INSTITUTE OF MATHEMATICS
of the
Polish Academy of Sciences

ul. Śniadeckich 8, P.O.B. 21, 00-956 Warszawa 10, Poland   http://www.impan.pl

IM PAN Preprint 702 (2009)

Krzysztof Szajowski

On a Random Number of Disorders

Published as manuscript

Received 10 February 2009

On a random number of disorders

Krzysztof Szajowski

IM PAS, Warszawa

Abstract. We register a random sequence constructed based on Markov processes by switching between them. At two random moments θ_1, θ_2, where $0 \leqslant \theta_1 \leqslant \theta_2$, the source of observations is changed. In effect the number of homogeneous segments is random. The transition probabilities of each process are known and *a priori* distribution of the disorder moments is given. The various questions are formulated concerning the distribution changes in the model in the former research. The random number of distributional segments creates new problems in solutions with relation to analysis of the model with deterministic number of segments. Two cases are presented in details. In the first one the objectives is to stop on or between the disorder moments while in the second one our objective is to find the strategy which immediately detects the distribution changes. Both problems are reformulated to optimal stopping of the observed sequences. The detailed analysis of the problem is presented to show the form of optimal decision function.

2000 AMS Mathematics Subject Classification: Primary: 60G40 60K99; Secondary: 90D60.

Key words and phrases: disorder problem, sequential detection, optimal stopping, Markov process, change point, double optimal stopping

1. INTRODUCTION

Suppose that process $X = \{X_n, n \in \mathbb{N}\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, is observed sequentially. The process is obtained from three Markov processes by switching between them at two

* Affiliation: Institute of Mathematics, Polish Academy of Science, Śniadeckich 8, 00-956 Warszawa, Poland

random moments of time, θ_1 and θ_2. Our objective is to detect these moments based on observation of X.

Such model of data appears in many practical problem of the quality control (see [5], [2], [12]), traffic anomalies in networks [6], epidemiology [1]. In management of manufacture the plants which produce some details changes their parameters. It makes that the details change their quality. Production can be divided into three sorts. Assuming that at the beginning of production process the quality is highest, from some moment θ_1 the products should be classified to lower sort and beginning with θ_2 the details should be categorized as at lowest quality. The aim is to recognize the moments of these changes.

Shiryaev [13] has considered the disorder problem for independent random variables with one disorder where the mean distance between disorder time and the moment of its detection was minimized. The probability maximizing approach to the problem was used by [3] and the stopping time which is in a given neighborhood of the moment of disorder with maximal probability was found. The problem with two disorders was considered by Yoshida [17], the author [14, 15] and Sarnowski and the author [11]. In [17] the problem of optimal stopping the observation of process X so as to maximize the probability that the distance between θ_i, $i = 1, 2$, and the moment of disorder will not exceed a given number (for each disorder independently). This question has been reformulated in [15] to simultaneous detection of both disorders under requirement that performance of procedure is globally measured for both detection and it has been extended to the case with unknown distribution between disorders (see [4]) in [11]. The methods of solution is based on reformulation of the question to the double optimal stopping problem (see [7], [9]) for markovian function of some statistics. In [14] the strategy which stops the process between the first and the second disorder with maximal probability has been constructed. The considerations are inspired by the problem regarding how can we protect ourselves against a second fault in a technological system after the occurrence of an initial fault or by the problem of detection the beginning and the end of an epidemic.

This paper is devoted to a generalization of the double disorder problem considered both in [14] and [15] in which immediate switch from the first preliminary distribution to the third one is possible with the positive probability that the random variables θ_1 and θ_2 are equal. It is also possible that we observe the homogeneous data without disorder when both disorder moments are equal to 0. The extension leads to serious difficulties in the construction of equivalent double optimal stopping models. The formulation of the problem can be found in section 2. The main results are subject of sections 4 (see Theorem 4.1) and 5.

2. FORMULATION OF DETECTION PROBLEMS

Let $(X_n, n \in \mathbb{N})$ be an observable sequence of random variables defined on the space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in $(\mathbf{E}, \mathcal{B})$, where $\mathbf{E}$ is a subset of $\mathbf{R}$. On $(\mathbf{E}, \mathcal{B})$ there are σ-additive measures $\{\mu_x\}_{x\in\mathbf{E}}$. Space $(\Omega, \mathcal{F}, \mathbf{P})$ supports variables θ_1, θ_2. They are $\mathcal{F}$-measurable variables with values in $\mathbb{N}$. We assume the following distributions:

$$\mathbf{P}(\theta_1 = j) = \mathbb{I}_{\{j=0\}}(j)\pi + \mathbb{I}_{\{j>0\}}(j)(1-\pi)p_1^{j-1}q_1, \tag{2.1}$$

$$\mathbf{P}(\theta_2 = k \mid \theta_1 = j) = \mathbb{I}_{\{k=j\}}(k)\rho + \mathbb{I}_{\{k>j\}}(k)(1-\rho)p_2^{k-j-1}q_2 \tag{2.2}$$

where $j = 0, 1, 2, ..., k = j, j+1, j+2, ...$. Additionally we consider Markov processes $(X_n^i, \mathcal{G}_n^i, \mu_x^i)$ on $(\Omega, \mathcal{F}, \mathbf{P})$, $i = 0, 1, 2$, where σ-fields $\mathcal{G}_n^i$ are the smallest σ-fields for which (X^i), $i = 0, 1, 2$, are adapted, respectively. Let us define process $(X_n, n \in \mathbb{N})$ in the following way:

$$X_n = X_n^0 \cdot \mathbb{I}_{\{\theta_1 > n\}} + X_n^1 \cdot \mathbb{I}_{\{\theta_1 \leqslant n < \theta_2\}} + X_n^2 \cdot \mathbb{I}_{\{\theta_2 \leqslant n\}}. \tag{2.3}$$

We make inference based on the observable sequence $(X_n,\, n \in \mathbb{N})$ only. It should be emphasized that the sequence $(X_n,\, n \in \mathbb{N})$ is not markovian under admitted assumption as it has been mentioned in [14], [16] and [6]. However, the sequence satisfies the Markov property given θ_1 and θ_2 (see [15] and [8]). Thus for further consideration we define filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, where $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$, related to real observation. Variables θ_1, θ_2 are not stopping times with respect to $\mathcal{F}_n$ and σ-fields $\mathcal{G}_n^{\bullet}$. Moreover, we assume that θ_1, θ_2 are independent of $(X^i_n,\, n \in \mathbb{N})$. Measures $\mu^\bullet_x$ satisfy the relations: $\mu^i_x(dy) = f^i_x(y)\mu_x(dy)$, $i = 0,1,2$, where the functions $f^i_x(.)$ are different and $f^i_x(y)/f^{(i+1)\mathrm{mod}3}_x(y) < \infty$ for $i = 0,1,2$ and all $x,y \in \mathbf{E}$. We assume that the measures μ^i_x, $x \in \mathbf{E}$ are known in advance and we have that $\mathbf{P}(X^i_1 \in A \mid X^i_0 = x) = \int_A f^i_x(y)\mu_x(dy) = \mu^i_x(A)$ for every $A \in \mathcal{B}$ and $i \in \{0,1,2\}$.

The model presented has the following heuristic justification: two disorders take place in the observed sequence (X_n). They affect distributions by changing their parameters. Disorders occur at two random moments of time θ_1 and θ_2, $\theta_1 \leqslant \theta_2$. They split the sequence of observations into segments, at most three ones. The first segment is described by (X^0_n), the second one - for $\theta_1 \leqslant n < \theta_2$ - by (X^1_n). The third is given by (X^2_n) and is observed when $n \geqslant \theta_2$. When the first disorder takes the place there is a "switch" from the initial distribution to distribution with density f^i_x with respect of measure μ_x, where $i = 1$ or $i = 2$. It depends on if $\theta_1 < \theta_2$ or $\theta_1 = \theta_2$. Next, if $\theta_1 < \theta_2$, at the random time θ_2 the distribution of observations becomes μ^2_x. We assume that the variables θ_1, θ_2 are unobservable.

Let $\mathcal{S}$ denote the set of all stopping times with respect to the filtration $(\mathcal{F}_n)$, $n = 0,1,\ldots$ and $\mathcal{T} = \{(\tau,\sigma) : \tau \leqslant \sigma,\ \tau,\sigma \in \mathcal{S}\}$. Two problems with three distributional segments are recalled to investigate them under weaker assumption that there are at most three homogeneous segments.

2.1. Detection of change. Our aim is to stop the observed sequence between the two disorders.This can be interpreted as a strategy for protecting against a second failure when the first has already happened. The mathematical model of this is to control the probability $\mathbf{P}_x(\tau < \infty, \theta_1 \leqslant \tau < \theta_2)$ by choosing the stopping time $\tau^* \in \mathcal{S}$ for which

$$\mathbf{P}_x(\theta_1 \leqslant \tau^* < \theta_2) = \sup_{\tau \in \mathcal{T}} \mathbf{P}_x(\tau < \infty, \theta_1 \leqslant \tau < \theta_2). \tag{2.4}$$

2.2. Disorders detection. Our aim is to indicate the moments of switching with given precision d_1, d_2 (Problem $\mathrm{D}_{d_1 d_2}$). We want to determine a pair of stopping times $(\tau^*, \sigma^*) \in \mathcal{T}$ such that for every $x \in \mathbb{E}$

$$\mathbf{P}_x(|\tau^* - \theta_1| \leqslant d_1, |\sigma^* - \theta_2| \leqslant d_2) = \sup_{\substack{(\tau,\sigma)\in\mathcal{T}\\ 0\leqslant\tau\leqslant\sigma<\infty}} \mathbf{P}_x(|\tau - \theta_1| \leqslant d_1, |\sigma - \theta_2| \leqslant d_2). \tag{2.5}$$

The problem has been considered in [15] under natural simplification that there are three segments of data (*i.e.* there is $0 < \theta_1 < \theta_2$). In the section 5 the problem D_{00} is analyzed.

3. ON SOME *A POSTERIORI* PROCESSES

The formulated problems will be translated to the optimal stopping problems for some Markov processes. The important part of the reformulation process is choice of the *statistics* describing knowledge of the decision maker. The *a posteriori* probabilities of some events play the crucial role. Let us define following *a posteriori* processes (cf. [17], [14]).

$$\Pi_n^i = \mathbf{P}_x(\theta_i \leqslant n|\mathcal{F}_n), \tag{3.1}$$

$$\Pi_n^{12} = \mathbf{P}_x(\theta_1 = \theta_2 > n|\mathcal{F}_n) = P_x(\theta_1 = \theta_2 > n|\mathcal{F}_{mn}), \tag{3.2}$$

$$\Pi_{mn} = \mathbf{P}_x(\theta_1 = m, \theta_2 > n|\mathcal{F}_{mn}), \tag{3.3}$$

for $m, n = 1, 2, \ldots$, $m < n$, $i = 1, 2$. For recursive representation of (3.1)–(3.3) we need following functions:

$$\begin{aligned}
\Pi^1(x,y,\alpha,\beta,\gamma) &= 1 - \frac{p_1(1-\alpha)f_x^0(y)}{\mathbf{H}(x,y,\alpha,\beta,\gamma)}\\
\Pi^2(x,y,\alpha,\beta,\gamma) &= \frac{(q_2\alpha + p_2\beta + q_1\gamma)f_x^2(y)}{\mathbf{H}(x,y,\alpha,\beta,\gamma)}\\
\Pi^{12}(x,y,\alpha,\beta,\gamma) &= \frac{p_1\gamma f_x^0(y)}{\mathbf{H}(x,y,\alpha,\beta,\gamma)}\\
\Pi(x,y,\alpha,\beta,\gamma,\delta) &= \frac{p_2\delta f_x^1(y)}{\mathbf{H}(x,y,\alpha,\beta,\gamma)}
\end{aligned}$$

where $\mathbf{H}(x,y,\alpha,\beta,\gamma) = (1-\alpha)p_1 f_x^0(y) + [p_2(\alpha-\beta) + q_1(1-\alpha-\gamma)]f_x^1(y) + [q_2\alpha + p_2\beta + q_1\gamma]f_x^2(y)$. In the sequel we adopt the following denotations

$$\vec{\alpha} = (\alpha, \beta, \gamma) \tag{3.4}$$

$$\overrightarrow{\Pi}_n = (\Pi_n^1, \Pi_n^2, \Pi_n^{12}). \tag{3.5}$$

The basic formulae used in the transformation of the disorder problems to the stopping problems are given in the following

LEMMA 3.1. *For each $x \in \mathbb{E}$ and each Borel function $u : \Re \to \Re$ the following formulae for $m, n = 1, 2, \ldots, m < n$ hold:*

$$\Pi^i_{n+1} = \Pi^i(X_n, X_{n+1}, \Pi^1_n, \Pi^2_n, \Pi^{12}_n) \tag{3.6}$$

$$\Pi^2_{n+1} = \Pi^2(X_n, X_{n+1}, \Pi^1_n, \Pi^2_n, \Pi^{12}_n) \tag{3.7}$$

$$\Pi^{12}_{n+1} = \Pi^{12}(X_n, X_{n+1}, \Pi^1_n, \Pi^2_n, \Pi^{12}_n) \tag{3.8}$$

$$\Pi_{m\,n+1} = \Pi(X_n, X_{n+1}, \Pi^1_n, \Pi^2_n, \Pi^{12}_n, \Pi_{m\,n}) \tag{3.9}$$

with boundary condition $\Pi^1_0 = \pi$, $\Pi^2_0(x) = \pi\rho$, $\Pi_{m\,m} = (1-\rho)\frac{q_1 f^1_{X_{m-1}}(X_m)}{p_1 f^0_{X_{m-1}}(X_m)}(1 - \Pi^1_m)$.

PROOF. The case of (3.6), (3.7) and (3.9), when $0 < \theta_1 < \theta_2$, has been proved in [17] and [14]. Let us assume $0 \leqslant \theta_1 \leqslant \theta_2$ and suppose that $A_i \in \mathcal{F}_i$, $i \leqslant n+1$. Denote $D = \{\omega : X_0 = x, X_i(\omega) \in A_i, 1 \leqslant i \leqslant n\}$.

Ad. (3.6) Let us consider the probability

$$\mathbf{P}_x(\theta_1 > n+1 \mid X_i \in A_i, i \leqslant n+1) = \frac{\mathbf{P}_x(\theta_1 > n+1, X_{n+1} \in A_{n+1} \mid D)}{\mathbf{P}_x(X_{n+1} \in A_{n+1} \mid D)}$$

This follows from Bayes' formula. Let us transform the probability appearing in the numerator:

$$\begin{aligned}\mathbf{P}_x(\theta_1 >\ & n+1, X_{n+1} \in A_{n+1} \mid X_i \in A_i, i \leqslant n) \\ &= \mathbf{P}_x\left(\theta_1 > n \mid X_i \in A_i, i \leqslant n\right) \cdot p_1 \cdot \mu^1_{X_n}(A_{n+1})\end{aligned}$$

Now we split the probability in the denominator into the following parts

$$\mathbf{P}_x(X_{n+1} \in A_{n+1} \mid D) = \mathbf{P}_x(\theta_2 > \theta_1 > n, X_{n+1} \in A_{n+1} \mid D) \tag{3.10}$$

$$+ \mathbf{P}_x(\theta_1 \leqslant n < \theta_2, X_{n+1} \in A_{n+1} \mid D) \tag{3.11}$$

$$+ \mathbf{P}_x(n < \theta_1 = \theta_2, X_{n+1} \in A_{n+1} \mid D) \tag{3.12}$$

$$+ \mathbf{P}_x(\theta_1 \leqslant \theta_2 \leqslant n, X_{n+1} \in A_{n+1} \mid D) \tag{3.13}$$

In (3.10) we have:

$$
\begin{aligned}
\mathbf{P}_x(\theta_1 > n \ , \ \ X_{n+1} \in A_{n+1} \mid D)& \\
&= \mathbf{P}_x(\theta_1 > n, \theta_1 = n+1, X_{n+1} \in A_{n+1} \mid D) \\
&\quad + \mathbf{P}_x(\theta_1 > n, \theta_1 \neq n+1, X_{n+1} \in A_{n+1} \mid D) \\
&= \mathbf{P}_x(\theta_1 > n \mid D)[\mu^0_{X_n}(A_{n+1})p_1 + q_1\mu^1_{X_n}(A_{n+1})]
\end{aligned}
$$

In (3.11) we get:

$$
\begin{aligned}
\mathbf{P}_x(\theta_1 \leqslant n < \theta_2 \ , \ \ X_{n+1} \in A_{n+1} \mid D)& \\
&= \mathbf{P}_x(\theta_1 \leqslant n < \theta_2, \theta_2 = n+1, X_{n+1} \in A_{n+1} \mid D) \\
&\quad + \mathbf{P}_x(\theta_1 \leqslant n < \theta_2, \theta_2 \neq n+1, X_{n+1} \in A_{n+1} \mid D) \\
&= (\mathbf{P}_x(\theta_1 \leqslant n \mid D) - \mathbf{P}_x(\theta_2 \leqslant n \mid D)) \\
&\quad \times [q_2\mu^2_{X_n}(A_{n+1}) + p_2\mu^1_{X_n}(A_{n+1})]
\end{aligned}
$$

In (3.13) the conditional probability is equal

$$
\begin{aligned}
\mathbf{P}_x(\theta_1 = \theta_2 > n \ , \ \ X_{n+1} \in A_{n+1} \mid D)& \\
&= \mathbf{P}_x(\theta_1 = \theta_2 > n, \theta_2 = n+1, X_{n+1} \in A_{n+1} \mid D) \\
&\quad + \mathbf{P}_x(\theta_1 = \theta_2 > n, \theta_2 \neq n+1, X_{n+1} \in A_{n+1} \mid D) \\
&= \mathbf{P}_x(\theta_1 = \theta_2 > n \mid D)[q_1\mu^2_{X_n}(A_{n+1}) + p_1\mu^0_{X_n}(A_{n+1})]
\end{aligned}
$$

In (3.12) this part has form:

$$
\mathbf{P}_x(\theta_2 \leqslant n, X_{n+1} \in A_{n+1} \mid D) = \mathbf{P}_x(\theta_2 \leqslant n \mid D)\mu^2_{X_n}(A_{n+1})
$$

Thus, taking into account (3.1) we have $\Pi^1_{n+1} = 1 - P_x\left(\theta_1 > n+1 \mid \mathcal{F}_{n+1}\right)$ and by (3.10)-(3.13) we get

$$
\Pi^1_{n+1} = 1 - \left[(1-\Pi^1_n)p_1\right]\mathbf{H}^{-1}(X_n, X_{n+1}, \overrightarrow{\Pi}_n)
$$

Using (3.1), it can be seen that (3.6) is satisfied.

Ad. (3.7) Applying similar reasoning and transformations to the process Π^2_{n+1} we get:

$$\begin{aligned}\Pi^2_{n+1} &= \mathbf{P}_x(\theta_2 \leqslant n+1 \mid \mathcal{F}_{n+1}) = \frac{\mathbf{P}_x(\theta_2 \leqslant n+1, X_{n+1} \in A \mid \mathcal{F}_n)}{\mathbf{P}_x(X_{n+1} \in A \mid \mathcal{F}_n)} \\ &= \left[(\Pi^1_n - \Pi^2_n)q_2 + \Pi^2_n\right] f^2_{X_n}(X_{n+1})\mathbf{H}^{-1}(X_n, X_{n+1}, \overrightarrow{\Pi}_n)\end{aligned}$$

which leads to formula (3.7).

✠

REMARK 3.1. *Let us assume that the considered Markov processes have the finite state space and* $\vec{x}_n = (x_0, x_1, \ldots, x_n)$ *is given. In this case the formula (3.9) follows from the Bayes formula:*

$$\mathbf{P}_x(\theta_1 = j, \theta_2 = k|\mathcal{F}_n) = \begin{cases} p^\theta_{jk} \prod_{s=1}^n f^0_{x_{s-1}}(x_s) S_n^{-1}(\vec{x}_n) & \text{if } j > n, \\ p^\theta_{jk} \prod_{s=1}^{j-1} f^0_{x_{s-1}}(x_s) & \\ \quad \times \prod_{t=j}^n f^1_{x_{t-1}}(x_t)(S_n^{-1}(\vec{x}_n))^{-1} & \text{if } j \leqslant n < k, \\ p^\theta_{jk} \prod_{s=1}^n f^0_{x_{s-1}}(x_s) \prod_{t=j}^{k-1} f^1_{x_{t-1}}(x_t) & \\ \quad \times \prod_{u=k}^n f^2_{x_{u-1}}(x_u) S_n^{-1}(\vec{x}_n) & \text{if } k \leqslant n, \end{cases}$$

where $p^\theta_{jk} = \mathbf{P}(\theta_1 = j, \theta_2 = k)$, $S_0(x_0) = 1$ *and for* $n \geqslant 1$

$$\begin{aligned} S_n(\vec{x}_n) &= (1-\pi)(1-\rho)\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\{p_1^{j-1}q_1p_2^{k-j-1}q_2\prod_{s=1}^{j-1}f^0_{x_{s-1}}(x_s)\prod_{t=j}^{k-1}f^1_{x_{t-1}}(x_t) \\ &\quad \times \prod_{u=k}^{n} f^2_{x_{u-1}}(x_u)\} + (1-\pi)\rho\sum_{j=1}^{n}\{p_1^{j-1}q_1\prod_{s=1}^{j-1}f^0_{x_{s-1}}(x_s)\prod_{t=j}^{n}f^2_{x_{t-1}}(x_t)\} \\ &\quad +(1-\pi)(1-\rho)\sum_{j=1}^{n}\{p_1^{j-1}q_1p_2^{n-j}\prod_{s=1}^{j-1}f^0_{x_{s-1}}(x_s)\prod_{t=j}^{n}f^1_{x_{t-1}}(x_t)\} \\ &\quad +(1-\pi)p_1^n\prod_{s=1}^{n}f^0_{x_{s-1}}(x_s). \end{aligned}$$

Moreover

$$\Pi_{m\,n+1}(x) = p_2 f^2_{X_n}(X_{n+1})\Pi_{m\,n}(x) S_n(\vec{x}_{n+1}) S^{-1}_{n+1}(\vec{x}_n)$$

and $S_{n+1}(\vec{x}_{n+1}) = \mathbf{H}(X_n, X_{n+1}, \overrightarrow{\Pi}_n)S_n(\vec{x}_n)$. *Hence*

$$\Pi_{m\,n+1}(x) = \frac{p_2 f^1_{X_n}(X_{n+1})\Pi_{m\,n}(x)}{\mathbf{H}(X_n, X_{n+1}, \overrightarrow{\Pi}_n)}.$$

LEMMA 3.2. *For each* $x \in \mathbf{E}$ *and each Borel function* $u : \mathbf{R} \longrightarrow \mathbf{R}$ *the following equations are fulfilled*

$$\mathbf{E}_x\left(u(X_{n+1})(1 - \Pi^1_{n+1}) \mid \mathcal{F}_n\right) = (1 - \Pi^1_n - \Pi^{12}_n)p_1 \int_{\mathbb{E}} u(y) f^0_{X_n}(y)\mu_{X_n}(dy), \tag{3.14}$$

$$\begin{aligned} &\mathbf{E}_x\left(u(X_{n+1})(\Pi^1_{n+1} - \Pi^2_{n+1}) \mid \mathcal{F}_n\right) \\ &\quad = \left[q_1(1 - \Pi^1_n - \Pi^{12}_n) + p_2(\Pi^1_n - \Pi^2_n)\right] \int_{\mathbb{E}} u(x) f^1_{X_n}(y)\mu_{X_n}(dy), \end{aligned} \tag{3.15}$$

$$\mathbf{E}_x\left(u(X_{n+1})\Pi^2_{n+1}) \mid \mathcal{F}_n\right) = \left[q_2\Pi^1_n + p_2\Pi^2_n + q_1\Pi^{12}_n\right] \int_{\mathbb{E}} u(y) f^2_{X_n}(y)\mu_{X_n}(dy), \tag{3.16}$$

$$\mathbf{E}_x\left(u(X_{n+1})\Pi^{12}_{n+1}) \mid \mathcal{F}_n\right) = \left[p_1\Pi^{12}_n\right] \int_{\mathbb{E}} u(y) f^0_{X_n}(y)\mu_{X_n}(dy) \tag{3.17}$$

$$\mathbf{E}_x(u(X_{n+1})|\mathcal{F}_n) = \int_{\mathbb{E}} u(y)\mathbf{H}(X_n, y, \overrightarrow{\Pi}_n(x))\mu_{X_n}(dy). \tag{3.18}$$

PROOF. The relations (3.14)-(3.17) are consequence of suitable division of Ω defined by (θ_1, θ_2) and properties established in Lemma 6.2. Let us prove the equation (3.16). To do this we need to define first σ-field $\widetilde{\mathcal{F}}_n = \sigma(\theta_1, \theta_2, X_0, ..., X_n)$. Notice that $\mathcal{F}_n \subset \widetilde{\mathcal{F}}_n$. We have:

$$\begin{aligned} &\mathbf{E}_x(u(X_{n+1})\Pi^2_{n+1} \mid \mathcal{F}_n) = \mathbf{E}_x(u(X_{n+1})\mathbf{E}_x(\mathbb{I}_{\{\theta_2 \leqslant n+1\}} \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n) \\ &\quad = \mathbf{E}_x(u(X_{n+1})\mathbb{I}_{\{\theta_2 \leqslant n+1\}} \mid \mathcal{F}_n) = \mathbf{E}_x(\mathbf{E}_x(u(X_{n+1})\mathbb{I}_{\{\theta_2 \leqslant n+1\}} \mid \widetilde{\mathcal{F}}_n) \mid \mathcal{F}_n) \\ &\quad = \mathbf{E}_x\left(\int_{\mathbb{E}} u(y)\mathbf{P}_x(dy \mid \widetilde{\mathcal{F}}_n, \theta_2 \leqslant n+1)\mathbf{P}_x(\theta_2 \leqslant n+1 \mid \widetilde{\mathcal{F}}_n) \mid \mathcal{F}_n\right) \\ &\quad = \int_{\mathbb{E}} u(y) f^2_{X_n}(y)\mu_{X_n}(dy)\mathbf{P}_x(\theta_2 \leqslant n+1 \mid \mathcal{F}_n) \\ &\quad = \int_{\mathbb{E}} u(y) f^2_{X_n}(y)\mu_{X_n}(dy)(\mathbf{P}_x(\theta_2 = n+1, \theta_1 \leqslant n < \theta_2 \mid \mathcal{F}_n) + \mathbf{P}_x(\theta_2 \leqslant n|\mathcal{F}_n)) \\ &\quad \stackrel{L.6.2}{=} \left(q_2\Pi^1_n + p_2\Pi^2_n + q_1\Pi^{12}_n\right) \int_{\mathbb{E}} u(y) f^2_{X_n}(y)\mu_{X_n}(dy) \end{aligned}$$

We used the properties of conditional expectation here. Similar transformations give us equations (3.14), (3.17) and (3.15). The sum of (3.14)-(3.17) gives (3.18). This proves Lemma 3.2.

✠

4. DETECTION OF NEW HOMOGENEOUS SEGMENT

4.1. Equivalent optimal stopping problem. For $X_0 = x$ let us define: $Z_n = \mathbf{P}_x(\theta_1 \leqslant n < \theta_2 \mid \mathcal{F}_n)$ for $n = 0, 1, 2, \ldots$. We have

$$Z_n = \mathbf{P}_x(\theta_1 \leqslant n < \theta_2 \mid \mathcal{F}_n) = \Pi_n^1 - \Pi_n^2 \tag{4.1}$$

$Y_n = \text{esssup}_{\{\tau \in \mathcal{T},\, \tau \geqslant \mathrm{n}\}} \mathbf{P}_{\mathrm{x}}(\theta_1 \leqslant \tau < \theta_2 \mid \mathcal{F}_{\mathrm{n}})$ for $n = 0, 1, 2, \ldots$ and

$$\tau_0 \quad = \quad \inf\{n : Z_n = Y_n\} \tag{4.2}$$

Notice that, if $Z_\infty = 0$, then $Z_\tau = \mathbf{P}_x(\theta_1 \leqslant \tau < \theta_2 \mid \mathcal{F}_\tau)$ for $\tau \in \mathcal{T}$. Since $\mathcal{F}_n \subseteq \mathcal{F}_\tau$ (when $n \leqslant \tau$) we have

$$\begin{aligned} Y_n &= \operatorname*{ess\,sup}_{\tau \geqslant \mathrm{n}} \mathbf{P}_{\mathrm{x}}(\theta_1 \leqslant \tau < \theta_2 \mid \mathcal{F}_{\mathrm{n}}) = \operatorname*{ess\,sup}_{\tau \geqslant \mathrm{n}} \mathrm{E}_{\mathrm{x}}(\mathbb{I}_{\{\theta_1 \leqslant \tau < \theta_2\}} \mid \mathcal{F}_{\mathrm{n}}) \\ &= \operatorname*{ess\,sup}_{\tau \geqslant \mathrm{n}} \mathrm{E}_{\mathrm{x}}(\mathrm{Z}_\tau \mid \mathcal{F}_{\mathrm{n}}) \end{aligned}$$

LEMMA 4.1. *The stopping time τ_0 defined by formula (4.2) is the solution of problem (2.4).*

PROOF. From the theorems presented in [3] it is enough to show that $\lim_{n \to \infty} Z_n = 0$. For all natural numbers n, k, where $n \geqslant k$ for each $x \in \mathbf{E}$ we have:

$$Z_n \quad = \quad E_x(\mathbb{I}_{\{\theta_1 \leqslant n < \theta_2\}} \mid \mathcal{F}_n) \leqslant E_x(\sup_{j \geqslant k} \mathbb{I}_{\{\theta_1 \leqslant j < \theta_2\}} \mid \mathcal{F}_n)$$

From Levy's theorem $\limsup_{n \to \infty} Z_n \leqslant E_x(\sup_{j \geqslant k} \mathbb{I}_{\{\theta_1 \leqslant j < \theta_2\}} \mid \mathcal{F}_\infty)$ where $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^\infty \mathcal{F}_n\right)$. It is true that: $\limsup_{j \geqslant k,\, k \to \infty} \mathbb{I}_{\{\theta_1 \leqslant j < \theta_2\}} = 0$ *a.s.* and by the dominated convergence theorem we get

$$\lim_{k \to \infty} E_x(\sup_{j \geqslant k} \mathbb{I}_{\{\theta_1 \leqslant j < \theta_2\}} \mid \mathcal{F}_\infty) = 0 \;\; a.s.$$

what ends the proof of the Lemma.

✠

The reduction of the disorder problem to optimal stopping of Markov sequence is the consequence of the following lemma.

LEMMA 4.2. *System* $X^x = \{X_n^x\}$*, where* $X_n^x = (X_{n-1}, X_n, \Pi_n^1, \Pi_n^2, \Pi_n^{12})$ *forms a family of random Markov functions.*

PROOF. Define a function:

$$\varphi(x_1, x_2, \vec{\alpha}\,; z) \quad = (x_2, z, \Pi^1(x_2, z, \vec{\alpha}), \Pi^2(x_2, z, \vec{\alpha}), \Pi^{12}(x_2, z, \vec{\alpha})) \tag{4.3}$$

Observe that

$$X_n^x = \varphi(X_{n-2}, X_{n-1}, \overrightarrow{\Pi}_{n-1}; X_n) = \varphi(X_{n-1}^x; X_n)$$

Hence X_n^x can be interpreted as function of previous state X_{n-1}^x and random variable X_n. Moreover, applying (3.18), we get that conditional distribution of X_n given σ-field $\mathcal{F}_{n-1}$ depends only on X_{n-1}^x. According to [13] (pp. 102-103) system X^x is a family of random Markov functions.

✠

This fact implies that we can reduce initial problem (2.4) to the problem of optimal stopping five-dimensional process $(X_{n-1}, X_n, \Pi_n^1, \Pi_n^2, \Pi_n^{12})$ with reward

$$h(x_1, x_2, \vec{\alpha}) = \alpha - \beta \tag{4.4}$$

The reward function results from equation (4.1). Thanks to Lemma 4.2 we construct the solution using standard tools of optimal stopping theory (cf [13]), as we do below.

Let us define two operators for any Borel function $v : \mathbf{E}^2 \times [0,1]^3 \longrightarrow [0,1]$ and the set $D = \{\omega : X_{n-1} = y, X_n = z, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_n^{12} = \gamma\}$:

$$\begin{aligned} T_x v(y, z, \vec{\alpha}) &= E_x(v(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1}) \mid D) \\ \mathbf{Q}_x v(y, z, \vec{\alpha}) &= \max\{v(y, z, \vec{\alpha}), \mathbf{T}_x v(y, z, \vec{\alpha})\} \end{aligned}$$

From well known theorems of optimal stopping theory ([13]), we infer that the solution of the problem (2.4) is the Markov time τ_0:

$$\tau_0 = \inf\{n \geqslant 0 : h(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1}) \geqslant h^*(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1})\}$$

where:

$$h^*(y, z, \vec{\alpha}) = \lim_{k\to\infty} \mathbf{Q}_x^k h(y, z, \vec{\alpha})$$

Of course

$$\mathbf{Q}_x^k v(y, z, \vec{\alpha}) = \max\{\mathbf{Q}_x^{k-1} v, \mathbf{T}_x \mathbf{Q}_x^{k-1} v\} = \max\{v, \mathbf{T}_x \mathbf{Q}_x^{k-1} v\}$$

To obtain a clearer formula for τ_0, we formulate (cf (3.5) and (3.4)):

THEOREM 4.1. **(a)** *The solution of problem (2.4) is given by:*

$$\tau^* = \inf\{n \geqslant 0 : (X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1}) \in B^*\} \tag{4.5}$$

Set B^* is of the form:

$$\begin{aligned} B^* &= \{(y, z, \vec{\alpha}) : (\alpha - \beta) \geqslant (1 - \alpha) \\ &\times \left[p_1 \int_{\mathbb{E}} R^*(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})) f_y^0(u) \mu_y(du) \right. \\ &+ \left. q_1 \int_{\mathbb{E}} S^*(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})) f_y^1(u) \mu_y(du) \right] \\ &+ \left. (\alpha - \beta) p_2 \int_{\mathbb{E}} S^*(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})) f_y^1(u) \mu_y(du) \right\} \end{aligned}$$

Where:

$$R^*(y, z, \vec{\alpha}) = \lim_{k\to\infty} R^k(y, z, \vec{\alpha}) \; , \quad S^*(y, z, \vec{\alpha}) = \lim_{k\to\infty} S^k(y, z, \vec{\alpha})$$

Functions R^k and S^k are defined recursively: $R^1(y,z,\vec{\alpha})=0$, $S^1(y,z,\vec{\alpha})=1$ and

$$
\begin{aligned}
R^{k+1}(y,z,\vec{\alpha}) &= (1-\mathbb{I}_{\mathcal{R}_k}(y,z,\vec{\alpha})) \\
&\times\left(p_1\int_{\mathbb{E}} R^k(y,u,\overrightarrow{\Pi}_1(y,u,\vec{\alpha}))f_y^0(u)\mu_y(du)\right. \\
&\left.+q_1\int_{\mathbb{E}} S^k(y,u,\overrightarrow{\Pi}_1(y,u,\vec{\alpha}))f_y^1(u)\mu_y(du)\right),
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
S^{k+1}(y,z,\vec{\alpha}) &= \mathbb{I}_{\mathcal{R}_k}(y,z,\vec{\alpha})+(1-\mathbb{I}_{\mathcal{R}_k}(y,z,\vec{\alpha})) \\
&\times p_2\int_{\mathbb{E}} S^k(y,u,\overrightarrow{\Pi}_1(y,u,\vec{\alpha}))f_y^1(u)\mu_y(du)
\end{aligned}
\tag{4.7}
$$

Where the set $\mathcal{R}_k$ is:

$$
\begin{aligned}
\mathcal{R}_k &= \left\{(y,z,\vec{\alpha}) : h(y,z,\vec{\alpha}) \geqslant \mathbf{T}_x\mathbf{Q}_x^{k-1}h(y,z,\vec{\alpha})\right\} \\
&= \{(y,z,\vec{\alpha}) : (\alpha-\beta) \geqslant (1-\alpha) \\
&\times\left[p_1\int_{\mathbb{E}} R^k(y,u,\overrightarrow{\Pi}_1(y,u,\vec{\alpha}))f_y^0(u)\mu_y(du)\right. \\
&\left.+\, q_1\int_{\mathbb{E}} S^k(y,u,\overrightarrow{\Pi}_1(y,u,\vec{\alpha}))f_y^1(u)\mu_y(du)\right] \\
&\left.+\,(\alpha-\beta)p_2\int_{\mathbb{E}} S^k(y,u,\overrightarrow{\Pi}^1(y,u,\vec{\alpha}))f_y^1(u)\mu_y(du)\right\}
\end{aligned}
$$

(b) *The value problem. The optimal value for (2.4) is given by the formula*

$$
\begin{aligned}
V(\tau^*) &= p_1\int_{\mathbb{E}} R^*(x,u,\overrightarrow{\Pi}_1(x,u,\pi,\rho\pi,\rho(1-\pi)))f_x^0(u)\mu_x(du) \\
&+q_1\int_{\mathbb{E}} S^*(x,u,\overrightarrow{\Pi}_1(x,u,\pi,\rho\pi,\rho(1-\pi)))f_x^1(u)\mu_x(du).
\end{aligned}
$$

PROOF. Part (a) results from Lemma 3.2 - the problem reduces to the problem of optimal stopping of the Markov process $(X_{n-1},X_n,\Pi_n^1,\Pi_n^2,\Pi_n^{12})$ with payoff function $h(y,z,\vec{\alpha})=\alpha-\beta$. Given (3.15) with the function u equal to unity we get on

$D = \{\omega : X_{n-1} = y, X_n = z, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_n^{12} = \gamma\}$:

$$\begin{aligned}
&\mathbf{T}_x h(y,z,\vec{\alpha}) = \mathbf{E}_x\left(\Pi_{n+1}^1 - \Pi_{n+1}^2 \mid \mathcal{F}_n\right)|_D \\
= &\left[(\Pi_n^1 - \Pi_n^2)p_2 \int_{\mathbb{E}} f_{X_n}^1(u)\mu_{X_n}(du) + (1-\Pi_n^1)q_1 \int_{\mathbb{E}} f_{X_n}^0(u)\mu_{X_n}(du)\right]|_D \\
= &(1-\alpha)q_1 + (\alpha-\beta)p_2
\end{aligned}$$

From the definition of R^1 and S^1 it is clear that

$$h(y,z,\vec{\alpha}) = \alpha - \beta = (1-\alpha)R^1(y,z,\vec{\alpha}) + (\alpha-\beta)S^1(y,z,\vec{\alpha})$$

Also $\mathcal{R}_1 = \{(y,z,\vec{\alpha}) : h(y,z,\vec{\alpha}) \geqslant \mathbf{T}_x h(y,z,\vec{\alpha})\}$. From the definition of $\mathbf{Q}_x$ and the facts above we obtain

$$\mathbf{Q}_x h(y,z,\vec{\alpha}) \quad = \quad (1-\alpha)R^2(y,z,\vec{\alpha}) + (\alpha-\beta)S^2(y,z,\vec{\alpha})$$

where $R^2(y,z,\vec{\alpha}) = q_1(1 - \mathbb{I}_{\mathcal{R}_1}(y,z,\vec{\alpha}))$ and $S^2(y,z,\vec{\alpha}) = p_2 + (1-p_2)\mathbb{I}_{\mathcal{R}_1}(y,z,\vec{\alpha}))$. Suppose the following induction hypothesis holds

$$\mathbf{Q}_x^{k-1} h(y,z,\vec{\alpha}) = (1-\alpha)R^k(y,z,\vec{\alpha}) + (\alpha-\beta)S^k(y,z,\vec{\alpha})$$

where R^k and S^k are given by equations (4.6), (4.7), respectively. We will show

$$\mathbf{Q}_x^k h(y,z,\vec{\alpha}) = (1-\alpha)R^{k+1}(y,z,\vec{\alpha}) + (\alpha-\beta)S^{k+1}(y,z,\vec{\alpha})$$

From the induction assumption and equations (3.14), (3.15) we obtain:

$$\begin{aligned}
\mathbf{T}_x b Q_x^{k-1} h(y,z,\vec{\alpha}) &= \mathbf{T}_x(1-\alpha)R^k(y,z,\vec{\alpha}) + \mathbf{T}_x(\alpha-\beta)S^k(y,z,\vec{\alpha}) \\
&= (1-\alpha)p_1 \int_{\mathbb{E}} R^k(y,u,\overrightarrow{\Pi}_1(y,u,\vec{\alpha}))f_y^0(u)\mu_y(du) \\
&\quad + [(1-\alpha)q_1 + (\alpha-\beta)p_2] \int_{\mathbb{E}} S^k(y,u,\overrightarrow{\Pi}_1(y,u,\vec{\alpha}))f_y^1(u)\mu_y(du) \\
&= (1-\alpha)\left[p_1 \int_{\mathbb{E}} R^k(y,u,\overrightarrow{\Pi}_1(y,u,\vec{\alpha}))f_y^0(u)\mu_y(du) + q_1 \int_{\mathbb{E}} S^k(y,u,\overrightarrow{\Pi}_1(y,u,\vec{\alpha}))\right. \\
&\times \quad \left. f_y^1(u)\mu_y(du)\right] + (\alpha-\beta)p_2 \int_{\mathbb{E}} S^k(y,u,\overrightarrow{\Pi}_1(y,u,\vec{\alpha}))f_y^1(u)\mu_y(du)
\end{aligned}$$

Notice that

$$(1-\alpha)R^{k+1}(y,z,\vec{\alpha})+(\alpha-\beta)S^{k+1}(y,z,\vec{\alpha})$$

is equal $\alpha-\beta = h(y,z,\vec{\alpha}) = \mathbf{Q}_x^k h(y,z,\vec{\alpha})$ for $(y,z,\vec{\alpha})\in\mathcal{R}_k$ and, taking into account (4.8), it is equal $\mathbf{T}_x\mathbf{Q}_x^{k-1}h(y,z,\vec{\alpha}) = \mathbf{Q}_x^k h(y,z,\vec{\alpha})$ for $(y,z,\vec{\alpha})\notin\mathcal{R}_k$, where $\mathcal{R}_k$ is given by (4.8). Finally we get

$$\mathbf{Q}_x^k h(y,z,\vec{\alpha}) = (1-\alpha)R^{k+1}(y,z,\vec{\alpha})+(\alpha-\beta)S^{k+1}(y,z,\vec{\alpha})$$

This proves (4.6) and (4.7). Using the monotone convergence theorem and theorems of optimal stopping theory ([13]) we conclude that the optimal stopping time τ^* is given by (cf 4.5). ✠

PROOF. Part (b). First, notice that Π_1^1, Π_1^2 and Π_1^{12} are given by (3.6)-(3.8) and the boundary condition formulated in Lemma 3.1. Under the assumption $\tau^*<\infty$ a.s. we get:

$$\begin{aligned}
&\mathbf{P}_x(\tau^*<\infty,\theta_1\leqslant\tau^*<\theta_2) = \sup_\tau \mathbf{E}Z_\tau \\
&\quad= \mathbf{E}\max\{h(x,X_1,\overrightarrow{\Pi}_1),\mathbf{T}_x h^*(x,X_1,\overrightarrow{\Pi}_1)\} = \mathbf{E}\lim_{k\to\infty}\mathbf{Q}_x^k h(x,X_1,\overrightarrow{\Pi}_1) \\
&\quad= \mathbf{E}\left[(1-\Pi_1^1)R^*(x,X_1,\overrightarrow{\Pi}_1)+(\Pi_1^1-\Pi_1^2)S^*(x,X_1,\overrightarrow{\Pi}_1)\right] \\
&\quad= p_1\int_{\mathbb{E}} R^*(x,u,\overrightarrow{\Pi}_1(x,u,\pi,\rho\pi,\rho(1-\pi)))f_x^0(u)\mu_x(du) \\
&+q_1\int_{\mathbb{E}} S^*(x,u,\overrightarrow{\Pi}_1(x,u,\pi,\rho\pi,\rho(1-\pi)))f_x^1(u)\mu_x(du)
\end{aligned}$$

We used Lemma 3.2 here and simple calculations for Π_1^1, Π_1^2 and Π_1^{12}. This ends the proof. ✠

4.2. Remarks. It is notable that the solution of formulated problem depends only on two-dimensional vector of posterior processes because $\Pi_n^{12} = \rho(1-\Pi_n^1)$. The formulas obtained are very general and for this reason - quite complicated. We simplify the model by assuming that $P(\theta_1>0)=1$ and $P(\theta_2>\theta_1)=1$. However, it seems that some further simplifications can be made in special cases. Further research should be carried out in this direction. From a practical point of view, computer algorithms are necessary to construct B^* – the set in which it is optimally to stop our observable sequence.

5. IMMEDIATE DETECTION OF THE FIRST AND THE SECOND DISORDER

5.1. Equivalent double optimal stopping problem. Let us consider the problem D_{00} formulated in (2.5). A *compound stopping variable* is a pair (τ, σ) of stopping times such that $0 \leqslant \tau \leqslant \sigma$ a.e.. The aim is to find a compund stopping variable $(\tau^\star, \sigma^\star)$ such that

$$\mathbf{P}_x((\theta_1, \theta_2) = (\tau^*, \sigma^*)) = \sup_{\substack{(\tau,\sigma)\in\mathcal{T} \\ 0\leqslant\tau\leqslant\sigma<\infty}} \mathbf{P}_x((\theta_1, \theta_2) = (\tau, \sigma)). \tag{5.1}$$

Denote $\mathcal{T}_m = \{(\tau, \sigma) \in \mathcal{T} : \tau \geqslant m\}$, $\mathcal{T}_{mn} = \{(\tau, \sigma) \in \mathcal{T} : \tau = m, \sigma \geqslant n\}$ and $\mathcal{S}_m = \{\tau \in \mathcal{S} : \tau \geqslant m\}$. Let us denote $\mathcal{F}_{mn} = \mathcal{F}_n$, $m, n \in \mathbb{N}$, $m \leqslant n$. We define two-parameter stochastic sequence $\xi(x) = \{\xi_{mn},\ m, n \in \mathbb{N},\ m < n,\ x \in \mathbb{E}\}$, where

$$\xi_{mn} = \mathbf{P}_x(\theta_1 = m, \theta_2 = n|\mathcal{F}_{mn}).$$

We can consider for every $x \in \mathbb{E}$, $m, n \in \mathbb{N}$, $m < n$, the optimal stopping problem of $\xi(x)$ on $\mathcal{T}^+_{mn} = \{(\tau, \sigma) \in \mathcal{T}_{mn} : \tau < \sigma\}$. A compound stopping variable (τ^*, σ^*) is said to be optimal in $\mathcal{T}^+_m$ (or $\mathcal{T}^+_{mn}$) if

$$\mathbf{E}_x\xi_{\tau^*\sigma^*} = \sup_{(\tau,\sigma)\in\mathcal{T}_m} \mathbf{E}_x\xi_{\tau\sigma} \tag{5.2}$$

(or $\mathbf{E}_x\xi_{\tau^*\sigma^*} = \sup_{(\tau,\sigma)\in\mathcal{T}^+_{mn}} \mathbf{E}_x\xi_{\tau\sigma}$). Let us define

$$\eta_{mn} = \operatorname*{ess\,sup}_{(\tau,\sigma)\in\mathcal{T}^+_{mn}} \mathbf{E}_x(\xi_{\tau\sigma}|\mathcal{F}_{mn}). \tag{5.3}$$

If we put $\xi_{m\infty} = 0$, then

$$\eta_{mn} = \operatorname*{ess\,sup}_{(\tau,\sigma)\in\mathcal{T}^+_{mn}} \mathbf{P}_x(\theta_1 = \tau, \theta_2 = \sigma|\mathcal{F}_{mn}).$$

From the theory of optimal stopping for double indexed processes (cf. [7],[10]) the sequence η_{mn} satisfies

$$\eta_{mn} = \max\{\xi_{mn}, \mathbf{E}(\eta_{mn+1}|\mathcal{F}_{mn})\}.$$

Moreover, if $\sigma^*_m = \inf\{n > m : \eta_{mn} = \xi_{mn}\}$, then (m, σ^*_n) is optimal in $\mathcal{T}^+_{mn}$ and $\eta_{mn} = \mathbf{E}_x(\xi_{m\sigma^*_n}|\mathcal{F}_{mn})$ a.e.. The case when there are no segment with distribution $f^1_x(y)$

appears with probability ρ. It will be taken into account. Define

$$\hat{\eta}_{mn} = \max\{\xi_{mn}, \mathbf{E}(\eta_{m\,n+1}|\mathcal{F}_{mn})\} \text{ for } n \geqslant m.$$

if $\hat{\sigma}^*_m = \inf\{n \geqslant m : \hat{\eta}_{mn} = \xi_{mn}\}$, then $(m, \hat{\sigma}^*_m)$ is optimal in $\mathcal{T}_{mn}$ and $\hat{\eta}_{mm} = \mathbf{E}_x(\xi_{m\sigma^*_m}|\mathcal{F}_{mm})$ a.e.. For further consideration denote

$$\eta_m = \mathbf{E}_x(\eta_{mm+1}|\mathcal{F}_m). \tag{5.4}$$

LEMMA 5.1. *Stopping time σ^*_m is optimal for every stopping problem (5.3).*

PROOF. It suffices to prove $\lim_{n\to\infty} \xi_{mn} = 0$ (cf. [3]). We have for $m, n, k \in \mathbb{N}$, $n \geqslant k > m$ and every $x \in \mathbb{E}$

$$\mathbf{E}_x(\mathbb{I}_{\{\theta_1=m,\theta_2=n\}}|\mathcal{F}_{mn}) = \xi_{mn}(x) \leqslant \mathbf{E}_x(\sup_{j\geqslant k} \mathbb{I}_{\{\theta_1=m,\theta_2=j\}}|\mathcal{F}_m),$$

where $\mathbb{I}_A$ is the characteristic function of the set A. By Levy's theorem

$$\limsup_{n\to\infty} \xi_{mn}(x) \leqslant \mathbf{E}_x(\sup_{j\geqslant k} \mathbb{I}_{\{\theta_1=m,\theta_2=j\}}|\mathcal{F}_{n\infty}),$$

where $\mathcal{F}_\infty = \mathcal{F}_{n\infty} = \sigma(\bigcup_{n=1}^\infty \mathcal{F}_n)$. We have $\lim_{k\to\infty} \sup_{j\geqslant k} \mathbb{I}_{\{\theta_1=m,\theta_2=j\}} = 0$ a.e. and by dominated convergence theorem

$$\lim_{k\to\infty} \mathbf{E}_x(\sup_{j\geqslant k} \mathbb{I}_{\{\theta_1=m,\theta_2=j\}}|\mathcal{F}_\infty) = 0.$$

✠

What is left is to consider the optimal stopping problem for $(\eta_{mn})_{m=0,n=m}^{\infty,\infty}$ on $(\mathcal{T}_{mn})_{m=0,n=m}^{\infty,\infty}$. Let us define

$$V_m = \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_m} \mathbf{E}_x(\eta_\tau|\mathcal{F}_m). \tag{5.5}$$

Then $V_m = \max\{\eta_m, \mathbf{E}_x(V_{m+1}|\mathcal{F}_m)\}$ a.e. and we define $\tau^*_n = \inf\{k \geqslant n : V_k = \eta_k\}$.

LEMMA 5.2. *The strategy τ^*_0 is the optimal strategy of the first stop.*

PROOF. To show that τ_0^* is the optimal first stop strategy we prove that $\mathbf{P}_x(\tau_0^* < \infty) = 1$. To this end, we argue in the usual manner i.e. we show $\lim_{m\to\infty}\eta_m = 0$.

We have

$$\begin{aligned}
\eta_m &= \mathbf{E}_x(\xi_{m\sigma_m^*}|\mathcal{F}_m) \\
&= \mathbf{E}_x(\mathbf{E}_x(\mathbb{I}_{\{\theta_1=m,\theta_2=\sigma_m^*\}}|\mathcal{F}_{m\sigma_m^*})|\mathcal{F}_m) \\
&= \mathbf{E}_x(\mathbb{I}_{\{\theta_1=m,\theta_2=\sigma_m^*\}}|\mathcal{F}_m) \\
&\leqslant \mathbf{E}_x(\sup_{j\geqslant k}\mathbb{I}_{\{\theta_1=j,\theta_2=\sigma_j^*\}}|\mathcal{F}_m).
\end{aligned}$$

Similarly as in proof of Lemma 5.1 we have got

$$\limsup_{m\to\infty}\eta_m(x) \leqslant \mathbf{E}_x(\sup_{j\geqslant k}\mathbb{I}_{\{\theta_1=j,\theta_2=\sigma_j^*\}}|\mathcal{F}_\infty).$$

Since

$$\lim_{k\to\infty}\sup_{j\geqslant k}\mathbb{I}_{\{\theta_1=k,\theta_2=\sigma_j^*\}} \leqslant \limsup_{k\to\infty}\mathbb{I}_{\{\theta_1=k\}} = 0,$$

it follows that

$$\lim_{m\to\infty}\eta_m(x) \leqslant \lim_{k\to\infty}\mathbf{E}_x(\sup_{j\geqslant k}\mathbb{I}_{\{\theta_1=j,\theta_2=\sigma_j^*\}}|\mathcal{F}_\infty) = 0.$$

✠

Lemmas 5.1 and 5.2 describe the method of solving the "disorder problem" formulated in Section 2 (see (5.1)).

5.2. Solution of the equivalent double stopping problem. For the sake of simplicity we shall confine ourselves to the case $d_1 = d_2 = 0$. It will be easily seen how to generalize the solution of the problem to solve $\mathrm{D}_{d_1 d_2}$ for $d_1 > 0$ or $d_2 > 0$. First of all we construct multidimensional Markov chains such that ξ_{mn} and η_m will be the functions of their states. By consideration of the section 3 concerning *a posteriori* processes we get $\xi_{00} = \pi\rho$ and for $m < n$

$$\begin{aligned}
\xi^x_{m\,n} &= \mathbf{P}_x(\theta_1 = m, \theta_2 = n|\mathcal{F}_{m\,n}) \\
&= (1-\pi)(1-\rho)\frac{p_1^{m-1}q_1p_2^{n-m-1}q_2\prod_{s=1}^{j-1}f^0_{x_{s-1}}(x_s)\prod_{t=j}^{n-1}f^1_{x_{t-1}}(x_t)f^2_{X_{n-1}}(X_n)}{S_n(x_0,x_1,\ldots,x_n)} \\
&= \frac{q_2}{p_2}\Pi_{m\,n}(x)\frac{f^2_{X_{n-1}}(X_n)}{f^1_{X_{n-1}}(X_n)}
\end{aligned}$$

and for $n = m$, by Lemma 6.2

$$\xi^x_{m\,m} = \mathbf{P}_x(\theta_1 = m, \theta_2 = m|\mathcal{F}_{m\,m}) = \rho\frac{q_1}{p_1}\frac{f^2_{X_{m-1}}(X_m)}{f^0_{X_{m-1}}(X_m)}(1-\Pi^1_m). \tag{5.6}$$

We can observe that $(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1}, \Pi_{m\,n+1})$ for $n = m+1, m+2, \ldots$ is a function of $(X_{n-1}, X_n, \overrightarrow{\Pi}_n, \Pi_{m\,n})$ and X_{n+1}. Besides the conditional distribution of X_{n+1} given $\mathcal{F}_n$ (cf. (3.18)) depends on X_n, $\Pi^1_n(x)$ and $\Pi^2_n(x)$ only. These facts imply that $\{(X_n, X_{n+1}, \overrightarrow{\Pi}_{n+1}, \Pi_{m\,n+1})\}_{n=m+1}^{\infty}$ form a homogeneous Markov process (see Chapter 2.15 of [13]). This allows us to reduce the problem (5.3) for each m to the optimal stopping problem of the Markov process $Z_m(x) = \{(X_{n-1}, X_n, \overrightarrow{\Pi}_n, \Pi_{m\,n}),\ m, n \in \mathbb{N},\ m < n,\ x \in \mathbb{E}\}$ with the reward function $h(t, u, \vec{\alpha}, \delta) = \frac{q_2}{p_2}\delta\frac{f^2_t(u)}{f^1_t(u)}$.

LEMMA 5.3. *A solution of the optimal stopping problem (5.3) for $m = 1, 2, \ldots$ has a form*

$$\sigma^*_m = \inf\{n > m : \frac{f^2_{X_{n-1}}(X_n)}{f^1_{X_{n-1}}(X_n)} \geqslant R^*(X_n)\} \tag{5.7}$$

where $R^*(t) = p_2\int_{\mathbb{E}} r^*(t,s)f^1_t(s)\mu_t(ds)$. The function $r^* = \lim_{n\to\infty} r_n$, where $r_0(t,u) = \frac{f^2_t(u)}{f^1_t(u)}$,

$$r_{n+1}(t,u) = \max\{\frac{f^2_t(u)}{f^1_t(u)}, p_2\int_{\mathbb{E}} r_n(u,s)f^1_u(s)\mu_u(ds)\}. \tag{5.8}$$

So $r^*(t,u)$ satisfies the equation

$$r^*(t,u) = \max\{\frac{f^2_t(u)}{f^1_t(u)}, p_2\int_{\mathbb{E}} r^*(u,s)f^1_u(s)\mu_u(ds)\}. \tag{5.9}$$

The value of the problem

$$\eta_m = \mathbf{E}_x(\eta_{m\,m+1}|\mathcal{F}_m) = \frac{q_1}{p_1}\frac{f^1_{X_{m-1}}(X_m)}{f^0_{X_{m-1}}(X_m)}(1-\Pi^1_m)R^\star_\rho(X_{m-1},X_m), \tag{5.10}$$

where

$$R^\star_\rho(t,u) = \max\{\rho\frac{f^2_t(u)}{f^1_t(u)}, \frac{q_2}{p_2}(1-\rho)R^\star(u)\}. \tag{5.11}$$

PROOF. For any Borel function $u:\mathbb{E}\times\mathbb{E}\times[0,1]^4\longrightarrow[0,1]$ and $D=\{\omega: X_{n-1}=t, X_n=u, \Pi^1_n(x)=\alpha, \Pi^2_n(x)=\beta, \Pi^{12}_n=\gamma, \Pi_{m\,n}(x)=\delta\}$ let us define two operators

$$\mathbf{T}_x u(t,u,\vec{\alpha},\delta) = \mathbf{E}_x(u(X_n,X_{n+1},\overrightarrow{\Pi}_{n+1}(x),\Pi_{m\,n+1}(x))|D)$$

and

$$\mathbf{Q}_x u(t,u,\vec{\alpha},\delta) = \max\{u(t,u,\vec{\alpha},\delta), \mathbf{T}_x u(t,u,\vec{\alpha},\delta)\}.$$

On the bases of the well-known theorem from the theory of optimal stopping (see [13], [10]) we conclude that the solution of (5.3) is a Markov time

$$\sigma^*_m = \inf\{n>m: h(X_{n-1},X_n,\overrightarrow{\Pi}_n,\Pi_{m\,n}) = h^*(X_{n-1},X_n,\overrightarrow{\Pi}_n(x),\Pi_{m\,n})\},$$

where $h^* = \lim_{k\to\infty}\mathbf{Q}^k_x h(t,u,\vec{\alpha},\delta)$. By (3.9) and (3.18) on $D=\{\omega: X_{n-1}=t, X_n=u, \Pi^1_n=\alpha, \Pi^2_n=\beta, \Pi^{12}_n=\gamma, \Pi_{m\,n}=\delta\}$ we have

$$\begin{aligned}
\mathbf{T}_x h(t,u,\vec{\alpha},\delta) &= \mathbf{E}_x(\frac{q_2}{p_2}\Pi_{m\,n+1}\frac{f^2_{X_n}(X_{n+1})}{f^1_{X_n}(X_{n+1})}|D)\\
&= \frac{q_2}{p_2}\delta p_2\mathbf{E}(\frac{f^1_u(X_{n+1})}{H(u,X_{n+1},\vec{\alpha})}\frac{f^2_u(X_{n+1})}{f^1_u(X_{n+1})}|\mathcal{F}_n)|_D\\
&\overset{(3.18)}{=} q_2\delta\int_{\mathbb{E}}\frac{f^2_u(s)}{H(u,s,\vec{\alpha})}H(u,s,\vec{\alpha})\mu_u(ds) = q_2\delta
\end{aligned}$$

and

$$\mathbf{Q}_x h(t,u,\vec{\alpha},\delta) = \frac{q_2}{p_2}\delta\max\{\frac{f^2_t(u)}{f^1_t(u)}, p_2\}. \tag{5.12}$$

Let us define $r_0(t,u)=1$ and

$$r_{n+1}(t,u)=\max\{\frac{f_t^2(u)}{f_t^1(u)},p_2\int_{\mathbb{E}} r_n(u,s)f_u^1(s)\mu_u(ds)\}.$$

We show that

$$\mathbf{Q}_x^\ell h(t,u,\vec{\alpha},\delta)=\frac{q_2}{p_2}\delta r_\ell(t,u) \tag{5.13}$$

for $\ell=1,2,\ldots$. We have by (5.12) $\mathbf{Q}_x h=\frac{q_2}{p_2}\gamma r_1$ and assume (5.13) for $\ell\leqslant k$. By (3.18) on $D=\{\omega: X_{n-1}=t, X_n=u, \Pi_n^1=\alpha,\Pi_n^2=\beta,\Pi_n^{12}=\gamma,\Pi_{mn}=\delta\}$ we have got

$$\begin{aligned}\mathbf{T}_x\mathbf{Q}_x^k h(t,u,\vec{\alpha},\delta) &= \mathbf{E}_x(\frac{q_2}{p_2}\Pi_{m\,k+1}r_k(X_n,X_{n+1})|D)\\ &= \frac{q_2}{p_2}\delta p_2\int_{\mathbb{E}} r_k(u,s)f_u^1(s)\mu_u(ds).\end{aligned}$$

It is easy to show (see [13]) that

$$\mathbf{Q}_x^{k+1}h=\max\{h,\mathbf{T}_x\mathbf{Q}_x^k h\},\ \text{for } k=1,2,\ldots.$$

Hence we have got $\mathbf{Q}_x^{k+1}h=\frac{q_2}{p_2}\delta r_{k+1}$ and (5.13) is proved for $\ell=1,2,\ldots$. This gives

$$h^*(t,u,\vec{\alpha},\delta)=\frac{q_2}{p_2}\delta\lim_{k\to\infty}r_k(t,u)=\frac{q_2}{p_2}\delta r^*(t,u) \tag{5.14}$$

and

$$\eta_{m\,n}=\operatorname*{ess\,sup}_{(\tau,\sigma)\in\mathcal{T}_{m\,n}}\mathbf{E}_x(\xi_{\tau,\sigma}|\mathcal{F}_{m\,n})=h^*(X_{n-1},X_n,\overrightarrow{\Pi}_n,\Pi_{m\,n}).$$

We have by (5.14) and (3.9)

$$\mathbf{T}_x h^*(t,u,\vec{\alpha},\delta)=\frac{q_2}{p_2}\delta p_2\int_{\mathbb{E}} r^*(u,s)f_u^1(s)\mu_u(ds)=\frac{q_2}{p_2}\delta R^*(u)$$

and σ_m^* has form (5.7). By (5.4), (5.6) and (3.18) we obtain

$$\begin{aligned}\eta_m &= \max\{\xi_{mm}^x,\mathbf{E}(\eta_{m\,m+1}|\mathcal{F}_m)\}=f(X_{m-1},X_m,\overrightarrow{\Pi}_m,\Pi_{mm})\\ &= \max\{\rho\frac{q_1}{p_1}\frac{f_{X_{m-1}}^2(X_m)}{f_{X_{m-1}}^0(X_m)}(1-\Pi_m^1),\frac{q_2}{p_2}(1-\Pi_{mm})R^\star(X_m)\}\\ &\overset{L.3.1}{=} \frac{q_1}{p_1}\frac{f_{X_{m-1}}^1(X_m)}{f_{X_{m-1}}^0(X_m)}(1-\Pi_m^1)R_\rho^\star(X_{m-1},X_m).\end{aligned} \tag{5.15}$$

✠

REMARK 5.1. *Based on the results of Lemma 5.3 and properties of the* a posteriori *process* Π_{nm} *we have that the expected value of success for the second stop when the observer stops immediately at* $n=0$ *is* $\pi\rho$ *and when at least one observation has been made* $\mathbf{E}(\eta_1|\mathcal{F}_0)=\frac{q_1}{p_1}\mathbf{E}((1-\Pi_1^1)\frac{f_x^1(X_1)}{f_x^0(X_1)}R_\rho^\star(x,X_1)|\mathcal{F}_0)=\frac{q_1}{p_1}(1-\pi)p_1\int_{\mathbb{E}}f_x^1(u)R_\rho^\star(x,u)\mu_x(du)$. *As a consequence we have optimal second moment*

$$\hat{\sigma}_0^\star=\begin{cases}0 & \text{if } \pi\rho\geqslant q_1(1-\pi)\int\limits_{\mathbb{E}}f_x^1(u)R_\rho^\star(x,u)\mu_x(du),\\ \sigma_0^\star & \text{otherwise.}\end{cases}$$

By lemmas 5.3 and 3.1 (formula (3.9)) the optimal stopping problem (5.5) has been transformed to the optimal stopping problem for the homogeneous Markov process

$$W=\{(X_{m-1},X_m,\overrightarrow{\Pi}_m,\Pi_m^{12}),\ m\in\mathbb{N},\ x\in\mathbb{E}\}$$

with the reward function

$$f(t,u,\vec{\alpha})=\frac{q_1}{p_1}\frac{f_t^1(u)}{f_t^0(u)}(1-\alpha)R_\rho^\star(t,u). \tag{5.16}$$

THEOREM 5.1. *A solution of the optimal stopping problem* (5.5) *for* $n=1,2,\ldots$ *has a form*

$$\tau_n^*=\inf\{k\geqslant n:(X_{k-1},X_k,\overrightarrow{\Pi}_k,)\in B^*\} \tag{5.17}$$

where $B^*=\{(t,u,\vec{\alpha}):\frac{f_t^2(u)}{f_t^1(u)}R_\rho^\star(t,u)\geqslant p_1\int_{\mathbb{E}}v^*(u,s)f_u^0(s)\mu_u(ds)\}$. *The function* $v^*(t,u)=\lim_{n\to\infty}v_n(t,u)$*, where* $v_0(t,u)=R_\rho^\star(t,u)$*,*

$$v_{n+1}(t,u)=\max\{\frac{f_t^2(u)}{f_t^1(u)}R_\rho^\star(t,u),p_1\int\limits_{\mathbb{E}}v_n(u,s)f_u^1(s)\mu_u(ds)\}. \tag{5.18}$$

So $v^*(t,u)$ *satisfies the equation*

$$v^*(t,u)=\max\{\frac{f_t^2(u)}{f_t^1(u)}R_\rho^\star(t,u),p_1\int\limits_{\mathbb{E}}v^*(u,s)f_u^1(s)\mu_u(ds)\}. \tag{5.19}$$

The value of the problem $V_n=v^*(X_{n-1},X_n)$.

PROOF. For any Borel function $u:\mathbb{E}\times\mathbb{E}\times[0,1]^3\to[0,1]$ and $D=\{\omega: X_{n-1}=t, X_n=u, \Pi_n^1(x)=\alpha, \Pi_n^2(x)=\beta, \Pi_n^{12}=\gamma\}$let us define two operators

$$\mathbf{T}_x u(t,u,\vec{\alpha}) \quad = \quad \mathbf{E}_x(u(X_n,X_{n+1},\overrightarrow{\Pi}_{n+1})|D)$$

and $\mathbf{Q}_x u(t,u,\vec{\alpha})=\max\{u(t,u,\vec{\alpha}),\mathbf{T}_x u(t,u,\vec{\alpha})\}$. Similarly as in the proof of Lemma 5.3 we conclude that the solution of (5.5) is a Markov time

$$\tau_m^* = \inf\{n>m: f(X_{n-1},X_n,\overrightarrow{\Pi}_n)=f^*(X_{n-1},X_n,\overrightarrow{\Pi}_n)\},$$

where $f^*=\lim_{k\to\infty}\mathbf{Q}_x^k f(t,u,\vec{\alpha})$. By (3.18) and (5.16) on $D=\{\omega: X_{n-1}=t, X_n=u, \Pi_n^1=\alpha, \Pi_n^2=\beta, \Pi_n^{12}=\gamma\}$ we have

$$\begin{aligned}
\mathbf{T}_x f(t,u,\vec{\alpha}) \quad &= \quad \mathbf{E}_x\Big(\frac{q_1}{p_1}(1-\Pi_{n+1}^1)\frac{f_{X_n}^1(X_{n+1})}{f_{X_n}^0(X_{n+1})}R_\rho^\star(X_n,X_{n+1})|D\Big)\\
&= \quad \frac{q_1}{p_1}(1-\alpha)p_1\mathbf{E}\Big(\frac{f_u^0(X_{n+1})}{H(u,X_{n+1},\alpha,\beta)}\frac{f_u^1(X_{n+1})}{f_u^0(X_{n+1})}R_\rho^\star(X_n,X_{n+1})|\mathcal{F}_n\Big)|_D\\
&\stackrel{(3.18)}{=} \quad \frac{q_1}{p_1}(1-\alpha)p_1\int_{\mathbb{E}}\frac{f_u^1(s)}{H(u,s,\alpha,\beta)}H(u,s,\alpha,\beta)R_\rho^*(u,s)\mu_u(ds)\\
&= \quad \frac{q_1}{p_1}(1-\alpha)p_1\int_{\mathbb{E}}R_\rho^*(u,s)f_{X_n}^1(s)\mu_u(ds)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{Q}_x f(t,u,\vec{\alpha}) \quad &= \quad \frac{q_1}{p_1}(1-\alpha)\max\{\frac{f_t^1(u)}{f_t^0(u)}R_\rho^\star(t,u), p_1\int_{\mathbb{E}}R_\rho^\star(u,s)f_u^1(s)\mu_u(ds)\}\\
&= \quad \frac{q_1}{p_1}\alpha v_1(t,u).
\end{aligned} \tag{5.20}$$

Let us define $v_1(t,u)=\max\{\frac{f_t^1(u)}{f_t^0(u)}R_\rho^\star(t,u), p_1\int_{\mathbb{E}}R_\rho^*(u,s)f_u^1(s)\mu_u(ds)$ and

$$v_{n+1}(t,u)=\max\{\frac{f_t^1(u)}{f_t^0(u)}R_\rho^\star(t,u), p_1\int_{\mathbb{E}}v_n(u,s)f_u^0(s)\mu_u(ds)\}.$$

We show that

$$\mathbf{Q}_x^\ell f(t,u,\vec{\alpha}) = \frac{q_1}{p_1}(1-\alpha)v_\ell(t,u) \tag{5.21}$$

for $\ell = 1,2,\ldots$. We have by (5.20) $\mathbf{Q}_x f(t,u,\vec{\alpha}) = \frac{q_1}{p_1}(1-\alpha)v_1(t,u)$ and assume (5.21) for $\ell \leqslant k$. By (3.18) on $D = \{\omega : X_{n-1} = t, X_n = u, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_n^{12} = \gamma\}$we have got

$$\begin{aligned}\mathbf{T}_x\mathbf{Q}_x^k f(t,u,\vec{\alpha}) &= \mathbf{E}_x(\frac{q_1}{p_1}(1-\Pi_{k+1}^1)v_k(X_n,X_{n+1})|D)\\ &= \frac{q_1}{p_1}(1-\alpha)p_1\int_{\mathbb{E}} v_k(u,s)f_u^0(s)\mu_u(ds).\end{aligned}$$

Hence we have got $\mathbf{Q}_x^{k+1}f = \frac{q_1}{p_1}(1-\alpha)v_{k+1}$ and (5.21) is proved for $\ell = 1,2,\ldots$. This gives

$$f^*(t,u,\vec{\alpha}) = \frac{q_1}{p_1}(1-\alpha)\lim_{k\to\infty} v_k(t,u) = \frac{q_1}{p_1}\alpha v^*(t,u)$$

and

$$V_m = \frac{q_1}{p_1}(1-\Pi_m^1)v^*(X_{m-1},X_m).$$

We have

$$\mathbf{T}_x f^*(t,u,\vec{\alpha}) = \frac{q_1}{p_1}(1-\alpha)p_1\int_{\mathbb{E}} v^*(u,s)f_u^0(s)\mu_u(ds).$$

Define $B^* = \{(t,u,\vec{\alpha}) : \frac{f_t^1(u)}{f_t^0(u)}R_\rho^\star(t,u) \geqslant p_1\int_{\mathbb{E}} v^*(u,s)f_u^0(s)\mu_u(ds)\}$ then τ_n^* for $n \geqslant 1$ has a form (5.17). The value of the problem (5.2), (5.5) and (2.5) is equal

$$v_0(x) = \max\{\pi, \mathbf{E}_x(V_1|\mathcal{F}_0)\} = \max\{\pi, \frac{q_1}{p_1}(1-\pi)p_1\int_{\mathbb{E}} v^*(u,s)f_u^0(s)\mu_u(ds)\}$$

and

$$\hat{\tau}_0^* = \begin{cases} 0 & \text{if } \pi \geqslant q_1(1-\pi)\int_{\mathbb{E}} v^*(u,s)f_u^0(s)\mu_u(ds),\\ \tau_0^* & \text{otherwise.}\end{cases}$$

✠

Based on Lemmas 5.3 and 5.1 the solution of the problem D_{00} can be formulated as follows.

THEOREM 5.2. *A compound stopping time* $(\tau^*, \sigma^*_{\tau^*})$, *where* σ^*_m *is given by (5.7) and* $\tau^* = \hat{\tau}^*_0$ *is given by (5.17) is a solution of the problem* D_{00}. *The value of the problem*

$$\mathbf{P}_x(\tau^* < \sigma^* < \infty, \theta_1 = \tau^*, \theta_2 = \sigma^*_{\tau^*}) = \max\{\pi, q_1(1-\pi)\int_{\mathbb{E}} v^*(u,s) f^0_u(s)\mu_u(ds)\}.$$

REMARK 5.2. *The problem can be extended to optimal detection of more than two successive disorders. The distribution of* θ_1, θ_2 *may be more general. The general a priori distributions of disorder moments leads to more complicated formulae, since the corresponding Markov chains are not homogeneous.*

6. APPENDICES

APPENDIX 1 — USEFUL RELATIONS

6.1. Conditional probability of various event defined by disorder moments. According to definition of Π^1_n, Π^2_n, Π^{12}_n we get

LEMMA 6.1. *For the model discribed in the section 2 the following formulae are valied.*

1. $\mathbf{P}_x(\theta_2 \geqslant n > \theta_1 | \mathcal{F}_n) = \Pi^1_n - \Pi^2_n$;
2. $\mathbf{P}_x(\theta_2 > \theta_1 > n | \mathcal{F}_n) = 1 - \Pi^1_n - \Pi^{12}_n$.

PROOF.

1. Let $\theta_1 \leqslant \theta_2$. Since $\{\omega : \theta_2 \leqslant n\} \subset \{\omega : \theta_1 \leqslant n\}$ it follows that $\mathbf{P}_x(\{\omega : \theta_1 \leqslant n < \theta_n\} | \mathcal{F}_n) = \mathbf{P}_x(\{\omega : \theta_1 \leqslant n\} \setminus \{\omega : \theta_2 \leqslant n\} | \mathcal{F}_n) = \Pi^1_n - \Pi^2_n$.
2. We have

$$\begin{aligned} \Omega &= \{\omega : n < \theta_1 < \theta_2\} \cup \{\omega : \theta_1 \leqslant n < \theta_2\} \\ &\quad \cup \{\omega : \theta_1 \leqslant \theta_2 \leqslant n\} \cup \{\omega : \theta_1 = \theta_2 > n\} \end{aligned} \tag{6.1}$$

hence $1 = \mathbf{P}_x(\omega : n < \theta_1 < \theta_2 | \mathcal{F}_n) + (\Pi^1_n - \Pi^2_n) + \Pi^2_n + \Pi^{12}_n$ and

$$\mathbf{P}_x(\omega : n < \theta_1 < \theta_2 | \mathcal{F}_n) = 1 - \Pi^1_n - \Pi^{12}_n.$$

✠

6.2. Some recursive formulae. In derivation of the formulae in Theorem 3.1 the form of the distribution of some random vectors is taken into account.

LEMMA 6.2. *For the model discribed in the section 2 the following formulae are valied.*

1. $\mathbf{P}_x(\theta_2 = \theta_1 > n+1|\mathcal{F}_n) = p_1\Pi_n^{12} = p_1\rho(1-\Pi_n^1)$;
2. $\mathbf{P}_x(\theta_2 > \theta_1 > n+1|\mathcal{F}_n) = p_1(1-\Pi_n^1-\Pi_n^{12})$;
3. $\mathbf{P}_x(\theta_1 \leqslant n+1|\mathcal{F}_n) = \mathbf{P}_x(\theta_1 \leqslant n+1 < \theta_2|\mathcal{F}_n) + \mathbf{P}_x(\theta_2 \leqslant n+1|\mathcal{F}_n)$;
4. $\mathbf{P}_x(\theta_1 \leqslant n+1 < \theta_2|\mathcal{F}_n) = q_1(1-\Pi_n^1-\Pi_n^{12}) + p_2(\Pi_n^1-\Pi_n^2)$;
5. $\mathbf{P}_x(\theta_2 \leqslant n+1|\mathcal{F}_n) = q_2\Pi_n^1 + p_2\Pi_n^2 + q_1\Pi_n^{12}$.

PROOF.

1. On the set $D = \{\omega : X_0 = x, X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n\}$ we have

$$\begin{aligned}
\mathbf{P}_x(\theta_2 = \theta_1 > n+1|D) &= \frac{\rho(1-\pi)\sum\limits_{j=n+2}^{\infty} p_1^{j-1}q_1 \int\limits_{\times_{i=1}^n A_i} \prod_{i=1}^n f_{x_{i-1}}^0(x_i)dx_1\ldots dx_n}{\mathbf{P}(D)}\\
&= p_1\frac{\rho(1-\pi)p_1^n \int\limits_{\times_{i=1}^n A_i} \prod_{i=1}^n f_{x_{i-1}}^0(x_i)dx_1\ldots dx_n}{\mathbf{P}(D)} = p_1\Pi_n^{12},\\
\mathbf{P}_x(\theta_1 > n|D) &= \frac{(1-\pi)\sum\limits_{j=n+1}^{\infty} p_1^{j-1}q_1 \int\limits_{\times_{i=1}^n A_i} \prod_{i=1}^n f_{x_{i-1}}^0(x_i)dx_1\ldots dx_n}{\mathbf{P}(D)}\\
&= \frac{(1-\pi)p_1^n \int\limits_{\times_{i=1}^n A_i} \prod_{i=1}^n f_{x_{i-1}}^0(x_i)dx_1\ldots dx_n}{\mathbf{P}(D)} = \frac{1}{\rho}\Pi_n^{12}.
\end{aligned}$$

2. Similarly as above we get

$$\begin{aligned}
\mathbf{P}_x(\theta_2 > \theta_1 > n+1|D) &= p_1\frac{\rho(1-\pi)p_1^n p_2 \int\limits_{\times_{i=1}^n A_i} \prod_{i=1}^n f_{x_{i-1}}^0(x_i)dx_1\ldots dx_n}{\mathbf{P}(D)}\\
&= p_1\mathbf{P}_x(\theta_2 > \theta_1 > n+1|D) \stackrel{L.\,6.1}{=} p_1(1-\Pi_n^1-\Pi_n^{12}).
\end{aligned}$$

3. It is obvious by assumption $\theta_1 \leqslant \theta_2$.

4. On the set D we have

$$\begin{aligned}
\mathbf{P}_x(\theta_1 \leqslant n+1 < \theta_2|\mathcal{F}_n) &= \frac{\sum_{j=0}^{n+1}\mathbf{P}(\omega:\theta_1=j)\sum_{n+2}^{\infty}(1-\rho)p_2^{k-j}q_2}{\mathbf{P}(D)} \\
&\quad \times \int_{\times_{i=1}^{n}A_i}\prod_{s=1}^{j-1}f^0_{x_{s-1}}(x_s)\prod_{r=j}^{n}f^1_{x_{r-1}}(x_r)dx_1\ldots dx_n \\
&= \frac{(1-\pi)p_1^n q_1(1-\rho)p_2 + p_2\sum_0^n \mathbf{P}(\omega:\theta_1=j)p_2^{n+1-j}}{\mathbf{P}(D)} \\
&\quad \times \int_{\times_{i=1}^{n}A_i}\prod_{s=1}^{j-1}f^0_{x_{s-1}}(x_s)\prod_{r=j}^{n}f^1_{x_{r-1}}(x_r)dx_1\ldots dx_n \\
&\overset{(L.6.1)}{=} q_1\mathbf{P}_x(\theta_2>\theta_1>n|\mathcal{F}_n) + p_2\mathbf{P}_x(\theta_1\leqslant n<\theta_2|\mathcal{F}_n) \\
&= q_1(1-\Pi^1_n-\Pi^{12}_n) + p_2(\Pi^1_n-\Pi^2_n).
\end{aligned}$$

5. If we substitute n by $n+1$ in (6.1) than we obtain

$$\begin{aligned}
\mathbf{P}_x(\theta_2 \leqslant n+1|\mathcal{F}_n) &= 1-\mathbf{P}_x(n+1<\theta_1=\theta_2|\mathcal{F}_n) \\
&\quad -\mathbf{P}_x(n+1<\theta_1<\theta_2|\mathcal{F}_n) - \mathbf{P}_x(\theta_1\leqslant n+1<\theta_2|\mathcal{F}_n) \\
&= 1-p_1\Pi^{12}_n - p_1(1-\Pi^1_n-\Pi^{12}_n) - q_1(1-\Pi^1_n-\Pi^{12}_n) \\
&\quad +p_2(\Pi^2_n-\Pi^1_n) = q_2\Pi^1_n + p_2\Pi^2_n + q_1\Pi^{12}_n.
\end{aligned}$$

✠

Acknowledgements.. I have benefited from discussions with Wojciech Sarnowski and Anna Karpowicz, for which I am grateful. They provided numerous corrections to the manuscript.

REFERENCES

[1] M. Baron. Early detection of epidemics as a sequential change-point problem. In V. Antonov, C. Huber, M. Nikulin, and V. Polischook, editors, *Longevity, aging and degradation models in reliability, public health, medicine and biology, LAD 2004. Selected papers from the first French-Russian conference, St.*

Petersburg, Russia, June 7–9, 2004, volume 2 of *IMS Lecture Notes-Monograph Series*, pages 31–43, St. Petersburg, Russia, 2004. St. Petersburg State Politechnical University.

[2] M. Basseville and A. Benveniste, editors. *Detection of abrupt changes in signals and dynamical systems*, volume 77 of *Lecture Notes in Control and Information Sciences*, page 373. Springer-Verlag, Berlin, 1986.

[3] T. Bojdecki. Probability maximizing approach to optimal stopping and its application to a disorder problem. *Stochastics*, 3:61–71, 1979.

[4] T. Bojdecki and J. Hosza. On a generalized disorder problem. *Stochastic Processes Appl.*, 18:349–359, 1984.

[5] B.E. Brodsky and B.S. Darkhovsky. *Nonparametric Methods in Change-Point Problems*. Mathematics and its Applications (Dordrecht). 243. Dordrecht: Kluwer Academic Publishers. 224 p., Dordrecht, 1993.

[6] P. Dube and R. Mazumdar. A framework for quickest detection of traffic anomalies in networks. Technical report, Electrical and Computer Engineering, Purdue University, November 2001. citeseer.ist.psu.edu/506551.html.

[7] G.W. Haggstrom. Optimal sequential procedures when more then one stop is required. *Ann. Math. Statist.*, 38:1618–1626, 1967.

[8] G.V. Moustakides. Quickest detection of abrupt changes for a class of random processes. *IEEE Trans. Inf. Theory*, 44(5):1965–1968, 1998.

[9] M.L. Nikolaev. Obobshchennye posledovatel′nye procedury. *Litovskiĭ Matematicheskiĭ Sbornik*, 19:35–44, 1979.

[10] M.L. Nikolaev. O kriterii optimal'nosti obobshchennoi posledovatel'noj procedury. *Litov. Mat. Sb.*, 21:75–82, 1981. On the criterion of optimality of the extended sequential procedure (in Russian).

[11] W. Sarnowski and K. Szajowski. On-line detection of a part of a sequence with unspecified distribution. *Stat. Probab. Lett.*, 78(15):2511–2516, 2008. doi:10.1016/j.spl.2008.02.040.

[12] W.A. Shewhart. *Economic control of quality of manufactured products*. D. Van Nostrand, Yew York, 1931.

[13] A.N. Shiryaev. *Optimal Stopping Rules*. Springer-Verlag, New York, Heidelberg, Berlin, 1978.

[14] K. Szajowski. Optimal on-line detection of outside observation. *J.Stat. Planning and Inference*, 30:413–426, 1992.

[15] K. Szajowski. A two-disorder detection problem. *Appl. Math.*, 24(2):231–241, 1996.

[16] B. Yakir. Optimal detection of a change in distribution when the observations form a Markov chain with a finite state space. In E. Carlstein, H.-G. Müller, and D. Siegmund, editors, *Change-point Problems. Papers*

from the AMS-IMS-SIAM Summer Research Conference held at Mt. Holyoke College, South Hadley, MA, USA, July 11–16, 1992, volume 23 of *IMS Lecture Notes-Monograph Series*, pages 346–358, Hayward, California, 1994. Institute of Mathematical Statistics.

[17] M. Yoshida. Probability maximizing approach for a quickest detection problem with complicated Markov chain. *J. Inform. Optimization Sci.*, 4:127–145, 1983.

Institute of Mathematics and Computer Science
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: Krzysztof.Szajowski@pwr.wroc.pl

Received on 23rd of November 2008;
last revised version on xx.xx.xxxx