t) = \frac{1}{2}\int_0^t \psi''(X(s))\langle Q\Sigma(s), \Sigma(s)\rangle\, ds$$

and

$$\begin{aligned} I_3 &= \sum_{s\le t}\big[\psi(X(s)) - \psi(X(s-)) - \psi'(X(s-))\Delta X(s)\big] \\ &= \int_0^t \int_{\mathbb{R}^1} \big[\psi(X(s-) + y) - \psi(X(s-)) - \psi'(X(s-))y\big]\, \mu_X(ds, dy) \\ &= \int_0^t \int_U \big[\psi\big(X(s-) + \langle \Sigma(s), z\rangle\big) \\ &\quad - \psi(X(s-)) - \psi'(X(s-))\langle \Sigma(s), z\rangle\big]\, \mu(ds, dz). \end{aligned}$$

Consequently,

$$\begin{aligned} \psi(X(t)) &= \psi(X(0)) + M_1(t) \\ &\quad + \int_0^t \psi'(X(s-))\big[A(t) + \langle \Sigma(s), a\rangle\big]\, ds \\ &\quad + \frac{1}{2}\int_0^t \psi''(X(s))\langle Q\Sigma(s), \Sigma(s)\rangle\, ds \\ &\quad + \int_0^t \int_U \big[\psi\big(X(s-) + \langle \Sigma(s), z\rangle\big) \\ &\qquad - \psi(X(s-)) - \mathbf{1}_{\{|z|\le 1\}}(z)\psi'(X(s-))\langle \Sigma(s), z\rangle\big]\, \mu(ds, dz). \end{aligned} \tag{18}$$

The HJM postulate requires that $\psi(X(t))$, $t \ge 0$, is a local martingale for $\psi(x) = e^{-x}$, $x \in \mathbb{R}$. Thus, there exists an increasing sequence $(\tau_n)$ of stopping times such that the integrals

$$\begin{aligned} &\int_0^t \int_U \mathbf{1}_{[0,\tau_n]}(s)\big[\psi\big(X(s-) + \langle \Sigma(s), z\rangle\big) \\ &\quad - \psi(X(s-)) - \mathbf{1}_{\{|z|\le 1\}}(z)\psi'(X(s-))\langle \Sigma(s), z\rangle\big]\, \mu(ds, dz) \\ &= \int_0^t \int_U \xi_n(s, z)\mu(ds, dz) \end{aligned}$$

are random variables with finite expectation. In particular also

$$\int_0^t \int_U \xi_n^+(s, z)\mu(ds, dz), \qquad \int_0^t \int_U \xi_n^-(s, z)\mu(ds, dz)$$

are integrable random variables. Since the random fields $\xi_n^+, \ \xi_n^-$ are predictable, it follows that

$$\boldsymbol{E}\int_0^t\int_U |\xi_n(s,z)|\mu(ds,dz) = \boldsymbol{E}\int_0^t\int_U |\xi_n(s,z)|ds\,\nu(dz) \ < \infty.$$

Consequently,

$$\boldsymbol{E}\int_0^t\int_U \mathbf{1}_{[0,\tau_n]}(s)\mathbf{1}_{\{|z|>1\}}(z)|\psi\big(X(s-)+\langle\Sigma(s),z\rangle\big)-\psi(X(s-))|\nu(dz)ds \ < \infty,$$

and thus,

$$\boldsymbol{E}\int_0^t \mathbf{1}_{[0,\tau_n]}(s)e^{-X(s-)}\big(\int_{|z|>1}|e^{-\langle\Sigma(s),z\rangle}-1|\nu(dz)\big)ds < \infty.$$

Hence, for each natural $n$, $\boldsymbol{P}$ -almost surely,

$$\int_0^{\tau_n} b(\Sigma(s))ds \ < \infty,$$

and assertion (i) of the theorem follows.

To prove (ii), consider formula (18). Note that, if $(H2)$ and (16) hold, then, using condition (6), for the function $\psi(x) = e^{-x}$,

$$\int_0^t\int_U |\psi\big(X(s-)+\langle\Sigma(s),z\rangle\big) \\ - \psi(X(s-)) - \mathbf{1}_{\{|z|\le 1\}}(z)\psi'(X(s-))\langle\Sigma(s),z\rangle|\,\nu(dz)ds < \infty.$$

Therefore

$$\begin{aligned}
&\int_0^t\int_U \big[\psi\big(X(s-)+\langle\Sigma(s),z\rangle\big) \\
&\quad - \psi(X(s-)) - \mathbf{1}_{\{|z|\le 1\}}(z)\psi'(X(s-))\langle\Sigma(s),z\rangle\big]\mu(ds,dz) \\
&= \int_0^t\int_U \big[\psi\big(X(s-)+\langle\Sigma(s),z\rangle\big) \\
&\quad - \psi(X(s-)) - \mathbf{1}_{\{|z|\le 1\}}(z)\psi'(X(s-))\langle\Sigma(s),z\rangle\big](\mu(ds,dz)-ds\,\nu(dz)) \\
&\quad + \int_0^t\int_U \big[\psi\big(X(s-)+\langle\Sigma(s),z\rangle\big) \\
&\quad - \psi(X(s-)) - \mathbf{1}_{\{|z|\le 1\}}(z)\psi'(X(s-))\langle\Sigma(s),z\rangle\big]\nu(dz)ds.
\end{aligned}$$

Consequently, formula (18) can be rewritten as

$$e^{-X(t)} = e^{-X(0)} + M_2(t) + \int_0^t \psi'(X(s-))\big[A(s) - J(\Sigma(s))\big]\,ds,$$

where $M_2(t)$ is a local martingale. This finishes the proof of the theorem. ∎

**Remark 2** Explicit formulation (17), in terms of the function $J$, indicates that the drift term is completely determined by the diffusion term. In the particular case when $a = 0$, $\mu = 0$, one arrives at the classical HJM condition (see [13]).

**Remark 3** Part (i) of Theorem 1 is in the spirit of Theorem 25.3 in Sato [19] which implies that for a finite dimensional Lévy process the conditions

$$\boldsymbol{E} e^{-\langle u, Z_t \rangle} < \infty \tag{19}$$

and

$$\int_{|y|>1} e^{-\langle u, y \rangle} \nu(dy) < \infty \tag{20}$$

are equivalent. This equivalence can be generalized to the infinite dimensional case (see [17]).
In Theorem 1 we generalized the implication (19) $\Rightarrow$ (20) taking a stochastic integral with respect to a Lévy process $Z$ instead of $Z$. For a fixed $T$, we have proved that if for a bounded process $\Sigma$, the process

$$Y_t = \exp\Big( - \int_0^t \langle \Sigma(s), dZ_s \rangle - \int_0^t J(\Sigma(s))\, ds \Big) \tag{21}$$

is a local martingale, then

$$\Sigma(t) \in B \qquad dt \otimes dP \;\text{ almost surely.} \tag{22}$$

**Remark 4** Our condition, even in the finite dimensional case, is more general than that given in the previous papers by Eberlein et al. who assume that the moment generating function of the Lévy process has to be finite in the whole interval $[-M, M]$ (which is equivalent to condition (5)) and the volatility function takes values in that interval. Indeed, if a 1-dimensional Lévy process $Z$ has positive jumps and the volatility is non-negative, then condition (16) is always satisfied (condition (5) might not be satisfied).

In the next theorem we describe the dynamics of the forward rate $f$ under the HJM condition. It is an immediate but useful consequence of Theorem 1.

**Theorem 5** *Assume that*

$$\int_{|y|\geq 1} e^{-\langle u, y \rangle} \nu(dy) < \infty \tag{23}$$

*for all $u$ from some neighborhood of the set in which $\int_t^\theta \sigma(t,v)dv$ takes values. Then the HJM condition* (17) *implies that the dynamics of $f$ has the form*

$$df(t,\theta) = \Big\langle DJ\Big( \int_t^\theta \sigma(t,v)dv \Big), \sigma(t,\theta) \Big\rangle\, dt + \langle \sigma(t,\theta), dZ(t) \rangle, \tag{24}$$

*where $DJ$ is the gradient of $J$.*

*Proof.* Using assumption (23) one can check differentiability of $J$. So, by (17) we have

$$\alpha(t,\theta) = \Big\langle DJ\Big( \int_t^\theta \sigma(t,v)dv \Big), \sigma(t,\theta) \Big\rangle$$

and (24) follows. ■

Thus, under very mild assumptions, the HJM postulate holds if and only if

$$df(t,\theta) = \Big\langle DJ\Big( \int_t^\theta \sigma(t,v)\, dv \Big), \sigma(t,\theta) \Big\rangle\, dt + \langle \sigma(t,\theta), dZ(t) \rangle.$$

**Remark 6** In fact, Theorem 5 holds under the weaker assumption that the directional derivatives of $J$ in the directions $\int_t^\theta \sigma(t,v)\,dv$ exist.

## 0.4 Existence of exponential moments

In this section we derive several consequences of our basic Theorem 1. In particular, we show that under a natural condition on the volatility $\sigma$ the function $b$ is finite on a certain set.
We denote by $supp\ (X)$ the support of the distribution of the random variable $X$. Let $S(t,\theta) = supp\left(\int_t^\theta \sigma(t,\eta)d\eta\right)$. As a corollary from Theorem 1 (i) we obtain

**Proposition 7** *If the HJM postulate is satisfied and for some $\theta$ there exists a closed set $K$ such that $K \subset S(t,\theta)$ for all $t$ in a subset of $[0,\theta]$ of positive measure, then there exists a dense subset $K_0$ of $K$ such that $K_0 \subset B$.*

*Proof.* The random variable $X$ takes all values from some dense subset of $K \cap supp\ (X)$, provided $P(X \in K) = 1$ and $K$ is closed. The proposition is a simple consequence of this remark. ■

Hence, as a special case we obtain:

**Corollary 8** *Let $\sigma$ be a deterministic function. For fixed $\theta$ the function $\gamma(t) = \int_t^\theta \sigma(t,v)\,dv$ defines a curve in $U$. Let*

$$K = \{u = \gamma(t) : t \in [0,\theta]\}.$$

*If the HJM postulate is satisfied, then by* (16)*, $\gamma(t) \in B$ for almost all $t$, so there exists a dense subset $K_0$ of $K$ such that $K_0 \subset B$.*

**Corollary 9** *Let $U = \mathbb{R}^d$. Under the assumptions of Proposition 7, if $\mathrm{Int}K \neq \emptyset$, then*

$$b(u) < \infty \tag{25}$$

*for all $u \in IntK$, i.e. $IntK \subset B$.*

*Proof.* We first prove that if $U = \mathbb{R}^d$ and (25) is satisfied on a dense subset $D$ of the open ball $B(x,r)$, $x \in \mathbb{R}^d$, $r > 0$, then it holds for all $c \in B(x,r)$. Indeed, for every $c \in B(x,r)$ there exist $c_1, \dots, c_{d+1} \in D$ such that $c$ belongs to the simplex with vertices $c_1, \dots, c_{d+1}$, i.e. $c = \sum_{i=1}^{d+1} \lambda_i c_i$, $\lambda_i \in [0,1]$, $\sum_{i=1}^{d+1} \lambda_i = 1$. Hence, by convexity of the exponential function,

$$\int_{|y|>1} e^{-\langle c,y\rangle}\nu(dy) \leq \sum_{i=1}^{d+1} \lambda_i \int_{\{|y|>1\}} e^{-\langle c_i,y\rangle}\nu(dy) < \infty,$$

since $c_1, \dots, c_{d+1} \in D$.
Next, since $G = \mathrm{Int}K$ is open, for every $x \in G$ there exists $r > 0$ such that $B(x,r) \subset G$. By assumption and Proposition 7, (25) holds for a dense subset of $B(x,r)$, so by the previous considerations (25) holds for all $y \in B(x,r)$, in particular for $y = x$. ■

**Corollary 10** *If $\nu$ is a Lévy measure of the $\alpha$-stable symmetric process $Z$ in $\mathbb{R}$, then*

$$\nu(dy) = c|y|^{-1-\alpha}dy \quad \textit{and} \quad \forall u \neq 0 \int_{\{|y|>1\}} e^{\langle u,y\rangle}|y|^{-1-\alpha}dy = \infty.$$

*Therefore, as a consequence of Theorem 1, the HJM postulate is not satisfied for the $\alpha$-stable symmetric process $Z$, so $Z$ cannot be used to model the forward rate.*

**Theorem 11** *There exists a model of the form* (10) *for which (H1) holds, the HJM postulate is satisfied and*

$$\int_{|y|>1} e^{-\langle u,y\rangle}\nu(dy) < \infty \tag{26}$$

*for $u$ in a dense subset of $B(0,r)$ and*

$$\int_{|y|>1} e^{-\langle u_n,y\rangle}\nu(dy) = \infty \tag{27}$$

*for a sequence $u_n \to 0$ as $n \to \infty$.*

*Proof.* First, we construct a probability measure $\nu$ for which (26) and (27) are satisfied. Let $\xi = (\xi_k)$, where $(\xi_k)$ are independent random variables each taking two values $x_k > 1$ and 0 and $\boldsymbol{P}(\xi_k = x_k) = p_k > 0$, $k = 1, 2, \ldots$, with $\sum_{k=1}^{\infty} p_k < \infty$. By the Borell-Cantelli lemma $\xi \in \ell^2$ with probability 1. Putting $\nu = \mathcal{L}(\xi)$ we see that for $u = -(a_1, a_2, \ldots) \in \ell^2$,

$$\int_{|y|>1} e^{-\langle u,y\rangle}\nu(dy) = \boldsymbol{E}(e^{\sum_{k=1}^{\infty} a_k\xi_k}) = \prod_{k=1}^{\infty} \boldsymbol{E}(e^{a_k\xi_k}) = \prod_{k=1}^{\infty}(e^{a_kx_k}p_k+1-p_k).$$

Obviously, for a dense set of $u \in \ell^2$ consisting of $u$ such that $a_k = 0$ apart from finitely many $k$, the expression on the right hand side is finite.
On the other hand,

$$\int_{|y|>1} e^{-\langle u,y\rangle}\nu(dy) = \infty \qquad \text{if}$$

$$\sum_{k=1}^{\infty}(e^{a_kx_k} - 1)p_k = \infty. \tag{28}$$

For given $(a_k)$, $a_k > 0$, $k = 1, 2, \ldots$, there always exist $(x_k)$, $x_k > 1$, $k = 1, 2, \ldots$, such that (28) holds. (28) implies that for any $m$,

$$\sum_{k=m}^{\infty}(e^{a_kx_k} - 1)p_k = \infty,$$

so for $u_m = -(0, 0, \ldots, a_m, a_{m+1}, \ldots) \in \ell^2$,

$$\int_{|y|>1} e^{-\langle u_m,y\rangle}\nu(dy) = \infty,$$

i.e. (27) is satisfied.
Now we describe the desired HJM model. Let $Z$ be a Lévy process with a Lévy

measure $\nu$ and $a = 0$, $Q = 0$. Let $f$ be given by (10) with deterministic constant process $\sigma(t,\theta) \equiv (1,0,0,\ldots) = \sigma$ and with the process $\alpha(t,\theta)$ obtained from (17). So $\alpha$ is also deterministic. To prove that the HJM postulate is satisfied note that from the properties of $\nu$, for $t \leq \theta$,

$$\boldsymbol{E}(e^{-\int_0^\theta f(t,u)du}) = C\ \boldsymbol{E}(e^{-\int_0^\theta \int_0^t \langle \sigma(s,u), dZ_s\rangle du}) = C\ \boldsymbol{E}(e^{-\theta\langle \sigma, Z_t\rangle}) =$$

$$= C\ \boldsymbol{E}(e^{-\theta t\langle \sigma, Z_1\rangle}) = C\ (\int_{|y|>1} e^{\langle b,y\rangle}\nu(dy) + \int_{|y|\leq 1} e^{\langle b,y\rangle}\nu(dy)) < \infty,$$

where $C = e^{-\int_0^\theta \int_0^t \alpha(s,u)dsdu},\ b = (\theta t, 0, 0, \ldots)$. Using Itô's lemma in the same way as in the proof of Theorem 1 and taking into account that $\alpha$ was chosen for (17) to hold we get the result. ∎

## 0.5 Analytical comments on strong exponential moments

In this section we study the relationship between existence of exponential moments and finiteness of

$$\int_{|y|>1} e^{\gamma|y|}\nu(dy) < \infty$$

for some $\gamma > 0$.
Note that the exponential moments of the restricted measure exist:

$$\int_{|y|>1} e^{\langle u,\, y\rangle}\nu(dy) < \infty \tag{29}$$

for $u \in -B$. Therefore the results in this paper can be presented as results concerning exponential moments of the Lévy measure $\nu$. For example, assuming that $U = \mathbb{R}^d$ and that (29) holds for $u$ in a dense subset of an open set $G$ we infer that (29) is satisfied for all $u \in G$ (the proof is analogous to the proof of Corollary 9).
The next proposition indicates that in the finite dimensional case existence of exponential moments of type (29) on an open ball implies existence of "normal" exponential moments on some ball.

**Proposition 12** *If $U = \mathbb{R}^d$, $G = B(0,r)$ and $b(u) < \infty$ on $G$, then*

$$\int_{|y|>1} e^{\gamma|y|}\nu(dy) < \infty \tag{30}$$

*for $\gamma \in \mathbb{R}$ such that $|\gamma| < \frac{r}{\sqrt{d}}$.*

*Proof.* Fix $\gamma \in \mathbb{R}$ such that $|\gamma| < \frac{r}{\sqrt{d}}$. Since

$$\sum_{j=1}^{d} |y_j| = \sum_{j=1}^{d} y_j sign\ y_j,$$

for each orthant $A_h = \{sign\ y_1 = e_1, \ldots, sign\ y_d = e_d\}$ for $h = (e_1, \ldots, e_d)$, $e_i \in \{-1, 1\}$, taking $c_h = \gamma h$ we obtain by assumption

$$\int_{A_h\cap\{|y|>1\}} e^{\gamma|y|}\nu(dy) \leq \int_{A_h\cap\{|y|>1\}} e^{\langle c_h,y\rangle}\nu(dy)$$
$$\leq \int_{\{|y|>1\}} e^{\langle c_h,y\rangle}\nu(dy) < \infty,$$

because $\gamma|y| \leq \langle c_h, y\rangle$ and $|c_h| < r$. Hence

$$\int_{|y|>1} e^{\gamma|y|}\nu(dy) \leq \sum_h \int_{\{|y|>1\}} e^{\langle c_h, y\rangle}\nu(dy) < \infty.$$

■

**Remark 13** If (30) is satisfied for all $\gamma \in B(0, r)$, then (29) holds for all $u \in B(0, r)$.

The next proposition indicates a difference between models with finite dimensional and infinite dimensional noises.

**Proposition 14** *There exists a measure $\nu$ on $U = \ell^2$ such that* (29) *holds for a dense subset of $\ell^2$ but not for all $u \in \ell^2$, and for any $\gamma > 0$,*

$$\int_{|y|>1} e^{\gamma\ |y|}\nu(dy) = \infty. \tag{31}$$

*Proof.* The measure $\nu$ constructed in the proof of Theorem 11 satisfies condition (26) and for $u_m = (0, 0, \ldots, a_m, a_{m+1}, \ldots) \in \ell^2$,

$$\int_{|y|>1} e^{\langle u_m, y\rangle}\nu(dy) = \infty. \tag{32}$$

(32) implies that the sequence $\gamma_m := |u_m|$ satisfies (31) because $\exp(\gamma_m|y|) \geq \exp(\langle u_m, y\rangle) > 0$. Moreover $\gamma_m$ tends to 0 as $m \longrightarrow \infty$, which completes the proof. ■

## 0.6 General case

We now analyze the general case proposed in two papers by Björk et al. [2], [3]. Let

$$\begin{aligned} df(t,\theta) &= \alpha(t,\theta)dt + \sigma(t,\theta)\ dW(t) \\ &+ \int_{|y|\leq 1} \sigma_0(t,\theta,y)(\mu(dt,dy) - dt\,\nu(dy)) + \int_{|y|>1} \sigma_1(t,\theta,y)\mu(dt,dy) \end{aligned} \tag{33}$$

with $\mu$ a Poissonian random measure with intensity $\nu(dx) \otimes dt$ and $W$ a Wiener process with values in $U$ and covariance operator $Q$. We assume that the measure $\nu$ satisfies condition (6). Moreover, for each $\theta$ the processes $\alpha(\cdot, \theta)$, $\sigma(\cdot, \theta)$, $\sigma_0(\cdot, \theta, \cdot)$, $\sigma_1(\cdot, \theta, \cdot)$ for $t \leq \theta$ are assumed to be predictable and

$$\begin{aligned} &\boldsymbol{E}\int_0^T \int_{|y|\leq 1} \int_0^T \sigma_0^2(t,\theta,y)d\theta dt\nu(dy) \\ &\quad = \boldsymbol{E}\int_0^T \int_{|y|\leq 1} ||\sigma_0(t,\cdot,y)||_H^2 dt\nu(dy) < \infty, \end{aligned} \tag{34}$$

$$\int_0^T \boldsymbol{E}\Big|\int_{|y|>1} \sigma_1(t,\theta,y)\mu(dt,dy)\Big|^2 d\theta < \infty, \tag{35}$$

where, as usual, by $\|\cdot\|_H$ we denote the norm in $H = L^2[0,T]$. We also assume that with probability one the processes $\alpha$, $\sigma$, $\sigma_0$ and $\sigma_1$ have bounded trajectories. These conditions guarantee that the integrals in (33) are well defined and $f(t,\cdot)$ is an element of $H$. The Schwarz inequality implies that (35) is satisfied provided $\sigma_1$ is bounded on the set $\{|x|>1\}$. As before we assume that for $\theta < t$,

$$\alpha(t,\theta) = 0, \quad \sigma(t,\theta) = 0.$$

If

$$\begin{aligned}\sigma_0(t,\theta,y) &= \langle \sigma(t,\theta), y\rangle \mathbf{1}_{\{|y|\le 1\}}(y),\\ \sigma_1(t,\theta,y) &= \langle \sigma(t,\theta), y\rangle \mathbf{1}_{\{|y|>1\}}(y),\end{aligned}$$

then we obtain the model considered in the previous sections (see [12]).
Set, for $t \in [0,T]$, $\theta \le T$,

$$\begin{aligned}A(t,\theta) &= (\mathbf{1}_{[0,\theta]}, \alpha)_H = \int_t^\theta \alpha(t,\eta)\,d\eta, \quad \Sigma(t,\theta) = \int_t^\theta \sigma(t,\eta)\,d\eta,\\ \Sigma_0^\theta(t,y) &= \int_0^\theta \sigma_0(t,\eta,y)\,d\eta, \quad \Sigma_1^\theta(t,y) = \int_0^\theta \sigma_1(t,\eta,y)\,d\eta.\end{aligned}$$

Let

$$\begin{aligned}J_1(f) &= \frac{1}{2}\langle Qu, u\rangle,\\ J_2(f) &= \int_{\{|y|>1\}} (e^{-f(y)} - 1)\nu(dy),\\ J_3(f) &= \int_{\{|y|\le 1\}} (e^{-f(y)} - 1 - f(y))\nu(dy)\end{aligned}$$

for $f : U \longrightarrow \mathbb{R}$ and

$$\mathcal{N} = \{f : U \longrightarrow R : \int_{\{|y|>1\}} e^{-f(y)}\nu(dy) < \infty\}.$$

In this general case arguing in a similar way as for Lévy process $Z$ we obtain

**Theorem 15** (i) *If the HJM postulate holds then, for arbitrary $\theta \le T$, $\boldsymbol{P}$-almost surely,*

$$\Sigma_1^\theta(t,\cdot) \in \mathcal{N} \tag{36}$$

*for almost all $t \in [0,\theta]$.*

(ii) *Assume that for all $\theta \le T$, $\boldsymbol{P}$-almost surely* (36) *holds for almost all $t \in [0,\theta]$. Then the HJM postulate holds if and only if the following HJM condition holds:*

$$\int_t^\theta \alpha(t,v)\,dv = J_1\Big(\Sigma(t,\theta)\Big) + J_2\Big(\Sigma_0^\theta(t,y)\Big) + J_3\Big(\Sigma_1^\theta(t,y)\Big) \tag{37}$$

*for almost all $t \in [0,\theta]$.*

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