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Nonlinear separable equations in linear spaces and commutative Leibniz algebras

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NONLINEAR SEPARABLE EQUATIONS
IN LINEAR SPACES AND COMMUTATIVE LEIBNIZ ALGEBRAS

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Introduction.

The purpose of the present paper is to find solutions of some nonlinear equations in linear spaces and commutative algebras by a "separation of variables" obtained by means of the methods of Algebraic Analysis (cf. the author PR[1] and following papers).

Recall that the classical variables separation theorem for ordinary differential equations with separate variables can be stated as follows:

Theorem 0.1. (cf. Triebel Tr[1]) *Suppose that* $-\infty < a < b < +\infty$, $-\infty < c < d < +\infty$, $f_1 \in C^1[a,b]$, $f_2 \in C^1[c,d]$, $f_2(y) \neq 0$ *for* $y \in [c,d]$, $x_0 \in [a,b]$, $y_0 \in [c,d]$. *Then a unique solution of the equation*

$$y' = f_1(x)f_2(y) \tag{0.1}$$

with the initial condition

$$y(x_0) = y_0 \tag{0.2}$$

can be calculated from the equation

$$F(x,y) = 0, \quad where \quad F(x,y) = F_2(y) - F_1(x), \tag{0.3}$$

$$F_2(y) = \int_{y_0}^{y} \frac{dv}{f_2(v)}, \quad F_1(x) = \int_{x_0}^{x} f_1(u)du.$$

A basic example for this theorem is the following

Example 0.1. Consider in $C[a,1]$, $0 < a < 1$, the separable equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} \quad i.e. \quad \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}x}{x}. \tag{0.4}$$

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Here we have: $f_1(x) = \frac{1}{x}$, $f_2(y) = y$ for $x, y \in C^1[a,1]$. This implies $\ln y = \ln x + \ln c = \ln(cx)$, where $O < c \in \mathbb{R}$ is arbitrary. Then $y = cx$.

If we require Condition (0.2) to be satisfied for $x_0, y_0 \in [a,1]$ then we get $y_0 = cx_0$. If it is the case, then $c = \frac{x_0}{y_0}$. So that $F_1(y) = \ln y - \ln y_0 = \ln \frac{y}{y_0}$, $F_1(x) = \ln x - \ln x_0 = \ln \frac{x}{x_0}$ and $F(x,y) = \ln \frac{y}{y_0} - \ln \frac{x}{x_0}$. □

It is shown that the structures of linear spaces and commutative algebras (even if they are Leibniz algebras, i.e. such algebras that the product satisfies the Leibniz condition) are not rich enough for our purposes. Therefore, in order to generalize the method used for separable differential equations (cf. Theorem 0.1 and Example 0.1), we have to admit that in Leibniz algebras under consideration there exist logarithms (cf. PR[3] and following papers).

Section 1 contains some basic notions and results (without poofs) of Algebraic Analysis. In Section 2 there are considered equations in linear spaces. Section 3 contains results for commutative Leibniz algebras. In Section 4 basic notions and facts (without proofs) about logarithmic and antilogarithmic mappings are collected. Section 5 is devoted to separable nonlinear equations in commutative Leibniz algebras with logarithms.

Separable ordinary and partial nonlinear differential equations haves been considered by several mathematicians, from L. Euler (cf. E[1]) to (for instance) J. S. Ritt (cf. R[1], also R[2]), where the main tool was the implicit function theorem.

1. Basic notions of Algebraic Analysis

We recall here the following notions and theorems (without proofs; cf. PR[1], PR[2], PR[3], PR[4]).

Denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Q}$ the fields of positive integers, nonnegative integers, reals, complexes, integers and rational numbers, respectively, and by $\mathbb{F}$ any field of scalars. If $\mathbb{F}$ is a field of numbers then by $\mathbb{F}[t]$ is denoted the set of all polynomials in t with coefficients in $\mathbb{F}$.

Let X be a linear space (in general, without any topology) over a field $\mathbb{F}$ of scalars of the characteristic zero.

- $L(X)$ is the set of all linear operators with domains and ranges in X;
- dom A is the domain of an $A \in L(X)$;
- $\ker A = \{x \in \text{dom } A : Ax = 0\}$ is the kernel of an $A \in L(X)$;
- $L_0(X) = \{A \in L(X) : \text{dom } A = X\}$;
- $I(X)$ is the set of all invertible elements in X;
- $I_n(X) = \{x \in X : \exists_{y \in X}\ y^n = x\}$ $(n \in \mathbb{N})$; if $x \in I_n(X)$ and $x = y^n$ then $y = x^{1/n}$ is said to be an nth *root* of x.

An operator $D \in L(X)$ is said to be *right invertible* if there is an operator $R \in L_0(X)$ such that $RX \subset \text{dom } D$ and $DR = I$, where I denotes the identity operator. The operator R is called a *right inverse* of D. By $R(X)$ we denote the set of all right invertible operators in $L(X)$. Let $D \in R(X)$. Let $\mathcal{R}_D \subset L_0(X)$ be the set of all right inverses for D, i.e. $DR = I$ whenever $R \in \mathcal{R}_D$. We have $\mathrm{dom}\, D = RX \oplus \ker D$, independently of the choice of an $R \in \mathcal{R}_D$. Elements of $\ker D$ are said to be *constants*, since by definition, $Dz = 0$ if and only if $z \in \ker D$. The kernel of D is said to be the *space of constants*. We should point out that, in general, constants are different than scalars, since they are elements of the space X. If two right inverses commute each with another, then they are equal.

An element $y \in \mathrm{dom}\, D$ is said to be a *primitive* for an $x \in X$ if $y = Rx$ for an $R \in \mathcal{R}_D$. Indeed, by definition, $x = DRx = Dy$. Again, by definition, all $x \in X$ have primitives. Let

$$\mathcal{F}_D = \{F \in L_0(X) : F^2 = F; FX = \ker D \text{ and } \exists_{R \in \mathcal{R}_D}\ FR = 0\}.$$

Any $F \in \mathcal{F}_D$ is said to be an *initial* operator for D corresponding to R. One can prove that **any** projection F' onto $\ker D$ is an initial operator for D corresponding to a right inverse $R' = R - F'R$ independently of the choice of an $R \in \mathcal{R}_D$.

If two initial operators commute each with another, then they are equal. Thus this theory is essentially **noncommutative**. An operator F is initial for D if and only if there is an $R \in \mathcal{R}_D$ such that

$$F = I - RD \quad \text{on } \mathrm{dom}\, D. \tag{1.1}$$

Even more. Write $\mathcal{R}_D = \{R_\gamma\}_{\gamma \in \Gamma}$. Then, by (1.1), we conclude that $\mathcal{R}_D$ induces in a unique way the family $\mathcal{F}_D = \{F_\gamma\}_{\gamma \in \Gamma}$ of the corresponding initial operators defined by means of the equality $F_\gamma = I - R_\gamma D$ on $\mathrm{dom}\, D \quad (\gamma \in \Gamma)$. Formula (1.1) yields (by a two-lines induction) the *Taylor Formula*:

$$I = \sum_{k=0}^{n} R^n F D^n + R^n D^n \text{ on } \mathrm{dom}\, D^n \quad (n \in \mathbb{N}). \tag{1.2}$$

It is enough to know one right inverse in order to determine all right inverses and all initial operators. Note that a superposition of a finite number of right invertible operators is again a right invertible operator.

The equation $Dx = y \ \ (y \in X)$ has the general solution $x = Ry + z$, where $R \in \mathcal{R}_D$ is arbitrarily fixed and $z \in \ker D$ is arbitrary. However, if we put an *initial condition*: $Fx = x_0$, where $F \in \mathcal{F}_D$ and $x_0 \in \ker D$, then this equation has a unique solution $x = Rx + x_0$.

If $T \in L(X)$ belongs to the set $\Lambda(X)$ of all left invertible operators, then $\ker T = \{0\}$. If D is invertible, i.e. $D \in \mathcal{I}(X) = R(X) \cap \Lambda(X)$, then $\mathcal{F}_D = \{0\}$ and $\mathcal{R}_D = \{D^{-1}\}$.

If $P(t) \in \mathbb{F}[t]$ then all solutions of the equation

$$P(D)x = y, \qquad y \in X, \tag{1.3}$$

can be obtained by a decomposition of the rational function $1/P(t)$ into vulgar fractions. One can distinguish subspaces of X with the property that all solutions of Equation (1.3) belong to a subspace Y whenever $y \in Y$ (cf. von TROTHA vT[1], PR[2]).

If X is an algebra over $\mathbb{F}$ with a $D \in L(X)$ such that $x, y \in \text{dom } D$ implies $xy, yx \in \text{dom } D$, then we shall write $D \in \mathbf{A}(X)$. The set of all *commutative* algebras belonging to $\mathbf{A}(X)$ will be denoted by $\mathsf{A}(X)$. Let $D \in \mathbf{A}(X)$ and

$$f_D(x,y) = D(xy) - c_D[xDy + (Dx)y] \quad \textit{for} \quad x, y \in \text{dom } D, \tag{1.4}$$

where c_D is a scalar dependent on D only. Clearly, f_D is a bilinear (i.e. linear in each variable) form which is symmetric when X is commutative, i.e. when $D \in \mathsf{A}(X)$. This form is called a *non-Leibniz component* (cf. PR[1]). If $D \in \mathbf{A}(X)$ then the product rule in X can be written as follows:

$$D(xy) = c_D[xDy + (Dx)y] + f_D(x,y) \quad \textit{for} \quad x, y \in \text{dom } D.$$

If $D \in \mathbf{A}(X)$ and if D satisfies the *Leibniz condition*:

$$D(xy) = xDy + (Dx)y \quad \textit{for } x, y \in \text{dom } D, \tag{1.5}$$

then X is said to be a *Leibniz algebra*. It means that in Leibniz algebras $c_D = 1$ and $f_D = 0$. The Leibniz condition implies that $xy \in \text{dom } D$ whenever $x, y \in \text{dom } D$. If X is a Leibniz algebra with unit e then $e \in \ker D$, i.e. D is not left invertible.

Non-Leibniz components for powers of $D \in \mathbf{A}(X)$ are determined by recurrence formulae (cf. PR[1], PR[3]).

Suppose that $D \in \mathbf{A}(X)$ and $\lambda \neq 0$ is an arbitrarily fixed scalar. Then $\lambda D \in \mathbf{A}(X)$ and $c_{\lambda D} = c_D,\ f_{\lambda D} = \lambda f_D$.

If $D_1, D_2 \in \mathbf{A}(X)$, the superposition $D = D_1 D_2$ exists and $D_1 D_2 \in \mathbf{A}(X)$, then

$$c_{D_1 D_2} = c_{D_1} c_{D_2} \quad \textit{and for } x, y \in \text{dom } D = \text{dom } D_1 \cap D_2 \tag{1.6}$$

$$f_{D_1 D_2}(x,y) = f_{D_1}(x,y) + D_1 f_{D_2}(x,y) + + c_{D_1} c_{D_2}[(D_1 x) D_2 y + (D_2 x) D_1 y].$$

For higher powers of D in Leibniz algebras, by an easy induction from Formulae (1.6) and the Leibniz condition, we obtain *the Leibniz formula*:

$$D^n(xy) = \sum_{k=0}^{n} \binom{n}{k} (D^k x) D^{n-k} y \quad \textit{for } x, y \in \text{dom } D^n \quad (n \in \mathbb{N}). \tag{1.7}$$

2. Equations in linear spaces.

We begin with a theorem which may look on the first point of view rather artificial. However, it plays some role in our subsequent considerations.

Theorem 2.1. (cf. PR[1]) *Suppose that X is a linear space over a field $\mathbb{F}$ of scalars (with the characteristic zero) $D \in R(X)$,* dim ker *$D \neq 0$ and F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Let $\{H_x\}_{x\in X}$ be a family of mappings of the space X into itself (in general, nonlinear) with respect to x. Then*

(i) Every solution $x \in$ dom D of the equation

$$Dx = H_x y, \qquad where\ y \in X\ is\ given, \tag{2.1}$$

is a solution of the equation

$$x - RH_x y = z, \qquad where\ z \in \ker D\ is\ arbitrary. \tag{2.2}$$

Conversely,

(ii) if a solution x of Equation (2.2) belongs to dom D then it is a solution of Equation (2.1).

(iii) If x is a solution of Equation (2.1) with the initial condition

$$Fx = x_0, \qquad where\ x_0 \in \ker D\ is\ given, \tag{2.3}$$

then it is a solution of the equation

$$x - RH_x y = x_0. \tag{2.4}$$

Proof. If an $x \in$ dom D satisfies Equation (2.1) then, by our assumptions,

$$0 = Dx - H_x y = Dx - DRH_x = D(x - RH_x y),$$

which implies (2.2). Conversely, if an $x \in$ dom D satisfies (2.2) then

$$0 = Dz = D(x - RH_x) = Dx - DRH_x y = Dx - H_x y,$$

i.e. x satisfies (2.1).

(iii) By (i), an $x \in$ dom D satisfies the initial value problem for Equation (2.1) with Condition (2.3) if it satisfies Equation (2.2) with Condition (2.3). Since, by definitions, $FR = 0$ and $Fz = z$ whenever $z \in$ ker D, we find

$$z = Fz = F(x - RH_x y) = Fx - FRH_x y = x_0,$$

i.e. x satisfies Equation (2.4). ■

Example 2.1. Let $\mathbb{F} = \mathbb{R}$, $X = C[0, 1]$, $D = \frac{\mathrm{d}}{\mathrm{d}t}$, $R = \int_0^t$, $(Fx)(t) \equiv c$, $(c \in \mathbb{R})$, for $t \in [0, 1]$, $x \in X$. Then an ordinary differential equation with *separable variables*

$$x'(t) = ax(t), \quad where\ H_x = a,\ a \in \mathbb{F}\ is\ given, \tag{2.5}$$

is equivalent to a *Volterra integral equation*

$$x(t) - \int_0^t ax(s)ds = c, \quad \textit{where } c \in \mathbb{R} \textit{ is an arbitrary constant.} \tag{2.6}$$

It is well-known that this equation has a unique solution for an arbitrarily fixed $c \in \mathbb{R}$. On the other hand, $X = C[0,1]$ is an algebra with the respect to the pointwise multiplication of functions as a structure operation. Therefore, any solution $x \neq 0$ of (2.5) satisfies the equation

$$\frac{x'(t)}{x(t)} = a, \quad \textit{i.e.} \quad \frac{\mathrm{d}}{\mathrm{d}t} \ln x(t) = a, \tag{2.7}$$

which implies

$$\ln x(t) = a\int_0^t \; ds + \ln c = \ln\; \exp(a\int_0^t \; ds) + \ln c = \ln[c\exp(a\int_0^t \; ds)] = \ln\; \mathrm{e}^{at},$$

where $c \in \mathbb{R} \setminus \{0\}$ is arbitrary. Finally, we conclude that

$$x(t) = c\mathrm{e}^{at}, \quad \textit{where } c \in \mathbb{R} \setminus \{0\}.$$

(cf. also Triebel Tr[1] and Example 0.1). Note that for the operator $D = \frac{\mathrm{d}}{\mathrm{d}t}$ there is a much richer family of initial operators than that given here (determined by the values of functions at the given points (cf. PR[1], PR[3]). □

Example 2.2. (cf. PR[1]) Suppose that $D \in R(X)$, dim ker $D \neq 0$, $R_0,...,R_{M+N-1} \in \mathcal{R}_D$,

$$Q(D) = \sum_{k=0}^{N-1} Q_k D^k, \quad \textit{where } Q_0, ..., Q_{N-1} \in L_0(X), \tag{2.8}$$

$$Q^o = \sum_{k=1}^{N-1} Q_0 R_{M+k} ... R_{M+N-1} \tag{2.9}$$

and the operator $I + Q^o$ is invertible. Let $\{H_x\}_{x \in X}$ be defined as in Theorem 2.1, i.e. it is a family of nonlinear mappings of X into itself depending on $x \in X$. Then we have

$$D_1 = Q(D)D^M \in R(X), \quad R_1 = R_0...R_{M+N-1}(I + Q^o)^{-1} \in \mathcal{R}_D,$$

$$\ker\; D_1 = \{z = (I - R_1 D_1)x : x \in \mathrm{dom}\; D\} =$$

$$= \{z = R_0...R_{M+N-1}(I + Q^o)^{-1}\big(\sum_{m=0}^{N-1} Q_m \sum_{k=m+1}^{N-1} R_m...R_{k-1} z_{m+k} + z_{M+m}\big) +$$

$$+\sum_{k=0}^{N-1} R_0...R_{k-1}z_k : z_0,...,z_{M+N-1} \in \ker\ D\}.$$

We therefore conclude that any $x \in \text{dom } Q(D)D^M$ satisfies the equation

$$(2.10) \qquad Q(D)D^M x = H_x y, \quad \textit{where } y \in X \qquad (M \geq 0)$$

if and only if x satisfies the equation

$$(2.11) \qquad x - R_0...R_{M+N-1}(I+Q^o)^{-1}H_x y = z, \quad z \in \ker\ Q(D)D^M.$$

This is a generalization of Theorem 2.1 for operators of order greater than 1, □

Example 2.3. (cf. PR[1]) Let all assumptions of Example 2.2 be satisfied and let $R_0,...,R_{N-1} = R$. Then every solution x of Equation (2.10) belongs to dom $Q(D)D^M$ and satisfies the equation

$$(2.12) \qquad x - R^{N+M}[Q(I,R)]^{-1}H_x y = z,$$

$$(2.13) \qquad \textit{where} \quad z = R^{N+M}[Q(I,R)]^{-1}\Big(\sum_{m=0}^{N-1}\sum_{k=m}^{N-1} R^{k-m}z_k\Big)+$$

$$+\sum_{k=0}^{M+n-1} R^k z_k \in \ker\ D^{M+N} \quad \textit{whenever} \quad z_0,...,z_{M+N-1} \in \ker\ D,$$

$$(2.14) \qquad Q(t,s) = \sum_{k=0}^{N} Q_k t^k_{N-k}, \quad Q(I,R) = I + Q^o$$

is invertible. Conversely, every solution of Equation (2.12) belonging to dom $Q(D)D^M$ satisfies Equation (2.10). A similar result can be obtained for the operator $Q^M Q(D)$. □

Example 2.4. Let X be a linear space (over $\mathbb{F}$). Let $D \in R(X)$ and let $R \in \mathcal{R}_D$. Suppose that $a \in \ker D$, $b \in \mathbb{F} \setminus \{0\}$, $y \in X$ and the mapping f of X into itself (not necessarily linear) are given. By definitions, $Da = 0$, $D(bx) = bDx$ whenever $x \in X$. Consider the equation

$$(2.15) \qquad Dx = f(a+bx)y.$$

Observe that here we have a particular case of Equation (2.1), where we put $H_x = f(a+bx)$ for $x \in X$. Write: $u = a + bx$. Then $Du = D(a+bx) = bDx$ and $Dx = b^{-1}u$. Then Equation (2.15) may be rewritten as $b^{-1}Du = f(u)y$, i.e.

$$(2.16) \qquad Du = bf(u)y.$$

Since $bRx = R(bx)$ whenever $b \in \ker\ D$, $x \in X$, we conclude that $u \in \text{dom}\ D = \text{dom}\ X$ is a solution of Equation (2.16) if and only if it is a solution of the equation

$$(2.17) \qquad u - bR[f(u)y] = z, \quad \textit{where } z \in \ker\ D \textit{ is arbitrary.}$$

It means that $x = b^{-1}(u-a)$ is a solution of Equation (2.15) if and only if u is a solution of Equation (2.17) belonging to dom D.

Let F be an initial operator for D corresponding to R. By definition, $FR = 0$ and $Fz = z$ whenever $z \in \ker\ D$. Then for a given $x_0 \in \ker\ D$ and the initial value condition

$$(2.18) \qquad Fx = x_0$$

we find

$$z = Fz = F\{u - bR[f(u)y]\} = Fu - bFR[f(u)y] = F(a + bx) = Fa + bFx = a + bx_0,$$

i.e. Equation (2.17) with Condition (2.18) satisfies the equation

$$(2.19) \qquad u - bR[f(u)y] = x_0.$$

Since $u = a + bx$, Equation (2.19) can be rewritten as

$$(2.20) \qquad a + bx - bR[f(a + bx)y] = x_0.$$

We therefore conclude that $x \in \text{dom}\ D$ is a solution of the initial value problem (2,15), (2.17): $Dx = f(a + bx)y$, $Fx = x_0$, if and only if $x \in \text{dom}\ D$ is a solution of Equation (2.20). □

Clearly, equations considered in Examples 2.2-2.4 are, in a sense, analogues (in linear spaces) of the classical ordinary differential equations with separable variables. However, in order to solve their resolving equations in a close form, we need some more rich structures.

3. Equations in commutative Leibniz algebras.

Let a commutative algebra X (over a field $\mathbb{F}$ of scalars) with $D \in R(X)$, $e \in \text{dom}\ D$ satisfy the Leibniz condition (1.5):

$$(3.1) \qquad D(xy) = xDy + yDx \quad \textit{for } x, y \in \text{dom}\ D.$$

Then X is a D-algebra, since $xy \in \text{dom}\ D$ whenever $x, y \in \text{dom}\ D$. The set of all such algebras will be denoted by $\mathbf{L}(D)$. Recall that $e \in \ker\ D$ (cf. Example 6.11 in PR[1]), i.e. the unit e is a constant. Here and in the sequel we shall use the following properties (cf. also PR[1]):

Suppose that $X \in \mathbf{L}(D)$ and F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Write: $g = Re$ *). Then we have $Dg = DRe = e$. So that, by an easy induction, we obtain

$$D^n g = 0 \quad (n \in \mathbb{N} \setminus \{1\}), \quad Dg = e. \tag{3.2}$$

Since $e \in \ker\ D$, we have $Fe = e$. Since $FR = 0$, we find $Fg = FRe = 0$. If F is multiplicative then (again by an easy induction):

$$Fg^n = (Fg)^n = 0 \quad (n \in \mathbb{N}). \tag{3.3}$$

By Formula (3.3), we have

$$Fp(g) = p(0) = p_0 e \quad for\ p(t) \in \mathbb{F}[t], \tag{3.4}$$

(3.5), $$Fw(g) = w(0) = \frac{p_0}{\tilde{p}_0} \quad for\ w(t) \in \mathbb{Q}[t],\ w = \frac{p}{\tilde{p}},\ \tilde{p}_0 \neq 0,\ i.e.\ \tilde{p}(0) \in I(X).$$

where

$$p(t) = \sum_{k=0}^{n} p_k t^k, \quad \tilde{p}(t) = \sum_{k=0}^{n} \tilde{p}_k t^k \in \mathbb{F}[t]. \tag{3.6}$$

Indeed, $p(0) = p_0 e$ and

$$Fp(g) = F \sum_{k=0}^{n} p_k g^k = \sum_{k=0}^{N} Fg^k = \sum_{k=0}^{n} p_k (Fg)^k = p_0 e = p(0),$$

i.e. $p(0)$ is invertible if and only if $p_0 \neq 0$. Similarly, $\tilde{p}(0) = \tilde{p}_0 e$.

Proposition 3.1. *Suppose that $X \in \mathbf{L}(D)$, F is a multiplicative initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $g = Re \in I(X)$. Then Fg^{-1} is not well-defined (does not exists).*

Proof. By (3.3) and (3.5), we have $Fg^{-1} = (Fg)^{-1}$. However, $Fg = 0$. ■

Example 3.1. Suppose that all conditions of Example 2.1 are satisfied. If we consider $X = C[0, 1]$ as an algebra (over $\mathbb{R}$) with respect to the pointwise multiplication of functions as the structure operation then $X = \mathbf{L}(\frac{\mathrm{d}}{\mathrm{d}t})$, since the Leibniz condition holds:

$$(xy)' = xy' + yx' \quad for\ x, y \in C^1[0, 1], \quad where \tag{3.1}$$

*) Elements of the form $g = Re$, where $R \in \mathcal{R}_D$, play the role of an *argument*, since in the case considered in Example 2.1 we have $g(t) = \int_0^t 1\ ds = t$ for $t \in [0, 1]$.

$$Dx = \frac{\mathrm{d}}{\mathrm{d}t}x = x', \quad \text{dom } D = C^1[0,1].$$

Let $(Fx)(t) = x(0)$ for $x \in X$. Then $Fg^{-1} = (Fg)^{-1}$ does not exists. Clearly, also g^{-1} does not exists for $g(0) = 0$.

Note that the algebra X described above is not a unique Leibniz algebra known in the classic Analysis. We mention here only two examples:

(**a**) Let $X = C[\Omega]$, where $\Omega = \{(t,s) : a \le t \le b,\ c \le s \le d\}$ with the pointwise multiplication of functions as a structure operation. Let $D_1 = \frac{\partial}{\partial t}$, $D_2 = \frac{\partial}{\partial s}$. Then X is a Leibniz D_1-algebra and, simultaneously, a Leibniz D_2-algebra, i.e. $X \in \mathbf{L}(D_i)$ $(i = 1, 2$ (cf. PR[1]).

(**b**) The space $X_a = \{x \in C[0,T] : x(t) = 0 \ \mathit{for}\ 0 \le t \le a < T\}$ $(a \in \mathbb{R}$ is arbitrarily fixed) with the multiplication defined by *convolution* :

$$(x * y)(t) = \int_0^t x(s)y(t-s)ds \quad \mathit{for}\ x, y \in C[0,T]$$

and with D defined by means of the equality $(Dx)(t) = tx(t)$ for $x \in X_a$ is a Leibniz D-algebra, i.e. $X_a = \mathbf{L}(D)$ (without unit and with zero divisors) (cf. PR[1]). □

Example 3.2. Suppose that $X \in \mathbf{L}(D)$, $R \in \mathcal{R}_D$, $g = Re \in I(X)$ and f is a mapping of X into itself (in general, nonlinear). Consider the equation

$$Dx = f(g^{-1}x) \quad (x \in \text{dom } D). \tag{3.8}$$

Write: $x = gu$. Then $u = g^{-1}$ and, by the Leibniz condition, we have

$$f(u) = f(g^{-1}x) = Dx = D(gu) = uDg + gDu = ue + gDu = u + gDu, \quad i.e.$$

$$Du = h(u), \quad \mathit{where}\ h(u) = g^{-1}[f(u) - u] \quad (u \in \text{dom } D. \tag{3.9}$$

We therefore conclude that Equation (3.8) has a solution $x \in$ dom D if and only if Equation (3.9) (with *separable variables*) has a solution $u \in$ dom D. If it is the case, then solutions of (3.8) are of the form: $x = gu$.

In the classical case Equation (3.8) is the so-called *homogeneous* ordinary differential equation $y' = f(\frac{y}{x})$ (cf. Tr[1], also Example 0.1).

Let F be a multiplicative initial operator corresponding to the given $R \in \mathcal{R}_D$. Then $FR = 0$ and $Fu = F(g^{-1}x) = (Fg^{-1})Fx$. Hence, by Proposition 3.1, Fu is not well-defined. Then $Fu \notin \ker\ D$, which implies that $Fx = F(gu) = (Fg)(Fu) \notin \ker\ D$, i.e. $x \notin$ dom D.

Clearly, if $R_1 \ne R$, $R_1 \in \mathcal{R}_D$ then the corresponding initial operator $F_1 = (I - R_1D) \ne F$ and, by definitions, we have $F_1R = -FR_1$. So that, if F_1 is multiplicative and $F_1g = F_1Re \ne 0$ is invertible then the element $F_1g^{-1} = (F_1g)^{-1}$ is well-defined. □

Example 3.3. Suppose that X, D, F, R satisfy all conditions of Example 3.2. Let h be a mapping (in general, nonlinear) of X into itself. Then the equation

$$(3.10)\qquad Dx = h(x)y, \quad \textit{where } y \in X \textit{ is given} \quad (x \in \mathrm{dom}\ D)$$

with separable variables is equivalent to the equation

$$(3.11)\qquad x - R[h(x)y] = z, \quad \textit{where } z \in \ker\ D \textit{ is arbitrary.}$$

The proof is similar to that given in Example 3.2 for Equations (3.8), (3.9). However, in order to solve Equation (3.10), we may apply another way. Suppose now that $h(x) \in I(\mathrm{dom}\ D)$ whenever $x \in \mathrm{dom}\ D$. Again we obtain a separable equation equivalent to (3.10):

$$(3.12)\qquad [h(x)]^{-1}Dx = y.$$

The element $h_1(x) = R\{[h(x)]^{-1}Dx\}$ is a primitive for $[h(x)]^{-1}Dx$, whenever $x \in \mathrm{dom}\ D$. Indeed, Equation (3.12) implies that

$$Dh_1(x) = DR\{[h(x)]^{-1}Dx\} = [h(x)]^{-1}Dx = DRy, \quad \textit{i.e.} \quad D[h_1(x) - Ry] = 0.$$

Hence we obtain the equation

$$(3.13)\qquad h_1(x) = Ry + z, \quad \textit{where } z \in \ker\ D \textit{ is arbitrary.}$$

If $h_1(x)$ is a one-to-one mapping then we conclude that

$$(3.14)\qquad x = h_1^{-1}(Ry + z), \quad \textit{where } z \in \ker\ D \textit{ is arbitrary.}$$

□

Example 3.3 can be generalized as follows.

Proposition 3.2. *Suppose that* $X \in \mathbf{L}(D)$, F *is a multiplicative initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$ *and* h *is a mapping of* X *into itself (in general, nonlinear) such that* $h(x) \in I(\mathrm{dom}\ D^n)$ *whenever* $x \in \mathrm{dom}\ D^n$ $(n \in \mathbb{N})$. *Then the equation*

$$(3.15)\qquad D^n x = h(x)y, \quad \textit{where } y \in X \textit{ is given} \quad (x \in \mathrm{dom}\ D^n)$$

is equivalent to the equation

$$(3.16)\qquad x - R^n[h(x)y] = z, \quad \textit{where } z \in \ker\ D^n \textit{ is arbitrary.}$$

Write:

$$(3.17)\qquad h_1(x) = R^n\{[h(x)]^{-1}D^n x\} \quad \textit{for } x \in \mathrm{dom}\ D^n \quad (n \in \mathbb{N}).$$

If h_1 is a one-to-one mapping of X into itself then all solutions of Equation (3,16) are of the form;

$$(3.18) \qquad x = h_1^{-1}(R^n y + z), \quad \textit{where} \sum_{k=0}^{n-1} R^k z_k \in \ker\ D^n,$$

$$z_0, ..., z_{n-1} \in \ker\ D \textit{ are arbitrary},$$

Proof. Similarly, as before, Equation (3.15) may be rewritten as

$$y = [h(x)]^{-1} D^n x = D^n R^n [h(x)]^{-1} = D^n h_1^{-1}(x),$$

which implies Equation (3.16). ■

Corollary 3.1. *Suppose that all assumptions of Proposition 3.1 are satisfied, $g = Re$ and $y = g^m$ (where $\in \mathbb{N}$ is fixed. Then the equation*

$$(3.19) \qquad D^n x = h(x) g^m \qquad (m, n \in \mathbb{N})$$

has all solutions of the form:

$$(3.20) \qquad x = h_1^{-1}(\frac{g^{n+m}}{(n+1)...(n+m)} + z), \quad \textit{where } z = \sum_{k=0}^{n-1} R^k z_k \in \ker\ D^n,$$

$$z_0, ..., z_{n-1} \in \ker\ D \textit{ are arbitrary}.$$

Proof. By Proposition (3.1), Equation (3.19) has all solutions of the form (3.18) with $y = g^m$. Recall that constants are non zero divisors, because X is a Leibniz algebra. Since F is multiplicative, we have

$$(3.21) \qquad R^k e = \frac{(Re)^k}{k!} = \frac{g^k}{k!} \qquad (k \in \mathbb{N})$$

(cf. von Trotha PRvT[1], PR[1]), Then for all $m, n \in \mathbb{N}$ we have

$$R^n y = R^n g^n = R^n (Re)^m = R^n m! R^m e = m! R^{n+m} e = m! \frac{g^{n+m}}{(n+m)} =$$

$$= \frac{g^{n+m}}{(n+1)...(n+m)},$$

which yields Formula (3.21). ■

Equations of the form (3.20) will be solved in Section 5 in another way.

Proposition 3.2. *Suppose that $X \in \mathbf{L}(D)$, $R \in \mathcal{R}_D$ and $a \in X$. Then* *(i)* $x \in \mathrm{dom}\ D$ *is a solution of the equation*

$$(3.22) \qquad Dx = ax + y, \quad y \in X \text{ is given},$$

if and only if $x \in \mathrm{dom}\ D$ *satisfies the equation*

$$(3.23) \qquad x - R(ax) = Ry + z, \quad \text{where } z \in \ker\ D \text{ is arbitrary}.$$

(ii) If F *is an initial operator for* D *corresponding to* R *and we have an initial value condition*

$$(3.24) \qquad Fx = x_0, \quad x_0 \in \ker\ D \text{ is given},$$

then a solution of the initial value problem (3.23),(3.24) (if it exists) satisfies the equation

$$(3.25) \qquad x - R(ax) = Ry + x_0.$$

(iii) If the operator $I - Ra$ *is invertible then Equation (3.23) (hence also (3.22)) has all solutions of the form:*

$$(3.25) \qquad x = (I - Ra)^{-1}(Ry + z) \quad \text{where } z \subset \ker\ D \text{ is arbitrary}$$

and the initial value problem (3.22),(3,24) has a unique solution

$$x = (I - Ra)^{-1}(Ry = x_0)$$

(cf. PR[1], PR[3]).

Proof. If x satisfies (3.22) then $DRy = y = Dx - ax = Dx - DR(ax) = D[x - R(ax)]$, i.e. $D[x - R(ax) - y] = 0$, which implies (3.23). Conversely, if x satisfies Equation (3.23) then $y = D(Ry + z) = D[x - R(ax_0] = Dx - DR(ax) = Dx - ax$. If $I - Ra$ is an invertible mapping then (3.23) immediately implies (3.26). If F is an initial operator for D corresponding to R then, by (3.25), we have $z = Fz = F[x - R(ax) - Ry] = Fx - FR(ax) - FRy = Fx = x_0$ (cf. Propositions 5.2, 5.3 and Corollary 5.1). ■

Example 3.4. Suppose that $X \in \mathbf{L}(D)$ and $R \in \mathcal{R}_D$. Recall that

$$(3.28) \qquad Dx^n = nx^{n-1}Dx, \quad \text{whenever } n \in \mathbb{N},\ x \in \mathrm{dom}\ D$$

(cf. PR[1]). This formula holds also for negative integers, i.e. we have

$$(3.29) \qquad Dx^{-n} = -nx^{-n-1}Dx, \quad \text{whenever } n \in \mathbb{N},\ x \in \mathrm{dom}\ D \cap I(X)$$

Indeed, $Dx^{-1} = x^{-2}Dx$. Then for $n \geq 2$ we have

$$Dx^{-1} = (Dx^{-1})^n = D(x^{-1})^n = n(x^{-1})^{n-1}Dx^{-1} = -nx^{-n+1}x^{-2}Dx = -nx^{-n-1}Dx.$$

Consider the equation

$$(3.30) \qquad Dx = ax^m, \quad \textit{where } a \in X,\ m \in \mathbb{N}.$$

By (3.29), this equation can be rewritten as follows:

$$Dx^{-(m+1)} = -(m+1)x^{-m}Dx = -(m+1)x^{-m}ax^m = -(m+1)a,$$

which implies that

$$(3.31) \quad x^{-(m+1)} = R[-(m+1)a] + z = z - (m+1)Ra, \quad \textit{where } z \in \ker\ D \textit{ is arbitrary.}$$

Suppose now that there is a $z \in \ker\ D$ such that the element $z-(m+1)Ra$ is invertible. Then $x^{m+1} = [z-(m+1)Ra]^{-1}$. If $[z-(m+1)Ra]^{-1} \in I_{m+1}(X)$ then

$$x = \{[z-(m+1)Ra]^{-1}\}^{1/(m+1)} = [z-(m+1)Ra]^{-1/(m+1)}.$$

Similarly, in order to solve the equation

$$(3.32) \qquad Dx = ax^{-m} \qquad (m \in \mathbb{N})$$

we have to rewrite (3.31) as follows:

$$Dx^{m+1} = (m+1)x^m Dx = (m+1)x^m ax^{-m} = (m+1)a,$$

which implies $x^{m+1} = (m+1)Ra + z$, where $z \in \ker\ D$ is arbitrary. If there is a $z \in \ker\ D$ such that $(m+1)Ra + z \in I_{m+1}(X)$ then

$$(3.33) \qquad x = [z + (m+1)Ra]^{1/(m+1)}.$$

To summarize, we conclude that the equations *)

$$(3.34) \qquad Dx = ax^{\mp m} \qquad (m \in \mathbb{N})$$

have solutions of the form:

$$(3.35) \qquad x = [z \pm (m+1)Ra]^{\pm 1/(m+1)},$$

respectively, if there are $z \in \ker\ D$ such that elements of the form (3.34) exist (cf. also Example 5.3). □

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*) To be short, we write here and in the sequel $\pm n$ ($n \in \mathbb{N}$) for two different equations, i.e. either for $n \in \mathbb{N}$ or for $-n \in \mathbb{N}$, and so on. This means that, as a matter of fact, we consider two types of equations (cf. also Section 5).

4. Algebras with logarithms.

We start with

Definition 4.1. Suppose that $D \in \mathsf{A}(X)$. Let a multifunction $\Omega : \text{dom } D \longrightarrow 2^{\text{dom } D}$ be defined as follows:

$$\Omega u = \{x \in \text{dom } D : Du = uDx\} \quad for \ u \in \text{dom } D. \tag{4.1}$$

The equation

$$Du = uDx \quad for \ (u, x) \in \text{graph } \Omega \tag{4.2}$$

is said to be the *basic equation*. Clearly,

$$\Omega^{-1}x = \{u \in \text{dom } D : Du = uDx\} \quad for \ x \in \ \text{dom D}.$$

The multifunction Ω is well-defined and dom $\Omega \supset \ker D \setminus \{0\}$.

Suppose that $(u, x) \in$ graph Ω, L is a selector of Ω and E is a selector of Ω^{-1}. By definitions, $Lu \in$ dom Ω^{-1}, $Ex \in$ dom Ω and the following equations are satisfied: $Du = uDLu$, $DEx = (Ex)Dx$.

Any invertible selector L of Ω is said to be a *logarithmic mapping* and its inverse $E = L^{-1}$ is said to be a *antilogarithmic mapping*. By $G[\Omega]$ we denote the set of all pairs (L, E), where L is an invertible selector of Ω and $E = L^{-1}$. For any $(u, x) \in$ dom Ω and $(L, E) \in G[\Omega]$ elements Lu, Ex are said to be *logarithm* of u and *antilogarithm* of x, respectively. The multifunction Ω is examined in PR[3]. The assumption that X is a commutative algebra is admitted here for simplicity and the sake of brevity only. $\square$

Clearly, by definition, for all $(L, E) \in G[\Omega]$, $(u, x) \in$ graph Ω we have

$$ELu = u, \quad LEx = x; \qquad DEx = (Ex)Dx, \quad Du = uDLu. \tag{4.3}$$

A logarithm of zero is not defined. If $(L, E) \in G[\Omega]$ then $L(\ker D \setminus \{0\}) \subset \ker D$, $E(\ker D) \subset \ker D$. In particular, $E(0) \in \ker D$.

If $D \in R(X)$ then logarithms and antilogarithms are uniquely determined up to a constant.

Let $D \in \mathsf{A}(X)$ and let $(L, E) \in G[\Omega]$. A logarithmic mapping L is said to be of the *exponential type* if $L(uv) = Lu + Lv$ for $u, v \in$ dom Ω. If L is of the exponential type then $E(x+y) = (Ex)(Ey)$ for $x, y \in$ dom Ω^{-1}. We have proved that a logarithmic mapping L is of the exponential type if and only if X is a *commutative Leibniz algebra* (cf. PR[3]). Moreover, $Le = 0$, i.e. $E(0) = e$. In Leibniz commutative algebras with $D \in R(X)$ a necessary and sufficient conditions for $u \in$ dom Ω is that $u \in I(X)$ (cf. PR[3]).

By $\mathbf{Lg}(D)$ we denote the class of these commutative algebras with $D \in R(X)$ and with unit $e \in \mathrm{dom}\ \Omega$ for which there exist invertible selectors of Ω, i.e. there exist $(L, E) \in G[\Omega]$. By $\mathsf{L}(D)$ we denote the class of these commutative Leibniz algebras with unit $e \in \mathrm{dom}\ \Omega$ for which there exist invertible selectors of Ω. By these definitions, $X \in \mathbf{Lg}(D)$ is a Leibniz algebra if and only if $X \in \mathsf{L}(D)$ and $D \in R(X)$. This class we shall denote by $L(D)$. It means that $L(D)$ is the class of these commutative Leibniz algebras with $D \in R(X)$ and with unit $e \in \mathrm{dom}\ \Omega$ for which there exist invertible selectors of Ω, i.e. there exist $(L, E) \in G[\Omega]$.

If $\ker D = \{0\}$ then either X is not a Leibniz algebra or X has no unit. Thus, by our definition, if $X \in L(D)$ then $\ker D \neq \{0\}$, i.e. the operator D is right invertible but not invertible.

Theorem 4.1. *Suppose that $X \in L(D)$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$, $(L, E) \in G[\Omega]$ and A is an algebra isomorphism of X. Let $D' = A^{-1}DA$ and let $\Omega' : \mathrm{dom}\ D' \longrightarrow 2^{\mathrm{dom}\ D'}$ be defined as follows:*

$$\Omega' u = \{x \in \mathrm{dom}\ D' : D'u = uD'x\} \quad \textit{for } u \in \mathrm{dom}\ D'. \tag{4.4}$$

Then there are $(L', E') \in G[\Omega']$ and $L' = A^{-1}LA$, $E' = A^{-1}EA$.

5. Equations in Leibniz algebras with logarithms.

We start with

Proposition 5.1. *Suppose that $X \in L(D)$, $(L, E) \in G[\Omega]$ and $g = Re$ for an $R \in \mathcal{R}_D$. Then $g \in I(X)$ and*

$$DLg = g^{-1}. \tag{5.1}$$

Proof. By definitions, $g \in \mathrm{dom}\ \Omega^{-1}$. Since X is a Leibniz algebra, this implies that $g \in I(X)$. Moreover, $g \in \mathrm{dom}\ \Omega \subset \mathrm{dom}\ D$. We therefore conclude that $DLg = g^{-1}Dg$ $= g^{-1}DRe = g^{-1}e = g^{-1}$, since the basic equation (4.1) is satisfied by g. ■

Proposition 5.2. *Suppose that all assumptions of Proposition 5.1 are satisfied and $Ra + z \in \mathrm{dom}\ \Omega^{-1}$ for an $a \in X$ and arbitrary $z \in \ker\ D$. Then all invertible solutions of the equation*

$$Dx = ax \tag{5.2}$$

are of the form

$$x = zERa, \quad \textit{where } z \in \ker\ D \textit{ is arbitrary.} \tag{5.3}$$

If F is a multiplicative initial operator for D corresponding to R then the initial value problem

$$Dx = ax, \quad Fx = x_0, \quad \textit{where } x_0 \in \mathrm{dom}\ D \textit{ is given,} \tag{5.4}$$

has a unique solution which is of the form:

$$x = x_0 ERa. \tag{5.5}$$

Proof. By definition, if x is a solution of Equation (5.2) then $x \in I(X) \cap \text{dom } D$. Hence $DLx = x^{-1}Dx = a + DRa$, i.e. $D(Lx - Ra) = 0$. This implies that $Lx - Ra = z' \in \ker\ D$ (z' is arbitrary). Write: $z = Ez'$. Then we get $x = ELx = E(Ra + z') = (Ez')(ERa) = zERa$, since the Leibniz condition holds. Now, we shall prove that

$$EF = FE, \quad LF = FL. \tag{5.6}$$

Indeed, by definitions, $EL = LE = I$. Write; $A = FE$. Then $AL = FEL = F = LEF = LA$, which implies $FE = A = ELA = EF$. In order to prove the second equality, write: $B = FL$. Then $BE = FLE = F = ELF = EB$ and $F = ELF = EB = BE$, which implies $FL = BEL = B = LF$.

If $Fx = x_0 \in \ker\ D$ then $F^2x = Fx_0 = x_0$ and, by Formulae (5.6) and the multiplicativity of F, we find $x_0 = Fx = F^2x = F(zRa) = (Fz)(FERa) = zEFRa = zE(0) = ze = z$, which implies Formula (5.5). ■

An initial operator $F \in \mathcal{F}_D$ is said to be *almost averaging* if

$$F(zx) = zFx \quad \textit{whenever } x \in X,\ z \in \ker\ D. \tag{5.7}$$

Clearly, every multiplicative initial operator F is almost averaging for $F(zx) = (Fz)(Fx) = zFx$, but not conversely, even if $\dim \ker D = 1$ (cf. PR[1]).

Corollary 5.1. *Proposition 5.2 holds for almost averaging F.*

Indeed, for all $z \in \ker\ D$, we have $x_0 = Fx = F(zERa) = zFERa = zEFRa = zE(0) = ze = z$, i.e. x is of the form (5.5). ■

Proposition 5.3. *Suppose that $X \in L(D)$, $(L, E) \in G[\Omega]$, $\pm Ra \in \text{dom } \Omega^{-1}$ for an $a \in X$ and F is an almost averaging initial operator for an $R \in \mathcal{R}_D$. Then the equation*

$$Dx = ax + y, \quad \textit{where } y \in X \textit{ is given}, \tag{5.8}$$

has all solutions of form:

$$x = (ERa)RE(-Ra) + zERa, \quad \textit{where } z \in \ker\ D \textit{ is arbitrary}. \tag{5.9}$$

Proof. We are looking for solutions of Equation (5.8) which are of the form: $x = uv$, where $v = ERa$ and u is to be determined. By the Leibniz condition and our assumptions, we have $Dv = DERa = (ERa)DRa = aERa = av$, i.e. v is a solution of Equation (5.2). Then

$$y = Dx - ax = D(uv) - auv = uDv + vDu - auv = uvDu + vDu - auv =$$

$$= uva + vDu - auv = vDu.$$

Since $v = ERa \in I(X)$ and $v^{-1} = (ERa)^{-1} = E(-Ra)$, we find $Du = v^{-1}y = yE(-Ra)$, i.e.

$$u = R[yER(-a)] + z, \quad \textit{where } z \in \ker\ D \textit{ is arbitrary.} \tag{5.10}$$

Then

$$x = uv = (ERa)\{R[yE(-Ra)] + z\} = (ERa)R[yE(-Ra)] + zERa,$$

i.e. x is of the form (5.9) (cf. also PR[3]). ■

Corollary 5.2. *Suppose that all assumptions of Proposition 5.3 are satisfied and F is a multiplicative initial operator for D corresponding to R. Then an initial value problem for Equation (5.8) with the initial condition*

$$Fx = x_0, \quad \textit{where } x_0 \in \ker\ D \textit{ is given,} \tag{5.11}$$

has a unique solution

$$x = (ERa)R[yE(-Ra)] + x_0ERa. \tag{5.12}$$

Proof. By our assumptions and Formulae (5.6), (5.9), we find $Fv = FERa = EFRa = E(0) = e$, and $Fu = F\{R[yE(-Ra)] + z\} = FR[yE(-Ra)] + Fz = z$. This implies that $x_0 = Fx = F(uv) = (Fu)(Fv) = ez = z$, which was to be proved. ■

Note 5.1. We have seen that (according to Proposition 3.1), in order to solve equations in question in a closed form, we had to assume that the operator $I - Ra$ is invertible and to calculate its inverse. In several cases this way could be much more complicate than a use of logarithms and antilogarithms. Also it may appear a necessity of some metric properties of the space and operators under consideration. □

Example 5.1. (*Generalized Pearson equation*). Suppose that all assumptions of Proposition 5.3 are satisfied, $g = Re$ and $a = w(g)$, where $w(t) \in \mathbb{Q}[t]$. Consider the equation

$$Dx = w(g)x. \tag{5.13}$$

By definitions, if $x \in I(X) \cap \operatorname{dom} D$ is a solution of Equation (5.13) then $DLx = x^{-1}Dx = w(g)$, i.e. $Lx = Rw(g) + z'$, where $z' \in \ker\ D$ is arbitrary. Write: $z = Ez'$. Then $\in \ker\ D$ and $x = ELx = E[Rw(g) + z'] = (Ez')ERw(g) = zERw(g)$ (cf. also Example 3.4 for $m = -1$). Observe that again $[Rw(g)](t) \in \mathbb{Q}[t]$. The equation

$$Dx = w(g)x + y, \quad \textit{where } y \in X \textit{ is given,} \tag{5.14}$$

has all solutions of the form

$$x = [ERw(g)]R\{yE[-Rw(g)]\} + zERw(g), \quad \textit{where } 0 \neq z \in \ker\ D \textit{ is arbitrary.}$$

□

Example 5.2. Suppose that all assumptions of Proposition 5.3 are satisfied, $g = Re \in I(X)$ and $a = e$. Then the equation

$$Dx = g^{-1}xLx \tag{5.15}$$

has all solutions belonging to dom $\Omega \cap$ dom Ω^{-1} of the form:

$$x = E(zERg^{-1}), \quad \textit{where } z \in \ker\ D \textit{ is arbitrary.} \tag{5.16}$$

Indeed, by our assumptions, whenever $x \in$ dom $\Omega \cap$ dom Ω^{-1}, Equation (5.15) implies $DLx = x^{-1}Dx = g^{-1}x$, i.e $Lx = zERg^{-1}$, where $z \in \ker\ D$ is arbitrary (cf. Proposition 5.3). Then $x = ELx = E(zERg^{-1})$ (cf. Example 5.10).

Suppose that F is an almost averaging initial operator for D corresponding to R. By Formulae (5.6), for an arbitrary $z \in \ker\ D$ we have

$$Fx = FE(zERg^{-1}) = EF(zERg^{-1}) = E(zFERg^{-1}) = E(zEFRg^{-1} =$$

$$= E[zE(0)] = E(ze) = Ez \in \ker\ D$$

(cf. Corollary 5.1)). Observe that, by definition, a $z \in \ker\ D$ such that $Ez = 0$ does not exists. Hence we have to admit that $Fx \neq 0$, i.e. $x_0 \neq 0$. We therefore conclude that an initial condition $Fx = x_0$ (where $x_0 \in \ker\ D \setminus \{0\}$ is given) for Equation (5.15) holds if and only if $z = LEz = LFx = Lx_0 \in \ker\ D$. If it is the case, then the corresponding initial value problem has a unique solution $x = E[(Lx_0)ERg^{-1}]$. Observe that $Lg = Rg^{-1}$. Hence Formula (5.15) can be rewritten in the following way: $x = E(zERg^{-1}) = E(zELg) = E(zg)$, which implies that the unique solution of the initial value problem under question is $x = E(gLx_0)$. □

Example 5.3. Suppose that all assumptions of Proposition 5.3 are satisfied and a is a mapping of X into itself such that $a(x) \neq e$ whenever $x \in$ dom Ω. Consider the equation

$$Dx = a(x)xLx. \tag{5.17}$$

The case $a(x) \equiv e$ has been considered in Example 5.2, Similarly, as in that example, we obtain the equalities $DLx = x^{-1}Dx = a(x)Lx$, which implies $x = ELx = E\{R[a(x)Lx] + Lz'\} = E\{zR[a(x)Lx]\}$, where $z = Lz'$, $z' \in \ker\ D$ is arbitrary. Then, by Formulae (5.6), we find

$$Fx = FE\{zR[a(x)Lx]\} = EF\{zR[a(x)Lx]\} = E\{zFR[a(x)]Lx\} = E(0) = e.$$

We therefore conclude that a necessary condition for Equation (5.17) to have a solution x is that $Fx = e$. □

Theorem 5.1. *Suppose that $X \in L(D)$, $(L, E) \in G[\Omega]$, $\pm Ra \in \mathrm{dom}\,\Omega$ for an $a \in X$ and an $R \in \mathcal{R}_D$ and F is an almost averaging initial operator for D corresponding to R. For a given $m \in \mathbb{N}$ write*

$$v^{\pm} = z \mp (m+1)Ra, \tag{5.18}$$

whenever there exist $z \in \ker\ D$ such that

$$v^{\pm} \in I_{m+1}(X) \cap \mathrm{dom}\,\Omega. \tag{5.19}$$

If Condition (5.19) is satisfied then equations

$$Dx = ax^{\mp m} \tag{5.20}$$

have solutions of the form:

$$x^{\pm} = (v^{\pm})^{1/(m+1)} = E[\mp \frac{1}{m+1} Lv^{\pm}], \tag{5.21}$$

respectively (cf. footnote in Example 3.4.)

Proof. Let $m \in \mathbb{N}$. Consider the equation $Dx = ax^m$. By our assumptions, we have $Dx^{-(m+1)} = -(m+1)x^{-m}Dx = -(m+1)x^{-m}ax^m = -(m+1)a$, which implies $x^{-(m+1)} = v^+$, where v^+ is determined by Formula (5.18). This, and Condition (5.19) together imply that x^+ is a solution, we are looking for. A similar proof for $-m \in \mathbb{N}$. ■

Corollary 5.4. *Suppose that all assumptions of Theorem 5.1 and Condition (5.19) are satisfied. Then an initial condition*

$$Fx = x_0, \quad \textit{where } x_0 \in \ker\ D \textit{ is given}, \tag{5.22}$$

holds if and only if

$$x_0 = E(\mp \frac{1}{m+1} Lz) = z^{\mp 1/(m+1)}, \quad \textit{i.e.} \quad z = x_0^{\mp(m+1)} = E(\mp \frac{1}{m+1} Lx_0).$$

Proof. By our assumptions, Formulae (5.6) and Theorem 5.1, we have

$$Fv^{\pm} = F[z \pm (m+1)Ra] = Fz \pm (m+1)FRa = z,$$

$$x_0 = Fx^{\pm} = FE(\mp \frac{1}{m+1} Lv^{\pm}) = EF(\mp \frac{1}{m+1} Lv^{\pm}) =$$

$$= E(\mp \frac{1}{m+1} FLv^{\pm}) = E(\mp \frac{1}{m+1} LFv^{\pm}) = E(\mp \frac{1}{m+1} Lz) =$$

$$= ELz^{\mp 1/(m+)} = z^{\mp 1/(m+1)}, \quad \textit{i.e.} \quad x_0 = ELx_0 = E[\mp(m+1)Lz] = z^{\mp(m+1)}.$$

■

Corollary 5.2. *Suppose that all assumptions of Theorem 5.1 and Condition (5.19) are satisfied. Then solutions of the equations*

$$Dy = ay(Ly)^{\pm m} \tag{5.23}$$

are of the form:

$$y^{\pm} = Ex^{\pm}, \quad where\ x^{\pm}\ are\ defined\ by\ Formulae\ (5.21). \tag{5.24}$$

Proof. Let $m \in \mathbb{N}$. Let $x = Ly$. By (5.23), we have $y = ELy = Ex$, $Dx = DLy = y^{-1}Dy = a(Ly)^m = ax^m$. It means that x is a solution of Equation (5.21), hence it is of the form (5.21). We therefore conclude that a solution $y = Ex$ is of the form (5.24). A similar proof for $-m \in \mathbb{N}$. ■

Proposition 5.4. *Suppose that all assumptions of Proposition 5.3 are satisfied and $Rb \in \operatorname{dom} \Omega^{-1}$, where $b \in X$ is given. Then all solutions of the equation*

$$Dx = axLx + bx \tag{5.25}$$

are of the form:

$$x = Ew, \quad where\ w = (ERa)R[bE(-Ra)] + zERa,\ z \in \ker\ D\ is\ arbitrary. \tag{5.26}$$

Proof. Write: $w = Lx$. Then $x = Ew$ and, by (5.25), we have $Dw = x^{-1}Dx = aLx + b = aw + b$. This, and Proposition 5.3 together imply that w is defined by (5.26). ■

Propositions 5.3 and 5.4 together imply

Corollary 5.3. *Suppose that all assumptions of Proposition 5.3 are satisfied. Then the initial value problem for Equation (5.25) with the initial condition $Fx = x_0$ has a unique solution*

$$x = Ew_0, \quad where\ w_0 = (ERa)R[bE(-Ra)] + x_0ERa. \tag{5.27}$$

Theorem 5.2. *Suppose that $X \in \mathsf{L}(D)$, $(L, E) \in G[\Omega]$, $a \in \operatorname{dom} \Omega$, h is an invertible mapping of X into itself such that*

$$R[ah(x)] \subset \operatorname{dom} \Omega \cap \operatorname{dom} \Omega^{-1} \quad for\ an\ R \in \mathcal{R}_D\ whenever\ x \subset \operatorname{dom} D$$

and F is an initial operator for D corresponding to R. Then

(i) Equations

$$Dx = axh(x)(Lx)^{\pm n} \quad (n \in \mathbb{N}) \tag{5.28}$$

have solutions x if and only if the equations for $n \neq -1$

$$x = E^2\big\{ \mp \frac{1}{n+1} L\{\mp(n+1)R[ah(x)] + z\}\big\} \tag{5.29}$$

(respectively) have solutions for a $z \in \ker\ D$.

If $n = -1$ then the only solution of (5.28) is an $x \in \ker\ D$ (i.e. x is a constant). If it is the case then $Fx = x$, $FDx = 0$.

(iii) If x is a solution of Equation (5.29) for $n \neq -1$ Then the initial condition $Fx = x_0$ ($x_0 \in \ker\ D$) is satisfied if and only if

$$z = (Lx_0)^{\pm(n+1)} \quad \textit{provided that } x_0 \neq 0 \tag{5.30}$$

(i.e. there is no solutions such that $Fx = 0$.)

(iv) If for $n \neq -1$, Condition (5.30) is satisfied and F is multiplicative then FDx is not well-determined.

Proof. Write: $u = Lx$, $\tilde{h} = ahE$. Then $x = Eu$ and $\tilde{h}L = ahEL = ah$. This, Equation (5.28) and our assumptions together imply that for $\pm n$ ($n \in \mathbb{N}$)

$$Du^{\mp(n+1)} = \mp(n+1)u^{\mp n}Du = \mp Du = \mp(n+1)(Lx)^{\mp n}DLx =$$

$$= \mp(n+1)(Lx)^{\mp n}x^{-1}Dx = \mp(n+1)ah(x) = \mp(n+1)ahEu = \mp(n+1)\tilde{h}(u).$$

If $n \neq -1$ (i.e. $n + 1 \neq 0$) then

$$E(\mp(n+1)Lu) = u^{\mp(n+1)} = \mp(n+1)R\tilde{h}(u) + z, \quad \textit{where } z \in \ker\ D, \quad \textit{i.e.}$$

$$\mp(n+1)Lu = LE[\mp(n+1)Lu] = L[\mp(n+1)R\tilde{h}(u) + z].$$

This implies that

$$x = Eu = E^2Lu = E^2\{\mp\frac{1}{n+1}L[\mp(n+1)R\tilde{h}(u) + z]\} =$$

$$= E^2\big\{\{\mp\frac{1}{n+1}L[\mp(n+1)R[ahE(Lx)] + z\}\big\} =$$

$$= E^2\big\{ \mp \frac{1}{n+1}L\{\mp(n+1)R[ah(x)] + z]\}\big\} \quad \textit{where } z \in \ker\ D.$$

This means that Equation (5.29) should be satisfied for a $z \in \ker\ D$.

(ii) If $n = -1$ then we get $Du^{-1} = 0$, which implies that $u^{-1} = \tilde{z} \in \ker\ D$ and $x = ELx = Eu = E\tilde{z}^{-1} = z \in \ker\ D$, i.e. x is a constant. Hence $Fx = Fz = z = x$. Then $Dx = 0$. So that, $FDx = 0$.

(iii) Suppose that $n \neq -1$, $x^{\pm}$ satisfy Equations (5.29) and the initial value conditions

$$F^{\pm}x = x_0^{\pm}, \quad \textit{where } x_0^{\pm} \in \ker\ D \textit{ are given}, \tag{5.31}$$

respectively. This means that there are $z \in \ker\ D$ such that

$$x_0^{\pm} = Fx_0^{\pm} = FE^2\big\{\mp \frac{1}{n+1}FL \mp (n+1)R[ah(x)] + z\}\big\} =$$

$$= E^2\big\{\mp \frac{1}{n+1}FL\{\mp(n+1)R[ah(x)] + z\}\big\} =$$

$$= E^2\big\{\mp \frac{1}{n+1}L\{\mp(n+1)FR[ah(x)] + Fz\}\big\} =$$

$$= E^2[\mp \frac{1}{n+1}Lz] = E^2Lz^{\mp\frac{1}{n+1}} = Ez^{\mp\frac{1}{n+1}},$$

and $z = (Lx_0)^{\mp(n+1)}$.

(iv) If (iii) is satisfied and F is multiplicative then

$$FDx = F[ah(x)(Lx)^n] = (Fa)\cdot 0\cdot [Fh(x)](LFx)^n.$$

However, $LFx = L(0)$ is not well-determined, so FDx does. ■

Example 5.4. Suppose that all assumptions of Theorem 5.2 are satisfied. Then $x \in \mathrm{dom}\ D$ is a solution of the equation

$$Dx = axh(x) \tag{5.32}$$

if and only if $x \in \mathrm{dom}\ D$ is a solution of the equation

$$x = zER[ah(x)] \quad \textit{for a } z \in \ker\ D. \tag{5.33}$$

indeed, we have $DLx = x^{-1}Dx = axh(x)$. Hence $x = ELx = E\{R[ah(x)] + Lz\} = (ELz)ER[ah(x)] = zER[ah(x)]$, where $z \in \ker\ D$. If F is an almost averaging initial operator for D corresponding to R then the initial value condition $Fx = x_0$ ($x_0 \in \ker\ D$ is given) implies that

$$x_0 = F\{zER[ah(x)]\} = zFER[ah(x)] = zEFR[ah(x)] = zE(0) = z\cdot e = z,$$

i.e. x should satisfy the equation

$$x = x_0ER[ah(x)]. \tag{5.34}$$

□

Example 5.5. Suppose that all assumptions of Theorem 5.2 are satisfied, $g = Re \in I(X)$, $a = g^{-1}$, $Rg^{-1} \in \text{dom}\ \Omega$ and $h(x) = x(x-e)$. Observe that

$$x^{-1} + (x-e)^{-1} = 2x^{-1}(x-e)^{-1} = 2[x(x-e)]^{-1}, \quad \textit{whenever}\ x, x-e \in I(X).$$

Similarly, as before, we conclude that the equation

$$Dx = 2g^{-1}x(x-e) \tag{5.35}$$

has a solution $x \in \text{dom}\ D$ if and only $x \in \text{dom}\ D$ satisfies the equation

$$x(x-e) = zg^2 \quad for\ a\ z \in \ker\ D. \tag{5.36}$$

If it is the case and F is multiplicative, then the initial condition $Fx = x_0$ ($x_0 \in \ker\ D$ is given) leads to the equality $x_0(x_0 - e) = zFg^2 = z(Fg)^2 = 0$. This implies that either $x_0 = 0$ or $x_0 = e$ *⁾. If $Fx = e$ then we find $x - e = RD(x-e) = RDx$, i.e.

$$x = RDx + e = [2g^{-1}x(x-e)] + e = R[2g^{-1}zg^2] + e + 2zRg = 2zR^2e = zg^2.$$

We therefore conclude that $e = Fx = F(zg^2) = z(Fg)^2 = 0$, which is a contradiction. Hence $Fx \neq e$. So that $Fx = 0$. By similar arguments, as before, we find that FDx is not well defined, i.e. there is no $z \in \ker\ D$ such that $FDx = z$. Hence Equation (5.36) has no solutions. □

Example 5.6. Suppose that all assumptions of Theorem 5.2 are satisfied, $g = Re \in I(X)$, $a = g^{-1}$ and F is an almost averaging initial operator for D corresponding to R. Then the equation

$$Dx = g^{-1}xh(x)Lx \tag{5.37}$$

has a solution $x \in \text{dom}\ D$ if and only if the equation

$$x = E\{zER[g^{-1}h(x)]\} \tag{5.38}$$

has a solution $x \in \text{dom}\ D$ for a $z \in \ker\ D$. Indeed, write: $u = Lx$. Then $x = Eu$ and we have $Du = DLx = x^{-1}Dx = g^{-1}h(x)Lx = g^{-1}h(Eu)u$. Hence $DLu = u^{-1}Du = g^{-1}h(Eu)$, which implies that for $z \in \ker\ D$

$$x = E^2L^2x = E^2L\{zER[g^{-1}h(x)]\} = E\{zER[g^{-1}h(x)]\}.$$

If x is a solution of (5.38) satisfying the initial condition $Fx = x_0$, where $x_0 \in \ker\ D$, then

$$\begin{aligned} x_0 = FxFE\{zER[g^{-1}h(x)]\} &= EF\{zER[g^{-1}h(x)]\} = \\ = E\{zEFR[g^{-1}h(x)]\} &= E[zE(0)] = E(z \cdot e) = Ez. \end{aligned}$$

Let $x_0 = 0$. Since there is no $z \in \ker\ D$ such that $Ez = x_0 = 0$, we conclude that Fx is not well-determined, hence there is no solution belonging to dom D. Similarly, as in previous examples, we can show that in this case also FDx is not well-determined. □

*⁾ Note that in Leibniz algebras constants are not zero divisors (cf. PR[1]).

REFERENCES.

L.Euler

E[1]*Nova methodus innumerablis aequationes differetiales secundi gradus reducendi ad aequationes differentiales primi gradus.* Commentarii Acaemiae Scientiarum Imperialis Petropolitanae 3(1728), 124-137; ed.1732 (cf. also Acta Imper. Acad. Sci. Petersburg, 1932)

D. Przeworska-Rolewicz

PR[1] *Algebraic Analysis.* PWN-Polish Scientific Publishers and D. Reidel, Warszawa-Dordrecht, 1988.

PR[2] A priori determined solutions of linear equations. Math. Japonica, 2, 44(1996), 395-412.

PR[3] *Logarithms and Antilogarithms. An Algebraic Analysis Approach.* With Appendix by Z. Binderman. Kluwer Academic Publishers, Dordrecht, 1998.

PR[4] Postmodern Logarithmo-technia. International Journal of Computers and Mathematics with Applications. 41(2001), 1143-54.

PR[5] Algebraic analysis in structures with Kaplansky-Jacobson property. Studia Math., 168(2)(2005), 165-185.

PR[6] Some summations formulae in commutative Leibniz algebras with logarithms. Control and Cybernetics, 3, 36(2007), 841-857.

D. Przeworska-Rolewicz and H. von Trotha

PRvT[1] Right inverses in D-R algebras with unit. J. Integral Equations, 3(1981), 245-259.

J. S. Ritt

R[1] *Differential Equations from the Algebraic Stand Point. American Math. Society, New York, 1932.*

R[2] On the singular solutions of algebraic differential equations. Annals of Math., 37(1936).

H. Triebel

Tr[1] *Analysis and Mathematical Physics.* D. Riedel, Dordrecht, 1985.

H. von Trotha

vT[1] *Structure properties of D-R spaces.* Dissertationes Math., 180, Warszawa, 1981.

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NONLINEAR SEPARABLE EQUATIONS IN LINEAR SPACES AND COMMUTATIVE LEIBNIZ ALGEBRAS

Abstract. There are considered nonlinear equations in linear spaces and algebras which can be solved by a "separation of variables" obtained due to Algebraic Analysis. It is shown that the structures of linear spaces and commutative algebras (even if they are Leibniz algebras) are not rich enough for our purposes. Therefore, in order to generalize the method used for separable ordinary differential equations, we have to admit that in algebras under consideration there exist logarithmic mappings. Section 1 contains some basic notions and results of Algebraic Analysis. In Section 2 there are considered equations in linear spaces. Section 3 contains results for commutative Leibniz algebras. In Section 4 basic notions and facts about logarithmic and antilogarithmic mappings are collected. Section 5 is devoted to separable nonlinear equations in commutative Leibniz algebras with logarithms.