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Abstract

Credit spread is defined as a difference between forward rates corresponding to bond facing the default risk and forward rates corresponding to risk-free bond. We give conditions under which credit spread is a positive process, considering joint model for rates.

1 Introduction

Let $P(t, \theta)$ and $\overline{P}(t, \theta)$ be the prices, at time t, of risk free and of defaultable bond paying 1, at the maturity time $\theta \geq t$. Let $f(t, \theta)$, $\overline{f}(t, \theta)$, be the corresponding forward rates, then

$$P(t, \theta) = e^{-\int_t^\theta f(t,\eta) d\eta}, \quad \overline{P}(t, \theta) = e^{-\int_t^\theta \overline{f}(t,\eta) d\eta}, \ 0 \leq t \leq \theta \tag{1.1}$$

If τ denotes the time of default of the defaultable bond then, for $t < \tau$, one should have

$$P(t, \theta) \geq \overline{P}(t, \theta), \ 0 \leq t \leq \theta, \tag{1.2}$$

an inequality implied by:

$$f(t, \theta) \leq \overline{f}(t, \theta), \ 0 \leq t \leq \theta. \tag{1.3}$$

The difference

$$s(t, \theta) = \overline{f}(t, \theta) - f(t, \theta), \ 0 \leq t \leq \theta, \tag{1.4}$$

is called spread, see e.g. [12], and its positivity is thus a natural modeling assumption.

Modeling and properties of spread processes were recently discussed in[12], where a specific equation for s, implying positivity, was imposed. In the present note we derive sufficient conditions for positivity of spreads, close to necessary, for general bond processes satisfying HJM conditions and driven by Lévy type noise. In [12] perturbations were Gaussian.

We work with the parametrization proposed by Musiela [11]. Thus if ξ is the *time to maturity* then we set:

$$f_t(\xi) = -\frac{\partial}{\partial \xi} \ln P(t, t + \xi), \ \overline{f}_t(\xi) = -\frac{\partial}{\partial \xi} \ln \overline{P}(t, t + \xi) \quad t, \xi \geq 0. \tag{1.5}$$

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We assume that $f, \overline{f}$, satisfy general equations, on appropriate Hilbert space H of functions defined on $[0, +\infty)$,

$$\mathrm{d}f_t = \left(\frac{\partial}{\partial\xi} f_t + \alpha(f_t)\right) \mathrm{d}t + \sigma(f_t)\mathrm{d}L(t) + b(f_t)\mathrm{d}W(t), \tag{1.6}$$

$$\mathrm{d}\overline{f}_t = \left(\frac{\partial}{\partial\xi} \overline{f}_t + \overline{\alpha}(f_t, \overline{f}_t)\right) \mathrm{d}t + \overline{\sigma}(f_t, \overline{f}_t)\mathrm{d}L(t) + \overline{b}(f_t, \overline{f}_t)\mathrm{d}W(t), \tag{1.7}$$

where coefficients α, σ, b and $\overline{\alpha}, \overline{\sigma}$ $\overline{b}$ are mappings from $H \times H$ into H. Moreover L is a one-dimensional Lévy martingale and W a real Wiener process. We restrict our considerations to one dimensional stochastic perturbations to simplify presentation, but our methods are applicable to vector or even infinite dimensional noise processes.

Our main results are sufficient conditions on volatilities $b, \sigma, \overline{b}, \overline{\sigma}$ implying positivity of spreads, see Theorem 1, Theorem 2 and Theorem 3. To derive them we use a generalized version of a comparison result by Milian [10]. Special attention is paid to models with linear volatilities. As was discovered in [13], contrary to models with Gaussian perturbations, which explode, there exists a larg family of models with Lévy perturbation without explosions. For them we give necessary and sufficient conditions for positivity of spreads. We believe that they will have interesting applications for pricing.

In Section 2 we describe the model and formulate theorems. In Section 3 we give examples and in Section 4 proofs.

2 Model and results

2.1 HJM conditions

We will consider processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the Leÿy process L has no gaussian component and that $\mathbb{E}\, L(1) = 0$. It admits the representation

$$L(t) = \int_0^t \int_{\mathbb{R}} y \left(\pi(\mathrm{d}s, \mathrm{d}y) - \mathrm{d}s\, \nu(\mathrm{d}y)\right),$$

where π is the Poisson random measure corresponding to L and ν is the jump intensity measure of L. Moreover, it is well-known that

$$\int_{\mathbb{R}} \left(|y|^2 \wedge 1\right) \nu(\mathrm{d}y) < \infty.$$

The absence of arbitrage on the market implies, see [2],[5] and [9], that the following formulas, called HJM conditions, must hold for $\alpha : H \to H$, $\overline{\alpha} : H \times H \to H$,

$$\begin{aligned}\alpha(f)(\xi) &= b(f)(\xi)\left(\int_0^\xi b(f)(\eta)\mathrm{d}\eta\right)\\ &\quad + \sigma(f)(\xi)\int_{\mathbb{R}} y\left(1-\exp\left\{-y\int_0^\xi \sigma(f)(\eta)\mathrm{d}\eta\right\}\right)\nu(\mathrm{d}y),\\ \overline{\alpha}(f,\overline{f})(\xi) &= \overline{b}(f,\overline{f})(\xi)\left(\int_0^\xi \overline{b}(f,\overline{f})(\eta)\mathrm{d}\eta\right)\\ &\quad + \overline{\sigma}(f,\overline{f})(\xi)\int_{\mathbb{R}} y\left(1-\exp\left\{-y\int_0^\xi \overline{\sigma}(f,\overline{f})(\eta)\mathrm{d}\eta\right\}\right)\nu(\mathrm{d}y).\end{aligned}$$

Note that the above integrals are well-defined if ν has support in $[-m,+\infty)$ for some $m \geq 0$ and

$$\sigma(f) \geq 0, \quad \overline{\sigma}(f,\overline{f}) \geq 0.$$

Let L^2_w denote the space of all functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $\int_0^{+\infty} |f(\xi)|^2 w(\xi)\mathrm{d}\xi < +\infty$, where w is some positive weight function, with inner product

$$\langle f, g\rangle_{L^2_w} = \int_0^{+\infty} f(\xi)g(\xi)w(\xi)\mathrm{d}\xi.$$

For $f, h \in L^2_w$ we shall write $f \geq h$ instead of

$$f(\xi) \geq h(\xi), \quad \xi - a.e.$$

We shall consider equations (1.6) and (1.7) on space L^2_w with w given by

$$w(\xi) = \mathrm{e}^{-2\gamma\xi}\overline{w}(\xi), \qquad \overline{w}(\eta) \leq M^2\overline{w}(\xi), \qquad 0 \leq \eta \leq \xi. \tag{2.1}$$

for some $\gamma \in \mathbb{R}$ and $M \geq 1$.
For explicit conditions implying existence of solutions f_t to(1.6) we refer to paper [13]. In the same spirit one can derive conditions of Lipschitz type implying existence of solutions $f_t, \overline{f}_t$ to the pair of equations (1.6) and (1.7).

2.2 Theorems

We begin with a theorem that gives sufficient conditions for (2.9) to hold, under the assumption that the Lévy process L does not have negative jumps.

Theorem 1. *Assume that ν has support in $[0,+\infty)$. If for every $f,\overline{f} \in L^2_w$, such that $\overline{f} \geq f$, we have*

$$\overline{f}(\xi) = f(\xi) \implies \overline{\sigma}(f,\overline{f})(\xi) = \sigma(f)(\xi), \quad \xi - a.e. \tag{2.2}$$

$$\overline{f}(\xi) = f(\xi) \implies \overline{b}(f,\overline{f})(\xi) = b(f)(\xi), \quad \xi - a.e. \tag{2.3}$$

$$0 \leq \sigma(f) \leq \overline{\sigma}(f,\overline{f}), \tag{2.4}$$

$$0 \leq b(f) \leq \overline{b}(f,\overline{f}), \tag{2.5}$$

then

$$\overline{f}_0 \geq f_0 \implies \overline{f}_t \geq f_t.$$

If L has also negative jumps, condition (2.2) has to be replaced by a stronger one presented in the next theorem.

Theorem 2. *Assume that ν has support in $[-m,+\infty)$ for some $m > 0$. If for every $f,\overline{f} \in L^2_w$ such that $\overline{f} \geq f$, conditions* (2.3)-(2.5) *are satisfied, and*

$$\overline{\sigma}(f,\overline{f}) - \sigma(f) \leq m^{-1}\left(\overline{f} - f\right), \tag{2.6}$$

then

$$\overline{f}_0 \geq f_0 \implies \overline{f}_t \geq f_t.$$

Let us finally consider an important case of models with linear volatilities,

$$\mathrm{d}f_t(\xi) = \left(\frac{\partial}{\partial\xi}f_t(\xi) + J'\left(\int_0^\xi af(\eta)\mathrm{d}\eta\right)af(\xi)\right)\mathrm{d}t + af_t(\xi)\mathrm{d}L(t), \tag{2.7}$$

$$\begin{aligned}\mathrm{d}\overline{f}_t(\xi) = \left(\frac{\partial}{\partial\xi}\overline{f}_t(\xi) + J'\left(\lambda\int_0^\xi f(\eta)\mathrm{d}\eta + \overline{\lambda}\int_0^\xi \overline{f}(\eta)\mathrm{d}\eta\right)\left(\lambda f_t(\xi) + \overline{\lambda}\,\overline{f}_t(\xi)\right)\right)\mathrm{d}t \\ + \left(\lambda f_t(\xi) + \overline{\lambda}\,\overline{f}_t(\xi)\right)\mathrm{d}L(t),\end{aligned} \tag{2.8}$$

for some $a,\lambda,\overline{\lambda} \geq 0$. For existence purposes we assume that the equations do not have Gaussian component i.e. $b = \overline{b} = 0$. As in [13] one can show that if the jump measure ν has support in $[-m,+\infty)$,for some $m \geq 0$ and $a < m^{-1}$, $\overline{\lambda} < m^{-1}$, then there exist solutions to (2.7), (2.8) which, to some extent, can be expressed in an explicit way.

Theorem 3. *Assume that the jump measure ν has support in $[-m,+\infty)$, for some $m \geq 0$ and $a < m^{-1}$, $\overline{\lambda} < m^{-1}$. The following conditions are equivalent,*

$$\overline{f}_0 \geq f_0 \implies \overline{f}_t \geq f_t,\ t \geq 0, \tag{2.9}$$

$$\overline{P}_0 \leq P_0 \implies \overline{P}_t \leq P_t,\ t \geq 0, \tag{2.10}$$

$$\lambda + \overline{\lambda} = a \tag{2.11}$$

The proofs of Theorem 2 and Theorem 3 are left to the final section, and Theorem 1 may be proved in the same way as Theorem 2.

3 Examples

Note that conditions (2.2), (2.3) imply that for every $f \in L^2_w$ the coefficients $\overline{b}, \overline{\sigma}, \overline{\alpha}$ satisfy

$$\begin{aligned}
\overline{b}(f,f) &= b(f),\\
\overline{\sigma}(f,f) &= \sigma(f),\\
\overline{\alpha}(f,f) &= \alpha(f).
\end{aligned}$$

Example 1. *Suppose that ν has support in $[-m,+\infty)$ and $\sigma, \overline{\sigma}$ are given by*

$$\begin{aligned}
\sigma(f)(\xi) &= \left(\lambda_1 \frac{f(\xi)}{f(\xi)+1}\right) \mathbf{1}_{\{f(\xi)\geq 0\}},\\
\overline{\sigma}(f,\overline{f})(\xi) &= \left(\lambda_1 \frac{\overline{f}(\xi)}{f(\xi)+1} + \lambda_2 \ln\left(\frac{\overline{f}(\xi)+1}{f(\xi)+1}\right)\right) \mathbf{1}_{\{\overline{f}(\xi)\geq f(\xi)\geq 0\}},
\end{aligned}$$

for some $\lambda_1, \lambda_2 \geq 0$. Then (2.4), (2.6) *are satisfied if*

$$\lambda_1 + \lambda_2 \leq m^{-1}.$$

Example 2. *Assume that for every $f \in L^2_w$, we have $\sigma(f) \geq 0$, and $\overline{\sigma}$ is given by*

$$\overline{\sigma}(f,\overline{f})(\xi) = \sigma(f)(\xi) + \psi\left(\overline{f}(\xi) - f(\xi)\right),$$

for some $\psi : \mathbb{R} \to \mathbb{R}_+$. Then condition (2.2) *holds if*

$$\psi(0) = 0,$$

and condition (2.6) *holds if*

$$\psi(u) \leq m^{-1}u, \quad u \geq 0.$$

If the coefficients $\sigma, \overline{\sigma}$ are Niemytskii operators, i.e.

$$\begin{aligned}
\sigma(f)(\xi) &= g(\xi, f(\xi)),\\
\overline{\sigma}(f,\overline{f})(\xi) &= \overline{g}(\xi, f(\xi), \overline{f}(\xi)),
\end{aligned}$$

for some $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\overline{g} : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, then condition (2.2) implies that

$$\overline{g}(\xi, x, x) = g(\xi, x), \qquad \xi \geq 0, x \in \mathbb{R}$$

Example 3. *Assume that ν has support in $[-m,+\infty)$ for some $m > 0$ and*

$$\begin{aligned}
\sigma(f)(\xi) &= \overline{g}(\xi, f(\xi), f(\xi)),\\
\overline{\sigma}(f,\overline{f})(\xi) &= \overline{g}(\xi, f(\xi), \overline{f}(\xi)),
\end{aligned}$$

for some $\overline{g} : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then (2.4), (2.6) *are fulfilled if and only if for every $x \in \mathbb{R}$, $\xi, u \geq 0$ we have*

$$0 \leq \overline{g}(\xi, x, x) \leq \overline{g}(\xi, x, x+u) \leq \overline{g}(\xi, x, x) + m^{-1}u.$$

4 Proofs

4.1 Proof of Theorem 2

The proof of the theorem is based on an extension of a comparison theorem due to Milian [10]. The theorem gives sufficient conditions for the inequality $X \geq Y$, where the pair X, Y is the solution to the system of stochastic evolution equations

$$\begin{aligned}\mathrm{d}X &= (AX + D(t,X))\,\mathrm{d}t + B(t,X)\mathrm{d}W,\\ \mathrm{d}Y &= (AY + d(t,Y))\,\mathrm{d}t + b(t,Y)\mathrm{d}W.\end{aligned}$$

In the above equations, A stands for the generator of a strongly continuous semigroup $S_A(t)_{t\geq 0}$ on L^2_w, W is a real valued Wiener process and operators D, d, B, b act from $\mathbb{R}_+ \times L^2_w \to L^2_w$. However the proof from [10] works for more general equations

$$\mathrm{d}X = (AX + D(t,X,Y))\,\mathrm{d}t + B(t,X,Y)\mathrm{d}W, \tag{4.1}$$

$$\mathrm{d}Y = (AY + d(t,X,Y))\,\mathrm{d}t + b(t,X,Y)\mathrm{d}W, \tag{4.2}$$

where now $D, d, B, b : \mathbb{R}_+ \times L^2_w \times L^2_w \to L^2_w$. In fact the following result holds.

Theorem 4. ***(Milian)*** *Assume that:*

(i) *The semigroup S_A, generated by A, preserves positivity, i.e. $f \geq 0 \implies S_A(t)f \geq 0$.*

(ii) *There exists $C > 0$ such that for all $t, s > 0$ and $x, y, f, h \in L^2_w$,*

$$\begin{aligned}&\|D(t,x,y) - D(s,f,h)\|_{L^2_w} + \|B(t,x,y) - B(s,f,h)\|_{L^2_w}\\ &\quad\leq C\left(|t-s| + \|x-f\|_{L^2_w} + \|y-h\|_{L^2_w}\right),\end{aligned}$$

$$\begin{aligned}&\|d(t,x,y) - d(s,f,h)\|_{L^2_w} + \|b(t,x,y) - b(s,f,h)\|_{L^2_w}\\ &\quad\leq C\left(|t-s| + \|x-f\|_{L^2_w} + \|y-h\|_{L^2_w}\right).\end{aligned}$$

(iii) *For every $t \geq 0$, all non-negative continuous $h \in L^2_w$ and all $x, y \in L^2_w$ such that $x \geq y$ satisfying $\langle x, h\rangle_{L^2_w} = \langle y, h\rangle_{L^2_w}$ one has $\langle D(t,x,y), h\rangle_{L^2_w} \geq \langle d(t,x,y), h\rangle_{L^2_w}$ and $\langle B(t,x,y), h\rangle_{L^2_w} = \langle b(t,x,y), h\rangle_{L^2_w}$.*

Let $X(t), Y(t), t \geq 0$ be mild solutions to (4.1) *and* (4.2) *respectively, such that $X(0) \geq Y(0)$. Then*

$$\mathbb{P}\left\{X(t) \geq Y(t), \quad t \geq 0\right\} = 1.$$

Note that condition (iii) from Theorem 4 is equivalent to the following condition (iv):

(iv) For every $t \geq 0$ and all $x, y \in L^2_w$ such that $x \geq y$, we have

$$x(\xi) = y(\xi) \implies \begin{array}{ll} D(t,x,y)(\xi) & \geq d(t,x,y)(\xi), \\ B(t,x,y)(\xi) & = b(t,x,y)(\xi), \end{array} \quad \xi - a.e.$$

Indeed. Let $h, x, y \in L^2_w$ be such that $h \geq 0$, $x \geq y$ and

$$\langle x-y, h\rangle_{L^2_w} = \int\limits_{\mathbb{R}_+} \left(x(\xi) - y(\xi)\right) h(\xi) w(\xi) \mathrm{d}\xi = 0.$$

Then $h = 0$ a.e. on set $\{\xi \geq 0 : x(\xi) > y(\xi)\}$, thus

$$\begin{aligned}\langle B(t,x,y) - b(t,x,y), h\rangle_{L^2_w} &= \int\limits_{\mathbb{R}_+} \left(B(t,x,y)(\xi) - b(t,x,y)(\xi)\right) h(\xi) w(\xi)\mathrm{d}\xi \\ &= \int\limits_{\{\xi\geq 0: x(\xi)=y(\xi)\}} \left(B(t,x,y)(\xi) - b(t,x,y)(\xi)\right) h(\xi) w(\xi)\mathrm{d}\xi,\end{aligned}$$

and

$$\langle D(t,x,y) - d(t,x,y), h\rangle_{L^2_w} = \int\limits_{\{\xi\geq 0: x(\xi)=y(\xi)\}} \left(D(t,x,y)(\xi) - d(t,x,y)(\xi)\right) h(\xi) w(\xi)\mathrm{d}\xi.$$

To check that assumptions of the theorem are satisfied in our situation remark that $A = \frac{\partial}{\partial \xi}$ generates the semigroup of shift operators $S(t)_{t\geq 0}$ given by $(S(t)f)(\xi) = f(\xi + t)$. If w satisfies (2.1) then $f \in L^2_w$ implies that $S(t)f \in L^2_w$, since we have

$$\begin{aligned}\|S(t)f\|^2_{L^2_w} &= \int\limits_0^{+\infty} |f(\eta+t)|^2 \mathrm{e}^{-2\gamma\eta}\overline{w}(\eta)\mathrm{d}\eta \\ &\leq \int\limits_0^{+\infty} |f(\eta+t)|^2 \mathrm{e}^{-2\gamma\eta} M^2 \overline{w}(\eta+t)dx \\ &= e^{2\gamma t}\int\limits_t^{+\infty} |f(\xi)|^2 \mathrm{e}^{-2\gamma\xi}\overline{w}(\xi)\mathrm{d}\xi \\ &\leq M^2 \mathrm{e}^{2\gamma t}\|f\|^2_{L^2_w}.\end{aligned}$$

If $f(\cdot) \geq 0$ then also $f(t+\cdot) \geq 0$, thus condition (i) from Theorem 4 holds. Let us approximate L by a sequence $\{L_n\}$ of processes satisfying $|\Delta L_n(t)| \geq 1/n$, $t \geq 0$, $n \in \mathbb{N}$. We assume that L_n converges $\mathbb{P}$-a.s. to L uniformly on each compact time interval. The existence of such a sequence follows from the Lévy–Khinchin decomposition. Let the pair $f^{(n)}, \overline{f}^{(n)}$ be the solution to the problem

$$\begin{aligned}\mathrm{d}f^{(n)}_t &= \left(\frac{\partial}{\partial\xi} f^{(n)}_t + \alpha\left(f^{(n)}_t\right)\right)\mathrm{d}t + \sigma\left(f^{(n)}_t\right)\mathrm{d}L_n(t) + b\left(f^{(n)}_t\right)\mathrm{d}W(t), \\ \mathrm{d}\overline{f}^{(n)}_t &= \left(\frac{\partial}{\partial\xi}\overline{f}^{(n)}_t + \overline{\alpha}\left(f^{(n)}_t, \overline{f}^{(n)}_t\right)\right)\mathrm{d}t \\ &\quad + \overline{\sigma}\left(f^{(n)}_t, \overline{f}^{(n)}_t\right)\mathrm{d}L_n(t) + \overline{b}\left(f^{(n)}_t, \overline{f}^{(n)}_t\right)\mathrm{d}W(t).\end{aligned}$$

For every stochastic process $(X_t)_{t\geq 0}$ with values in L^2_w, we have

$$\sup_{0\leq t\leq T}\mathbb{E}\left\|\int_0^t S(t-s)\sigma\left(X_s\right)\mathrm{d}\left(L_n-L\right)(s)\right\| = \int_{0\leq y\leq\frac{1}{n}}|y|^2\,\nu(\mathrm{d}y)\sup_{0\leq t\leq T}\mathbb{E}\left\|\int_0^t S(t-s)\sigma\left(X_s\right)\right\|.$$

Thus $\int_0^{\cdot} S(\cdot-s)\sigma\left(X_s\right)\mathrm{d}L_n(s)$ converges to $\int_0^{\cdot} S(\cdot-s)\sigma\left(X_s\right)\mathrm{d}L(s)$ and by local inversion theorem (see e.g. Lemma 9.2 in [4]) we obtain that $f_t^{(n)}$ converges to f_t. In the same way we obtain that $\overline{f}_t^{(n)}$ converges to $\overline{f}_t$. Thus it is enough to show that $\overline{f}_t^{(n)}\geq f_t^{(n)}$, if $\overline{f}_0^{(n)}\geq f_0^{(n)}$. To do this note that L_n has only isolated jumps. Between the jumps the driving process is Wiener and $\overline{f}(\xi)=f(\xi)$ implies

$$\begin{aligned}
\overline{b}(f,\overline{f})(\xi)&=b(f)(\xi),\\
\overline{\sigma}(f,\overline{f})(\xi)&=\sigma(f)(\xi),\\
\overline{\alpha}(f,\overline{f})(\xi)&\geq\alpha(f)(\xi),
\end{aligned}$$

thus, by Milian Theorem, $\overline{f}_t^{(n)}\geq f_t^{(n)}$ till the first jump time τ of L_n. From the assumptions of our theorem, we have

$$\left(\sigma\left(f_{\tau-}^{(n)}\right)(\xi)-\overline{\sigma}\left(f_{\tau-}^{(n)},\overline{f}_{\tau-}^{(n)}\right)(\xi)\right)\left(L_n(\tau)-L_n(\tau-)\right)\leq\overline{f}_{\tau-}^{(n)}(\xi)-f_{\tau-}^{(n)}(\xi).$$

Indeed, if $L_n(\tau)-L_n(\tau-)\geq 0$, then the left hand side is non-positive, and if $L_n(\tau)-L_n(\tau-)<0$, then we know that $L_n(\tau-)-L_n(\tau)\leq m$, so

$$\begin{aligned}
&\left(L_n(\tau-)-L_n(\tau)\right)\left(\overline{\sigma}\left(f_{\tau-}^{(n)},\overline{f}_{\tau-}^{(n)}\right)(\xi)-\sigma\left(f_{\tau-}^{(n)}\right)(\xi)\right)\\
&\qquad\leq m\left(\overline{\sigma}\left(f_{\tau-}^{(n)},\overline{f}_{\tau-}^{(n)}\right)(\xi)-\sigma\left(f_{\tau-}^{(n)}\right)(\xi)\right)\leq\overline{f}_{\tau-}^{(n)}(\xi)-f_{\tau-}^{(n)}(\xi).
\end{aligned}$$

Since

$$\begin{aligned}
f_{\tau}^{(n)}(\xi)&=f_{\tau-}^{(n)}(\xi)+\sigma\left(f_{\tau-}^{(n)}\right)(\xi)\left(L_n(\tau)-L_n(\tau-)\right),\\
\overline{f}_{\tau}^{(n)}(\xi)&=\overline{f}_{\tau-}^{(n)}(\xi)+\overline{\sigma}\left(f_{\tau-}^{(n)},\overline{f}_{\tau-}^{(n)}\right)(\xi)\left(L_n(\tau)-L_n(\tau-)\right),
\end{aligned}$$

we get

$$\overline{f}_{\tau}^{(n)}(\xi)-f_{\tau}^{(n)}(\xi)\geq 0.$$

□

4.2 Proof of Theorem 3

Write

$$u_t(\xi)=\int_0^{\xi}f_t(\eta)\mathrm{d}\eta,\qquad\overline{u}_t(\xi)=\int_0^{\xi}\overline{f}_t(\eta)\mathrm{d}\eta.$$

If the pair $f_t, \overline{f}_t$ is the solution to (2.7), (2.8), then the pair $u_t, \overline{u}_t$ is the solution to the following problem:

$$\begin{aligned} \mathrm{d}u_t(\xi) &= \left(\frac{\partial}{\partial \xi} u_t(\xi) + J\left(a u_t(\xi)\right)\right) \mathrm{d}t + a u_t(\xi) \mathrm{d}L(t), \\ \mathrm{d}\overline{u}_t(\xi) &= \left(\frac{\partial}{\partial \xi} \overline{u}_t(\xi) + J\left(\lambda u_t(\xi) + \overline{\lambda}\overline{u}_t(\xi)\right)\right) \mathrm{d}t + \left(\lambda u_t(\xi) + \overline{\lambda}\overline{u}_t(\xi)\right) \mathrm{d}L(t). \end{aligned}$$

Inequality $\overline{u}_t \geq u_t$ will imply the inequality between prices of bonds

$$\overline{P}(t,\theta) \leq P(t,\theta), \quad 0 \leq t \leq \theta.$$

As in the proof of Theorem 2 the following conditions for $\overline{u}_t \geq u_t$ should hold:

$$\lambda + \overline{\lambda} = a, \; \overline{\lambda} \leq m^{-1}, \tag{4.3}$$

which imply also the stronger inequality

$$\overline{f}_0 \geq f_0 \implies \overline{f}_t \geq f_t.$$

□

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