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Regularity of Solutions to Nonlinear Nondiagonal Parabolic System

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REGULARITY OF SOLUTIONS TO NONLINEAR NONDIAGONAL PARABOLIC SYSTEM

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Abstract

Estimates of $H^{\alpha}-$norms of weak solutions has been obtained for a model nondiagonal parabolic system of nonlinear differential equations with matrix of coefficients satisfying special structure conditions. A technique based on estimating the certain functions of unknowns is employed to this end.
Key words: nondiagonal parabolic system, Hölder continuity, boundedness, Dirichlet problem.[1]

1 Introduction.

In the present paper we study boundedness and Hölder continuity of weak solutions to the nonlinear nondiagonal parabolic system of two equations in divergence form under special assumptions upon its structure.

It is well-known that the De Giorgi-Nash-Moser estimates are no longer valid in general for an elliptic system, the latter can be regarded as a special case of the parabolic version. An example of an unbounded solution to the linear elliptic system with bounded coefficients was built up by De Giorgi in [4]. There is yet another example due to J. Nečas and J. Souček of a nonlinear elliptic system with the coefficients sufficiently smooth, but the weak solution not belonging to $W^{2,2}$.

These two and many other examples illustrate that the regularity problem for elliptic systems proves to be far more complicated then that for second order elliptic equations.

Concerning systems of differential equations until now a priori estimates of De Giorgi type has been extended only to a special class of parabolic systems of equations, the so-called weakly coupled systems.

Therefore there constitutes an interest the question of finding strongly-coupled systems, whose solutions exhibit certain regularity.

The technique we are utilizing consists in switching to new functions, for each of which the estimate is established in a conventional way, wherefrom we are able to infer the conclusion about each component of the vector function solution. This technique has been and employed primarily by Tesei in [6] to semilinear diagonal systems but with

[1]2000 AMS Subject Classification: Primary 35B50; Secondary 35K40, 35K55

complicated righthand side (see also [3]). Then it was applied by Wiegner and his disciple Küfner in [7] and [5] to nondiagonal parabolic systems.

At the same time there is classical method in the theory of differential equations, of introducing local coordinate system, which is applied to determine the type of second order partial differential equation: elliptic, parabolic, hyperbolic. It turns out that the combination of two ideas allows to tackle nonlinear nondiagonal systems of equations.

If we consider for the sake of simplicity the system with constant coefficients our method would reduce to the eigenvalue problem for the matrix of coefficients. In the case of a variable coefficients we perform the procedure of diagonalization locally at each point of the domain calculating the derivatives with respect to the local basis. Our approach hinges upon switching to new functions of unknowns and local diagonalization of the system.

In the present paper, although restricting ourselves to systems of second order equations in divergence form possessing special structure, we demonstrate H^α regularity of solution to nonlinear parabolic systems of two equations in which coupling occurs in the leading derivatives and whose leading coefficients depend on x, u, and v.

2 Boundedness.

2.1 Basic notations and hypotheses.

We shall be concerned with a system of two equations of the form:

$$\begin{cases} u_t - \dfrac{\partial}{\partial x_i}\left(a_1(x,u,v)\nabla u + b_1(x,u,v)\nabla v\right) = f_1(x,t), \\ v_t - \dfrac{\partial}{\partial x_i}\left(a_2(x,u,v)\nabla u + b_2(x,u,v)\nabla v\right) = f_2(x,t), \\ \quad (x,t) \in Q, \end{cases} \tag{2.1}$$

$$f_j(x,t) \in L^\tau(Q), \quad \tau > \frac{n+2}{2}. \tag{2.2}$$

About the coefficients of the model system we suppose that

$$\forall u, v, x_i \in \mathbb{R}$$
$$[b_2(x,u,v) - a_1(x,u,v)]^2 + 4a_2(x,u,v)b_1(x,u,v) > 0$$

and there are two functions of (n+2) variables $\gamma_1(x,u,v)$ and $\gamma_2(x,u,v)$ such that there is satisfied the following system of equations, which we propose to call the characteristic system for the system on issue:

$$\forall u, v, x_i \in \mathbb{R} \tag{2.3}$$

$$\begin{cases} a_1(x,u,v)\gamma(x,u,v) + a_2(x,u,v) = \Lambda(x,u,v)\gamma(x,u,v), \\ b_1(x,u,v)\gamma(x,u,v) + b_2(x,u,v) = \Lambda(x,u,v); \end{cases} \tag{2.4}$$

where $\gamma(x,u,v)$ stands for $\gamma_1(x,u,v)$, $\gamma_2(x,u,v)$,

$$\omega_1 \le \gamma_1(x,u,v) \le A_1, \tag{2.5}$$

$$\omega_2 \le \gamma_2(x,u,v) \le A_2, \tag{2.6}$$

$$\omega_2 > A_1; \tag{2.7}$$

Λ is a measurable $\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ Caratheodory function such that

$$0 < L_1 \le \Lambda(x,u,v) \le L_2, \quad \forall u, v, x_i \in \mathbb{R}, \tag{2.8}$$

$L_{1,2}$, $\omega_{1,2}$, $A_{1,2}$ are numbers.

Remark 1. *It is easy to check that hypotheses* (2.3)-(2.8) *imply parabolicity. In fact,*

$$\begin{aligned} a_1|\nabla u|^2 + b_1\nabla u\nabla v + a_2\nabla u\nabla v + b_2|\nabla v|^2 = \\ = \Lambda_1(\gamma_1\nabla u + \nabla v)^2 + \Lambda_2(\gamma_2\nabla u + \nabla v)^2 \ge \\ \ge \frac{L_1(\omega_2 - A_1)^2}{2\max[A_1^2, A_2^2]}(|\nabla u|^2 + |\nabla v|^2) = \lambda(|\nabla u|^2 + |\nabla v|^2). \end{aligned}$$

In this respect conditions (2.3)-(2.8) *are narrower than ellipticity (parabolicity) condition (see below). There is every reason to speak of new type of parabolicity determined by hypotheses* (2.3)-(2.8).

The boundary conditions of the Dirichlet type are assigned:

$$\begin{cases} (u - g_1, v - g_2)(x,t) \in W_0^{1,2}(\Omega) & \text{a.e.} \quad t \in (0,T), \\ (u,v)(x,0) = (u_0, v_0)(x). \end{cases} \tag{2.9}$$

A solution to system (2.1) with Dirichlet data (2.9) is understood in the weak sense, as in [2].

Definition 2.1. *A measurable vector function* $(u^1, u^2) = (u, v)$ *is called a weak solution of problem* (2.1)-(2.9) *if*

$$u^j \in C(0,T; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega))$$

and for all $t \in (0,T]$

$$\int\limits_\Omega u^j\varphi_j(x,t)dx + \iint\limits_{\Omega\times(0,t]} \{-u^j\varphi_{j\,t} + a_j u_{x_i}\varphi_{j\,x_i} + b_j u_{x_i}\varphi_{j\,x_i}\}dxd\tau =$$

$$= \int\limits_\Omega u_0^j\varphi_j(x,0)dx + \iint\limits_{\Omega\times(0,t]} f^j\varphi_j dxd\tau$$

for all testing functions

$$\varphi \in W^{1,2}(0,T; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega)).$$

The boundary condition in (2.9) *is meant in the weak sense.*

Let us describe the notions, quantities and functions entering system (2.1) that will appear in this paper.

Here and onward we accept the following notations $Q = (0, T] \times \Omega$; $S = \partial\Omega \times (0, T]$; $\partial Q \equiv \{\Omega \times \{0\}\} \bigcup \{\partial\Omega \times (0, T]\}$; Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary; $x \in \Omega$; $T > 0$; $t \in (0, T]$; $n \geq 2$; $i = 1, ..., n$; $j = 1, 2$ and summation convention over repeated indices is assumed; $u, v \in C(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$; $W_0^{1,2}(\Omega)$ is a space of functions in $W^{1,2}(\Omega)$ vanishing on $\partial\Omega$ in the sense of traces for a.e $t \in (0, T]$.

By parabolicity of system (2.1) it is meant that the part without derivatives with respect to time is elliptic. The notion of ellipticity of a system of differential equations is understood in the sense introduced in [1].

The coefficients of system (2.1) satisfy growth conditions:

$$\exists\ \Lambda_2 > 0\ |\forall r^j \in \mathbb{R}^2,\ \forall x \in \mathbb{R}^n;$$
$$|a^j(x, r)|, |b^j(x, r)| \leq \Lambda_2. \tag{2.10}$$

On the functions $g_j(x, t)$, $(u_0, v_0)(x)$ in boundary data (2.9) we assume to be fulfilled the following assumptions:

$$g_j(x, t) \in L^\infty(S), \quad (u_0, v_0)(x) \in L^\infty(\overline{\Omega} \times \{0\}).$$

2.2 Estimates of L^∞-norms.

Let us now turn our attention to the question of boundedness of weak solutions to a system whose coefficients satisfy assumptions (2.3)-(2.7). Our main result is the following

Theorem 2.2. *Let (u, v) be a solution to system (2.1). For the functions $H_1 = u\gamma_1(x, u, v) + v$ and $H_2 = u\gamma_2(x, u, v) + v$ the following estimates hold*

$$||H_1||_{L_\infty(Q)} \leq C; \quad ||H_2||_{L_\infty(Q)} \leq C.$$

Hence it is easily seen that the same estimates take place for the components of the solution themselves:

$$||u||_{L_\infty(Q)} \leq C, \quad ||v||_{L_\infty(Q)} \leq C,$$

where constant C depends only on the data: n, f^j,, Λ_1, $mesQ$ $|g_{1,2}|_{\infty,(S)}$, $|u_0, v_0|_{\infty,(\Omega)}$; constants in the embedding theorems, constants $\omega_{1,2}$, $A_{1,2}$ and is independent of u and v.

To prove the Theorem we need the well-known Stampacchia's lemma:

Lemma 2.3. *Let $\psi(y)$ be a nonnegative nondecreasing function defined on $[k_0, \infty)$ which satisfies:*

$$\psi(m) \leq \frac{C}{(m-k)^\vartheta} \{\psi(k)\}^\delta \qquad for \quad m > k \geq k_0,$$

with $\vartheta > 0$ and $\delta > 1$. Then

$$\psi(k_0 + d) = 0,$$

where $d = C^{1/\vartheta}\{\psi(k_0)\}^{(\delta-1)/\vartheta} 2^{\delta/(\delta-1)}$.

For proof see [1, Lemma 4.1, p. 8]. We make also use of the following lemma (see [2, Prop. 3.1, p. 7]):

Lemma 2.4. *If $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega))$ then there holds the inequality:*

$$\int_0^T\int_\Omega u^q \le C\left(\int_0^T\int_\Omega |\nabla u|^2\right)\left(ess \sup_{0<t<T}\int_\Omega |u|^2\right)^{2/n}$$

with $q = 2\frac{n+2}{n}$ and constant C depending only on n.

Proof of Theorem 2.2. Fix the point of Q, (x_0,t_0). Introduce the local coordinate system $y_i = x_i - x_{0i}$, $s = t - t_0$ with the origin at (x_0,t_0). Let us change to this coordinate system locally at each point of the domain. We shall have

$$\frac{\partial}{\partial x_i} = \left(\frac{\partial y_i}{\partial x_i}\right)\frac{\partial}{\partial y_i} = \frac{\partial}{\partial y_i};$$
$$\frac{\partial}{\partial t} = \left(\frac{\partial s}{\partial t}\right)\frac{\partial}{\partial s} = \frac{\partial}{\partial s}.$$

After changing to these new coordinates the system will take the form

$$\begin{cases} u_s - \dfrac{\partial}{\partial y_i}\left(a_1\nabla_y u + b_1\nabla_y v\right) = f_1, \\ v_s - \dfrac{\partial}{\partial y_i}\left(a_2\nabla_y u + b_2\nabla_y v\right) = f_2, \end{cases} \tag{2.11}$$

where ∇_y stands for gradient with respect to y coordinates and $a_1,a_2,b_1,b_2 \equiv a_1,a_2,b_1,b_2(y,u(y,s),v(y,s))$. Multiply the first equation of (2.11) by $\gamma(x_0,u(x_0,t_0), v(x_0,t_0))$ (γ is either γ_1 or γ_2) and add the second one. Multiply the obtained relation by function $(\gamma u + v - k)_+$ with $k \ge k_0 = \max[||\gamma(x,g_1(x,t),g_2(x,t))g_1 + g_2||_{L^\infty(S)}, ||\gamma(x,u_0(x,t),v_0(x,t))u_0 + v_0||_{L^\infty(\Omega)}]$, $(x,t) \in \partial Q$, and integrate over the domain Q in x_0 and t_0. Thus we have (we can without misunderstanding denote (x_0,t_0) by (x,t))

$$\frac{1}{2}\int_0^t\int_\Omega \frac{d}{ds}(H-k)^2\chi_{A(k)} +$$
$$+\int_0^t\int_\Omega < ([a_1\gamma + a_2]/\gamma)\nabla_y\gamma u + [b_1\gamma + b_2]\nabla_y v > \nabla_y(H-k)\chi_{A(k)} -$$
$$-\int_0^t\int_\Omega \frac{\partial}{\partial y_i}(< ([a_1\gamma + a_2]/\gamma)\nabla_y\gamma u + [b_1\gamma + b_2]\nabla_y v > (H-k))\chi_{A(k)} =$$
$$= \int_0^t\int_\Omega (f_1\gamma + f_2)(H-k)\chi_{A(k)}.$$

Here $\chi_{A(k)}$ is a characteristic function of the set $A(k,t)=\{x\in\Omega|H-k\geq 0\}$, $a_1,a_2,b_1,b_2\equiv$

$a_1,a_2,b_1,b_2(y,u(y,s),v(y,s))$ and $\gamma=\gamma(x,u(x,t),v(x,t))$. After taking into account hypotheses (2.3) and (2.8), along with the fact that $a_j,b_j(y,u(y,s),v(y,s))|_{y=0,s=0}=a_j,b_j(x,u(x,t),v(x,t))$ at each point of the domain, this results in

$$\frac{1}{2}\int_0^t\int_\Omega \frac{d}{ds}(H-k)^2\chi_{A(k)}+\int_0^t\int_\Omega \Lambda_1|\nabla_y(H-k)|^2\chi_{A(k)}-$$
$$-\int_0^t\int_\Omega \frac{\partial}{\partial y_i}(<([a_1\gamma+a_2]/\gamma)\nabla_y\gamma u+[b_1\gamma+b_2]\nabla_y v>(H-k))\chi_{A(k)}\leq$$
$$\leq C_1\int_0^t\int_\Omega |f|(H-k)\chi_{A(k)},$$

where it has been denoted $|f|=|f_1|+|f_2|$. Now the point is to estimate the terms

$$\int_0^t\int_\Omega \frac{d}{ds}(H-k)^2\chi_{A(k)}$$

and

$$\int_0^t\int_\Omega \frac{\partial}{\partial y_i}(<([a_1\gamma+a_2]/\gamma)\nabla_y\gamma u+[b_1\gamma+b_2]\nabla_y v>(H-k))\chi_{A(k)}.$$

Remind that the derivatives in these expressions are with respect to local coordinate system at each point of the domain Q. Let us split the interval $(0,t]$ into N equal segments $(0,t_1),(t_1,t_2),\ldots,(t_{N-1},t]$. Choose at each interval the origin of the axis $0s$. Since the length of the interval tends to zero as N goes to infinity, we shall, obviously, have after integration by parts

$$\int_0^t\int_\Omega \frac{d}{ds}(H-k)^2\chi_{A(k)}=\lim_{N\to\infty}\sum_{m=1}^N\int_{t_{m-1}}^{t_m}\int_\Omega \frac{d}{ds}(H-k)^2\chi_{A(k)}=$$
$$=\lim_{N\to\infty}\sum_{m=1}^N\left\{\int_\Omega (H-k)^2\chi_{A(k)}|_{s=t_m}-\int_\Omega (H-k)^2\chi_{A(k)}|_{s=t_{m-1}}\right\}=$$
$$=\int_\Omega (H-k)^2\chi_{A(k)}|_{s=t}.$$

Analogously, let us split the domain Ω (this domain may be assumed to be an $n-$ dimensional cube and the function $H-k$ can be extended by zero up to its boundary) into N^n equal cubes. Choose at each cube the origin $\mathcal{O}$ of the local coordinate system. Since the diameter of the cube tends to zero as N goes to infinity, we shall, obviously, have after integration by parts (C_m stands for $m-$th cube)

$$\int\limits_{\Omega} \frac{\partial}{\partial y_i}(< ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (H-k))\chi_{A(k)} =$$

$$= \lim_{N\to\infty} \sum_{m=1}^{N^n} \int\limits_{C_m} \frac{\partial}{\partial y_i}(< ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma + [b_1\gamma + b_2]\nabla_y v > (H-k))\chi_{A(k)} =$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \int\limits_{\partial C_m} < ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (H-k)\chi_{A(k)}.$$

Consider the integrals over the neighboring faces (F_m stands for the $m-$th face):

$$\int\limits_{F_m} < ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (H-k)\chi_{A(k)} -$$

$$- \int\limits_{F_l} < ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (H-k)\chi_{A(k)} = 0,$$

and

$$\int\limits_{\partial C_m} < ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (H-k)\chi_{A(k)} = 0,$$

since the solution can be approximated by smooth functions over which the limit is to be taken afterwards, and when each face is approached from different sides, the integrands in these expressions tend to the same values, while the integrand on the outer surface of the domain is zero.

Thus we get

$$\frac{1}{2}\int\limits_{\Omega}(H-k)^2\chi_{A(k)}(t) + \int\limits_0^t\int\limits_{\Omega} \Lambda_1|\nabla_y(H-k)|^2\chi_{A(k)} \le C_1 \int\limits_0^t\int\limits_{\Omega} |f|(H-k)\chi_{A(k)}.$$

Hence, taking the supremum over t on the segment $(0,T]$ we haver the following

$$\frac{1}{2}\sup_{0<t<T}\int\limits_{\Omega}(H-k)^2\chi_{A(k)}(t) + \int\limits_{Q} \Lambda_1|\nabla_y(H-k)|^2\chi_{A(k)} \le C_1 \int\limits_{Q} |f|(H-k)\chi_{A(k)}.$$

On this step we introduce the set of increasing levels $\{k_m\}$:

$$k_m = \left(1 - \frac{1}{2^m}\right) d + k_0,$$

where the positive number is to be determined later. We shall show that $\chi_{A(d+k_0)} \equiv 0$. We maintain that

$$\int\limits_Q (H-k_m)^{2\frac{n+2}{n}} \chi_{A(k_m)} \leq C_2(u,v) \left(\sup_{0<t<T} \int\limits_\Omega (H-k_m)^2 \chi_{A(k_m)}(t) + \right. \left. + \int\limits_Q |\nabla_y (H-k_m)|^2 \chi_{A(k_m)} \right)^{\frac{n+2}{n}} \tag{2.12}$$

with constant C_2 dependent on the domain, righthand sides of the equations and the solutions of the problem on issue, since we don't impose uniqueness, and is independent of the levels from the set $\{k_m\}$. In fact, assume the opposite, i. e., that whatever large C_m we take, there exists k_m such that

$$\begin{aligned} (2.13)\quad & C_4 \left(\int\limits_Q \sup_{0<t<T} \int\limits_\Omega (u^2+v^2)(t) + \int\limits_Q (|\nabla u|^2 + |\nabla v|^2) \right) / C_m \geq \\ & \geq \left(C_3 \int\limits_Q (\gamma u - \gamma k_{01} + v - k_{02})^{2\frac{n+2}{n}} \chi_{A(k_m)} + \right. \\ & \left. + C_3 \int\limits_Q (\gamma k_{01} + k_{02})^{2\frac{n+2}{n}} \chi_{A(k_m)} \right) / C_m \geq \\ & \geq \left(\int\limits_Q (H)^{2\frac{n+2}{n}} \chi_{A(k_m)} \right) / C_m \geq \left(\int\limits_Q (H-k_m)^{2\frac{n+2}{n}} \chi_{A(k_m)} \right) / C_m > \\ & > \left(\sup_{0<t<T} \int\limits_\Omega (H-k_m)^2 \chi_{A(k_m)}(t) + \int\limits_Q |\nabla_y (H-k_m)|^2 \chi_{A(k_m)} \right)^{\frac{n+2}{n}}, \end{aligned}$$

where k_{01} and k_{02} stand for the supremum of u and v on the parabolic boundary respectively. Hence, since the integral on the left is uniformly bounded for all solutions (u,v), owing to energy estimate, we have that

$$\int\limits_Q |\nabla_y (H-k_m)|^2 \chi_{A(k_m)} \to 0$$

as $m \to \infty$ and, so, $A(d+k_0) \equiv 0$, i. e., everything is proven. Thus we come to

$$\left(\int\limits_Q (H-k_m)^{2\frac{n+2}{n}} \chi_{A(k_m)} \right)^{\frac{n}{n+2}} \leq C_5 \int\limits_Q |f| (H-k_m) \chi_{A(k_m)}. \tag{2.14}$$

Applying generalized Hölder's inequality to the right of (2.14) we obtain

$$\|H - k_m\|_{q,Q} \le C_6 \|f\|_{r,Q} \{\psi(k_m)\}^{1-1/q-1/r}; \tag{2.15}$$

here we've denoted:

$$\psi(k_m) = \int_0^T \mathrm{mes} A(k_m, t) dt,$$

and q is as in Lemma 2.4. Let us estimate:

$$(k_{m+1} - k_m)\{\psi(k_{m+1})\}^{1/q} = (k_{m+1} - k_m) \left(\int_0^T \int_\Omega \chi_{A(k_{m+1})} \right)^{1/q} <$$
$$< \left(\int_0^T \int_\Omega (H - k_m)^q \chi_{A(km+1)} \right)^{1/q} < \|H - k_m\|_{q,Q},$$

where $k_{m+1} > k_m \ge k_0$. Substituting this estimate into (2.15), we come down to

$$(k_{m+1} - k_m)^q \psi(k_m) \le C_7 \left\{\psi(k_m)\right\}^{q(1-1/q-1/r)} = C_7 \left\{\psi(k_m)\right\}^{\delta}. \tag{2.16}$$

From the hypotheses on f_j and by the choice of r

$$\tau > r > (n+2)/2, \tag{2.2}$$

hence

$$2\frac{(n+2)}{n}\left(1 - \frac{n}{2(n+2)} - \frac{1}{r}\right) > 1; \qquad \text{and thus} \quad \delta > 1.$$

On the strength of Lemma 2.3 from relation (2.16) we can conclude that

$$\psi(k_0 + d) = 0$$

for some d sufficiently large, but finite, depending only on n, f^j, Λ_1, $|g_{1,2}|_{\infty,(S)}$, $|u_0, v_0|_{\infty,(\Omega)}$; constants in the embedding theorems and the components of the solution u and v. It should be noted that the constant C_2 in (2.12) can't be infinite since this would imply that $d = \infty$ and, because of the energy estimate, the measure of the set where $(H - k_m)$ is different from zero would be zero and we couldn't get a strict inequality in alternative (2.13), that is, a contradiction to our assumption.

So, the constant C_2 and along with d are finite for each solution, but are dependent yet on the solution.

Now we are in a position to prove the uniform boundedness of H for all the solutions. To this end we start from the very beginning of the proof and go down to inequality (2.12), maintaining that the latter is held with constant C_2 uniform for all of the solutions (u, v) of the problem. If there is no such constant C_2, then we can for any C_m, whatever large, choose the level k_m and function H_m with the property $\sup_Q H_m \ge \sup_Q H_{m-1}$ (this can be taken without loss of generality), such that the revers strict inequality holds. Since all of the solutions of the problem are uniformly bounded in $C(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,2}(\Omega))$, owing to energy estimate, we can, because of the compact embedding of W into L^2 ($W \hookrightarrow\hookrightarrow L^2$), choose from this sequence of functions $(H_m - k_m)$ a subsequence which converges strongly in L^2 to some function $(\widetilde{H} - (d + k_0))$. From (2.13) it follows that $\widetilde{H} - (d + k_0) \equiv 0$, where d is chosen dependent on C_2. In virtue of the assumption that $\sup\limits_Q H_m \geq \sup\limits_Q H_{m-1}$, all the other functions H of solutions are also bounded by $d + k_0$. Repeating then the subsequent part of the proof of the case with nonuniform C_2, we come down to $\sup\limits_Q H \leq d + k_0$ with d dependent on C_2. By the argument of the same argument as in the "nonuniform" case we come to conclusion that C_2 and hence d, can not be infinite: the opposite would lead to contradiction with the strict inequality in alternative similar to (2.13).

Analogously can be proved the estimate for the infimum of H in Q. To this end we must consider the truncated functions $(k - (\gamma u + v))_+$ and test the system on them.

And thus, finally, we have that

$$||H_1||_\infty \leq C_8, \quad ||H_2||_\infty \leq C_9,$$

where $||\cdot||_\infty$ stands for $L_\infty(Q)$ norm. It is not difficult to resolve these estimates and to obtain that the same estimates hold for the components (u, v) of solution themselves. In fact

$$\inf_{\partial Q}(\gamma_1 u + v) \leq \gamma_1 u + v \leq \sup_{\partial Q}(\gamma_1 u + v);$$
$$\inf_{\partial Q}(\gamma_2 u + v) \leq \gamma_2 u + v \leq \sup_{\partial Q}(\gamma_2 u + v).$$

Subtracting the second estimate from the first one we get

$$|u| \leq \left(|\sup_{\partial Q}(\gamma_1 u + v) - \inf_{\partial Q}(\gamma_2 u + v)| + |\sup_{\partial Q}(\gamma_2 u + v) - \inf_{\partial Q}(\gamma_1 u + v)| \right) / |\omega_2 - A_1|.$$

Hence the estimate for the component v is self-evident. Thus we come down to

$$||u||_\infty \leq C_{10}, \quad ||v||_\infty \leq C_{11}.$$

□

Remark 2. *In the proof of Theorem 2.2 we have performed local, at each point, diagonalization of system* (2.1)*. It is clear, however, that the system can not be reduced to diagonal by changing to unknowns $\gamma_1 u + v$ and $\gamma_2 u + v$, let alone be solved by this substitution.*

2.3 Some examples.

Example 1. The system governing chemotaxis, i. e., movement of living organisms due to the presence of curtain chemicals:

$$\text{(2.17)} \qquad \begin{cases} u_t - \triangle u - \nabla\,(u \nabla v) = 0, \\ v_t - D\triangle v = -\beta v + u, \qquad (x,t) \in Q, \end{cases}$$

with Dirichlet data, $u, v \geq 0$, $\beta > 1$, $D < 1$. The terms $\triangle u$ and $\triangle v$ govern the random mobility, $\nabla(u\nabla v)$ - chemotaxis, $(-\beta v)$ - decay, u - production of species. We assume the energy estimate. For this problem the characteristic system (2.3) would be:

$$\begin{cases} \gamma + 0\delta = \Lambda\gamma, \\ \gamma u + D\delta = \Lambda\delta, \end{cases}$$

which is satisfied by $\gamma = (1 - D)$, $\delta = u$, $\Lambda = 1$. We can introduce the function $H(u, v) \equiv \gamma u + \delta v \equiv (1 - D)u + uv$ and in the spirit of the reasoning of the previous section establish the estimate of $L^\infty(Q)$ of function H. The second equation would give us the estimate for v. Hence follow the estimates of $\|u\|_\infty$ and $\|v\|_\infty$.

Example 2. The system describing the outbreak in the susceptibles-infectives-removed model:

$$\begin{cases} u_t - \theta\triangle u = -\alpha\beta u\triangle v - \beta uv, \\ v_t - \varphi\triangle v = \alpha\beta u\triangle v + \beta uv - \alpha v, \qquad (x,t) \in Q, \end{cases} \tag{2.18}$$

with Neumann data, $u, v \geq 0$, $\theta, \varphi, \lambda, \beta > 0$, $\varphi > \theta$. For this problem the characteristic system (2.3) would be:

$$\begin{cases} \delta\theta + 0 = \Lambda\delta, \\ \delta(-\alpha\beta u) + (\varphi + \alpha\beta u) = \Lambda, \end{cases}$$

which is satisfied by $\delta = \left(\dfrac{\varphi - \theta}{\alpha\beta}\right)\dfrac{1}{u} + 1$, $\Lambda = \theta$. We can introduce the function $H(u, v) \equiv \delta u + v \equiv \left(\dfrac{\varphi - \theta}{\alpha\beta}\right) + u + v$ and in the spirit of the reasoning of the previous section establish the estimate of $L^\infty(Q)$ of function H. Hence follow the maximum principle and the estimates of $\|u\|_\infty$ and $\|v\|_\infty$.

Remark 3. *These are the examples of triangular systems, that is why we have only one function H, which is sufficient.*

3 Hölder continuity.

3.1 Basic notations and hypotheses.

For any $x_0 \in \Omega$ and $t_0 \in [0, T]$ $\exists\alpha_1 > 0$ and $\exists\beta_1 > 0$ such that

$$f_{1,2}(x,t)/(|x - x_0|^{\alpha_1} + |t - t_0|^{\beta_1})^{1 + \frac{4}{n}\left(1 - \frac{1}{\kappa}\right)} \in L^r(Q), \tag{3.1}$$
$$r = \kappa\frac{n+2}{2}, \quad \kappa \in (1, \infty)\,.$$

When righthand sides are such as these, we say that $\|f_{1,2}\|_{\alpha_1,\beta_1,r,Q} = const < \infty$.

On the functions $g_{1,2}(x,t)$, $u_0(x)$, $v_0(x)$ in boundary data (2.9) we assume

$$g_{1,2}(x,t) \in H^{\alpha_g,\beta_g}(S), \quad u_0(x), v_0(x) \in H^{\alpha_0}(\overline{\Omega} \times \{0\}),$$

where H stands for a Hölder space.

3.2 Hölder continuity.

We construct the functions w and z:

$$w(x,x',t,t') = (u(x,t) - u(x',t'))/(|x-x'|^\alpha + |t-t'|^\beta),$$
$$z(x,x',t,t') = (v(x,t) - v(x',t'))/(|x-x'|^\alpha + |t-t'|^\beta),$$
$$(x,t),\ (x',t') \in Q.$$

Our proof of Hölder continuity hinges upon the following self-evident theorem:

Theorem 3.1. *If $w(x,t), z(x,t) \in L^\infty(Q)$ for a. e. $(x',t') \in Q$ then $u(x,t), v(x,t) \in H^{\alpha,\beta}(Q)$.*

We establish the boundedness of functions $H_1 \equiv \gamma_1(x,u,v)w+z$ and $H_2 \equiv \gamma_2(x,u,v)w+z$ of (x,t) for a. e. $(x',t') \in Q$, the boundedness of w and z will follow by subtraction.

Theorem 3.2. *For H_1 and H_2 as a functions of (x,t) the following estimate is valid*

$$\|H_1(x,t)\|_{\infty,Q}, \|H_2(x,t)\|_{\infty,Q} \leq C \qquad \textit{for almost all } (x',t') \in Q,$$

where were C depends only on the data of the problem, and not on the w, z, x' and t'.

Remark 4. *The fact that we don't impose any conditions on the domain boundary doesn't lead to contradiction with Wiener's criterion, as might seem, because in the latter the function on the boundary is C, whereas in our case it is C^α.*

Proof of Theorem 3.2. Without loss of generality we may assume that $x' = \mathcal{O}$, $t' = 0$ and $\gamma u(x',t') + v(x',t') = 0$.

Step 1. On the first step we shall assume that $(O,0) \in \partial Q$. To begin, with consider $H_1 \equiv H$ with $\gamma_1 \equiv \gamma$. Fix the point of Q, (x_0,t_0). Introduce the local coordinate system $y_i = x_i - x_{0i}$, $s = t - t_0$ with the origin at (x_0,t_0). Let us change to this coordinate system locally at each point of the domain. We shall have

$$\frac{\partial}{\partial x_i} = \left(\frac{\partial y_i}{\partial x_i}\right)\frac{\partial}{\partial y_i} = \frac{\partial}{\partial y_i};$$
$$\frac{\partial}{\partial t} = \left(\frac{\partial s}{\partial t}\right)\frac{\partial}{\partial s} = \frac{\partial}{\partial s}.$$

After changing to these new coordinates the system takes the form

$$\begin{cases} u_s - \dfrac{\partial}{\partial y_i}\left(a_1\nabla_y u + b_1 \nabla_y v\right) = f_1, \\ v_s - \dfrac{\partial}{\partial y_i}\left(a_2\nabla_y u + b_2 \nabla_y v\right) = f_2, \end{cases} \tag{3.2}$$

where ∇_y stands for gradient with respect to y coordinates. Multiply the first equation of (3.2) by $\gamma(x_0, u(x_0,t_0), v(x_0,t_0))$ (γ is either γ_1 or γ_2) and add the second one. Multiply the obtained relation by function $(|x_0|^\alpha + |t_0|^\beta)(H-k)_+$ with

$$\sup_{(x,t),(x',t')\in S} \frac{|(\gamma g_1(x,t)+g_2(x,t)) - (\gamma g_1(x',t')+g_2(x',t'))|}{(|x-x'|^{\alpha_g}+|t-t'|^{\beta_g})} \times$$
$$\times \sup_{(x,t),(x',t')\in S} \frac{(|x-x'|^{\alpha_g}+|t-t'|^{\beta_g})}{(|x-x'|^{\alpha}+|t-t'|^{\beta})} +$$
$$+ \sup_{(x,x')\in\Omega} \frac{|(\gamma u_0(x)+v_0(x)) - (\gamma u_0(x')+v_0(x'))|}{|x-x'|^{\alpha_0}} \cdot \sup_{(x,x')\in\Omega} \frac{|x-x'|^{\alpha_0}}{|x-x'|^{\alpha}} \le$$
$$\le \sup_{(x,t),(x',t')\in S} \frac{|(\gamma g_1(x,t)+g_2(x,t)) - (\gamma g_1(x',t')+g_2(x',t'))|}{(|x-x'|^{\alpha_g}+|t-t'|^{\beta_g})} \times$$
$$\times ((\mathrm{diam}\ \Omega)^{\alpha_g-\alpha} + T^{\beta_g-\beta}) +$$
$$+ \sup_{(x,x')\in\Omega} \frac{|(\gamma u_0(x)+v_0(x)) - (\gamma u_0(x')+v_0(x'))|}{|x-x'|^{\alpha_0}} \cdot (\mathrm{diam}\ \Omega)^{\alpha_0-\alpha} = k_0 \le k,$$

$\alpha \le \min[\alpha_g, \alpha_0, \alpha_1]$, $\beta \le \min[\beta_g, \beta_1]$, and integrate over the domain Q in x_0 and t_0. Thus we have (we can without misunderstanding denote (x_0, t_0) by (x,t))

$$\frac{1}{2}\int_0^t\int_\Omega \frac{d}{ds}(|x|^\alpha+|t|^\beta)^2(H-k)^2\chi_{A(k)} +$$
$$+\int_0^t\int_\Omega < ([a_1\gamma+a_2]/\gamma)\nabla_y\gamma u + +[b_1\gamma+b_2]\nabla_y v > (|x|^\alpha+|t|^\beta)\nabla_y(H-k)\chi_{A(k)} -$$
$$-\int_0^t\int_\Omega \frac{\partial}{\partial y_i}(< ([a_1\gamma+a_2]/\gamma)\nabla_y\gamma u + [b_1\gamma+b_2]\nabla_y v > (|x|^\alpha+|t|^\beta)(H-k))\chi_{A(k)} =$$
$$= \int_0^t\int_\Omega (f_1\gamma+f_2)\,(|x|^\alpha+|t|^\beta)(H-k)\chi_{A(k)}.$$

Here $\chi_{A(k)}$ is a characteristic function of the set $A(k,t) = \{x\in\Omega | H-k \ge 0\}$, $a_1, a_2, b_1, b_2 \equiv a_1, a_2, b_1, b_2(y, u(y,s), v(y,s))$ and $\gamma \equiv \gamma(x, u(x,t), v(x,t))$. Along with taking into account hypotheses (2.3) and (2.8), and the fact that $a_j, b_j(y, u(y,s), v(y,s))|_{y=0,s=0} = a_j, b_j(x, u(x,t), v(x,t))$ at each point of the domain this results in

$$\frac{1}{2}\int_0^t\int_\Omega \frac{d}{ds}(|x|^\alpha+|t|^\beta)^2(H-k)^2\chi_{A(k)} + \int_0^t\int_\Omega \Lambda_1(|x|^\alpha+|t|^\beta)^2|\nabla_y(H-k)|^2\chi_{A(k)} -$$
$$-\int_0^t\int_\Omega \frac{\partial}{\partial y_i}(< ([a_1\gamma+a_2]/\gamma)\nabla_y\gamma u + [b_1\gamma+b_2]\nabla_y v > (|x|^\alpha+|t|^\beta)(H-k))\chi_{A(k)} \le$$
$$\le C_1\int_0^t\int_\Omega (|x|^\alpha+|t|^\beta)|f|(H-k)\chi_{A(k)},$$

where it has been denoted $|f| = |f_1| + |f_2|$. Now the point is to estimate the terms

$$\int_0^t \int_\Omega \frac{d}{ds}(|x|^\alpha + |t|^\beta)^2 (H-k)^2 \chi_{A(k)}$$

and

$$\int_0^t \int_\Omega \frac{\partial}{\partial y_i}(< ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (|x|^\alpha + |t|^\beta)(H-k))\chi_{A(k)}.$$

Remind that the derivatives in these expressions are with respect to local coordinate system at each point of the domain Q. Let us divide the interval $(0,t]$ into N equal segments $(0,t_1),(t_1,t_2),\ldots,(t_{N-1},t]$. Choose at each interval the origin of the axis $0s$. Since the length of the interval tends to zero as N goes to infinity, we shall, obviously, have after integration by parts

$$\int_0^t \int_\Omega \frac{d}{ds}(|x|^\alpha + |t|^\beta)^2 (H-k)^2 \chi_{A(k)} = \lim_{N\to\infty} \sum_{m=1}^{N} \int_{t_{m-1}}^{t_m} \int_\Omega \frac{d}{ds}(|x|^\alpha + |t|^\beta)^2 (H-k)^2 \chi_{A(k)} =$$

$$= \lim_{N\to\infty} \sum_{m=1}^{N} \left\{ \int_\Omega (|x|^\alpha + |t|^\beta)^2 (H-k)^2 \chi_{A(k)}|_{s=t_m} - \right.$$

$$\left. - \int_\Omega (|x|^\alpha + |t|^\beta)^2 (H-k)^2 \chi_{A(k)}|_{s=t_{m-1}} \right\} =$$

$$= \int_\Omega (|x|^\alpha + |t|^\beta)^2 (H-k)^2 \chi_{A(k)}|_{s=t}.$$

Analogously, let us divide the domain Ω (this domain may be assumed to be a $n-$ dimensional cube and the function $H-k$ can be extended by zero) into N^n equal cubes. Choose at each cube the origin $\mathcal{O}$ of the local coordinate system. Since the diameter of the cube tends to zero as N goes to infinity, we shall, obviously, have after integration by parts (C_m stands for $m-$th cube)

$$\int_\Omega \frac{\partial}{\partial y_i}(< ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (|x|^\alpha + |t|^\beta)(H-k))\chi_{A(k)} =$$

$$= \lim_{N\to\infty} \sum_{m=1}^{N^n} \int_{C_m} \frac{\partial}{\partial y_i}(< ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (|x|^\alpha + |t|^\beta)(H-k))\chi_{A(k)} =$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \int_{\partial C_m} < [a_1\gamma + a_2]/\gamma \nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (|x|^\alpha + |t|^\beta)(H-k)\chi_{A(k)}.$$

Consider the integrals over the neighboring faces (F_m stands for the $m-$th face):

$$\int\limits_{F_m} < ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (|x|^\alpha + |t|^\beta)(H-k)\chi_{A(k)} -$$

$$- \int\limits_{F_l} < ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (|x|^\alpha + |t|^\beta)(H-k)\chi_{A(k)} = 0,$$

and

$$\int\limits_{\partial C_m} < ([a_1\gamma + a_2]/\gamma)\nabla_y \gamma u + [b_1\gamma + b_2]\nabla_y v > (|x|^\alpha + |t|^\beta)(H-k)\chi_{A(k)} = 0,$$

since the solution can be approximated by smooth functions over which a limit is to be taken afterwards, and since when each face is approached from different sides the integrands in these expressions tend to the same values, while the integrand on the outer surface of the domain vanishes.

Thus we get

$$\frac{1}{2}\int\limits_{\Omega} (|x|^\alpha + |t|^\beta)^2 (H-k)^2 \chi_{A(k)}(t) + \int\limits_0^t \int\limits_{\Omega} \Lambda_1 (|x|^\alpha + |t|^\beta)^2 |\nabla_y (H-k)|^2 \chi_{A(k)} \leq$$

$$\leq C_1 \int\limits_0^t \int\limits_{\Omega} (|x|^\alpha + |t|^\beta)|f|(H-k)\chi_{A(k)}.$$

Hence, taking the supremum over t on the segment $(0, T]$ we haver the following

$$\frac{1}{2}\sup_{0<t<T} \int\limits_{\Omega} (|x|^\alpha + |t|^\beta)^2 (H-k)^2 \chi_{A(k)}(t) + \int\limits_{Q} \Lambda_1 (|x|^\alpha + |t|^\beta)^2 |\nabla_y (H-k)|^2 \chi_{A(k)} \leq$$

$$\leq C_1 \int\limits_{Q} (|x|^\alpha + |t|^\beta)|f|(H-k)\chi_{A(k)}.$$

On this step we introduce the set of increasing levels $\{k_m\}$:

$$k_m = \left(1 - \frac{1}{2^m}\right) d + k_0,$$

where the positive number is to be determined later. We shall show that $\chi_{A(d+k_0)} \equiv 0$.

We maintain that

$$(3.3)\quad \int\limits_Q (|x|^\alpha+|t|^\beta)^{2\frac{n+2}{n}}(H-k_m)^{2\frac{n+2}{n}}\chi_{A(k_m)}\le$$

$$\le C_2(u,v)\left(\sup_{0<t<T}\int\limits_\Omega (|x|^\alpha+|t|^\beta)^2(H-k_m)^2\chi_{A(k_m)}(t)+\right.$$

$$\left.+\int\limits_Q (|x|^\alpha+|t|^\beta)^2|\nabla_y(H-k_m)|^2\chi_{A(k_m)}\right)^{\frac{n+2}{n}}$$

with constant C_2 dependent on the domain, righthand sides of the equations and the solutions of the problem on issue, since we don't impose uniqueness, and is independent of the levels from the set $\{k_m\}$. In fact, assume the opposite, i. e., that whatever large C_m we take, there exists k_m such that

$$(3.4)\quad C_4\left(\int\limits_Q \sup_{0<t<T}\int\limits_\Omega (u^2+v^2)(t)+\int\limits_Q(|\nabla u|^2+|\nabla v|^2)\right)/C_m\ge$$

$$\ge\left(C_3\int\limits_Q (|x|^\alpha+|t|^\beta)^{2\frac{n+2}{n}}(\gamma w-\gamma k_{01}+z-k_{02})^{2\frac{n+2}{n}}\chi_{A(k_m)}+\right.$$

$$\left.+C_3\int\limits_Q (|x|^\alpha+|t|^\beta)^{2\frac{n+2}{n}}(\gamma k_{01}+k_{02})^{2\frac{n+2}{n}}\chi_{A(k_m)}\right)/C_m\ge$$

$$\ge\left(\int\limits_Q (|x|^\alpha+|t|^\beta)^{2\frac{n+2}{n}}(H-k_0)^{2\frac{n+2}{n}}\chi_{A(k_m)}\right)/C_m\ge$$

$$\ge\left(\int\limits_Q (|x|^\alpha+|t|^\beta)^{2\frac{n+2}{n}}(H-k_m)^{2\frac{n+2}{n}}\chi_{A(k_m)}\right)/C_m>$$

$$>\left(\sup_{0<t<T}\int\limits_\Omega (|x|^\alpha+|t|^\beta)^2(H-k_m)^2\chi_{A(k_m)}(t)+\right.$$

$$\left.+\int\limits_Q (|x|^\alpha+|t|^\beta)^2|\nabla_y(H-k_m)|^2\chi_{A(k_m)}\right)^{\frac{n+2}{n}},$$

where k_{01} and k_{02} stand for the supremum of w and z on the parabolic boundary respectively. Hence, since the integral on the left is uniformly bounded for all solutions (u,v), owing to energy estimate, we have that

$$\int_Q (|x|^\alpha + |t|^\beta)^2 |\nabla_y (H - k_m)|^2 \chi_{A(k_m)} \to 0$$

as $m \to \infty$ and $A(d + k_0) \equiv 0$, so that everything is proven.

Thus we come to

$$\left(\int_Q (|x|^\alpha + |t|^\beta)^{2\frac{n+2}{n}} (H - k_m)^{2\frac{n+2}{n}} \chi_{A(k_m)} \right)^{\frac{n}{n+2}} \leq \tag{3.5}$$

$$\leq C_5 \int_Q (|x|^\alpha + |t|^\beta)|f|(H - k_m)\chi_{A(k_m)}.$$

Applying generalized Hölder's inequality to the right of (2.14) we obtain

$$\|H - k_m\|_{q,Q} \leq C_6 \|f_{1,2}\|_{\alpha_1,\beta_1,r,Q} \{\psi(k_m)\}^{1-1/q-1/r}; \tag{3.6}$$

here we've denoted:

$$\psi(k_m) = \int_0^T \int_\Omega (|x|^\alpha + |t|^\beta)^q \chi_{A(k_m)},$$

and q is as in Lemma 2.4. Let us estimate:

$$(k_{m+1} - k_m)\{\psi(k_{m+1})\}^{1/q} = (k_{m+1} - k_m) \left(\int_0^T \int_\Omega (|x|^\alpha + |t|^\beta)^q \chi_{A(k_{m+1})} \right)^{1/q} <$$

$$< \left(\int_0^T \int_\Omega (|x|^\alpha + |t|^\beta)^q (H - k_m)^q \chi_{A(km+1)} \right)^{1/q} < \|H - k_m\|_{q,Q},$$

where $k_{m+1} > k_m \geq k_0$. Substituting this into (3.6), we come down to

$$(k_{m+1} - k_m)^q \psi(k_{m+1}) \leq C_7 \{\psi(k_m)\}^{q(1-1/q-1/r)} = C_7 \{\psi(k_m)\}^\delta. \tag{3.7}$$

By the choice of r

$$r > (n+2)/2, \tag{3.1}$$

hence

$$2\frac{(n+2)}{n}\left(1 - \frac{n}{2(n+2)} - \frac{1}{r}\right) > 1; \qquad \text{and thus} \quad \delta > 1.$$

On the strength of Lemma 2.3 from relation (3.7) we can conclude that

$$\psi(k_0 + d) = 0$$

for some d sufficiently large, but finite, depending only on n, f^j, Λ_1, $|g_{1,2}|_{H^{\alpha_g,\beta_g}(S)}$, $|u_0, v_0|_{H^{\alpha_0}(\overline{\Omega}\times\{0\})}$; constants in the embedding theorems and the components of the solution u and v. It should be noted that the constant C_2 in (3.3) can't be infinite since this would imply that $d = \infty$ and, because of the energy estimate, the measure of the set where $(H - k_m)$ is different from zero would be zero and we couldn't get a strict inequality in alternative (3.4), which constitutes a contradiction to our assumption.

So, the constant C_2, along with d are finite for each solution, but dependent yet on the solution.

Now we are in a position to prove the uniform boundedness of H for all of the solutions. To this end we start from the very beginning of the proof and go down to inequality (3.3), maintaining that the latter holds with constant C_2 uniform for all of the solutions (u, v) of the problem. If there is no such constant C_2, then we can for any C_m, whatever large, choose the level k_m and the function H_m with the property $\sup\limits_Q H_m \geq \sup\limits_Q H_{m-1}$ (the function can be chosen such without loss of generality), such that the revers strict inequality takes place. Since all the solutions of the problem are uniformly bounded in $C(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$ due to energy estimate, we can, because of the compact embedding of W into L^2 ($W \hookrightarrow\hookrightarrow L^2$), choose from this sequence of functions $(H_m - k_m)$ a subsequence which converges strongly in L^2 to some function $(\widetilde{H} - (d + k_0))$. From (3.4) it follows that $\widetilde{H} - (d + k_0) \equiv 0$, where d is chosen dependent on C_2. Due to the assumption that $\sup\limits_Q H_m \geq \sup\limits_Q H_{m-1}$ all the other functions H of the solutions are also bounded by $d + k_0$. Repeating then the subsequent part of the proof of the case with nonuniform C_2 we come down to $\sup\limits_Q H \leq d + k_0$ with d dependent on C_2. By the argument of the same kind as in the "nonuniform" case we come to conclusion that C_2 and hence d, can not be infinite: the opposite would lead to contradiction with the strict inequality in alternative similar to (3.4).

Analogously can be proved the estimate for the infimum of H in Q. To this end we must consider the truncated functions $(k - H)_+$ and test the system on them.

Step 2. On this step we shall assume that $(O, 0) \in Q$. As appears quite obvious from geometrical considerations, we have to take $k_0 + d$ from the previous step for k_0 and repeat the whole argument of the previous step.

Considering $H_2 \equiv H$ with $\gamma_2 \equiv \gamma$, thus, finally, we have that

$$||H_1||_\infty \leq C_8, \quad ||H_2||_\infty \leq C_9,$$

where $|| \cdot ||_\infty$ stands for $L_\infty(Q)$ norm.

It is not difficult to resolve these estimates and to obtain that the same estimates hold for the components (w, z) of solution themselves. Indeed,

$$\inf_{\partial Q}(\gamma_1 w + z) \leq \gamma_1 w + z \leq \sup_{\partial Q}(\gamma_1 w + z);$$
$$\inf_{\partial Q}(\gamma_2 w + z) \leq \gamma_2 w + z \leq \sup_{\partial Q}(\gamma_2 w + z).$$

Subtracting the second estimate from the first one, we get

$$|w| \leq \left(|\sup_{\partial Q}(\gamma_1 w + z) - \inf_{\partial Q}(\gamma_2 w + z)| + |\sup_{\partial Q}(\gamma_2 w + z) - \inf_{\partial Q}(\gamma_1 w + z)| \right) / |\omega_2 - A_1|.$$

Hence the estimate for the component z is self-evident. Thus we come down to

$$||w||_\infty \le C_{10}, \quad ||z||_\infty \le C_{11}$$

for a. e. $(x', t') \in Q$. Hence follows the Hölder continuity. □

Remark 5. *In the proof of Theorem 3.2 we have performed local, at each point, diagonalization of system* (2.1)*. It is clear, however, that system can not be reduced to diagonal by changing to unknowns* $\gamma_1 w + z$ *and* $\omega_2 w + z$*, let alone be solved by this substitution.*

Example. The system governing chemotaxis:

$$\begin{cases} u_t - \triangle u - \nabla(u\nabla v) = 0, \\ v_t - D\triangle v = -\beta v + u, \qquad (x,t) \in Q, \end{cases} \tag{3.8}$$

with Dirichlet data, $u, v \ge 0$, $\beta > 1$, $D > 0$. We assume the energy estimate and boundedness of solutions. For this problem the characteristic system (2.3) would be:

$$\begin{cases} \gamma + 0\delta = \Lambda\gamma, \\ \gamma u + D\delta = \Lambda\delta, \end{cases}$$

which is satisfied by $\gamma = (1 - D)$, $\delta = u$, $\Lambda = 1$. We can introduce the function $H(u, v) \equiv \gamma w + \delta z \equiv (1 - D)w + uz$ and in the spirit of the reasoning of the previous section establish the estimate of $L^\infty(Q)$ of function H. The second equation would give us the estimate for z. Hence follow the estimates of $\|w\|_\infty$ and $\|z\|_\infty$.

Remark 6. *This is the example of a triangular system, that is why we have only one function* H*, which is sufficient.*

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