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IM PAN Preprint 712 (2010)

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Published as manuscript

Received 4 February 2010

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Abstract

The Cauchy problem on $[0,\infty[\,\times\mathbb{R}^n$ is considered for systems of PDE with constant coefficients. The spectral condition of I. G. Petrovskiĭ is proved to be necessary and sufficient for existence of a fundamental solution having the form of a convolution semigroup of distributions on $\mathbb{R}^n$ rapidly decreasing in the sense of L. Schwartz.

1 Introduction and main results

Denote by $\mathcal{S}(\mathbb{R}^n)$ the space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$, and by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. If $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and $T\in\mathcal{S}'(\mathbb{R}^n)$ then the convolution $T*\varphi$ makes sense (and is an infinitely differentiable slowly increasing function). Therefore the set

$$\mathcal{O}'_C(\mathbb{R}^n)=\{T\in\mathcal{S}'(\mathbb{R}^n): T*\varphi\in\mathcal{S}(\mathbb{R}^n) \text{ for every } \varphi\in\mathcal{S}(\mathbb{R}^n)\}$$

is well defined. The elements of $\mathcal{O}'_C(\mathbb{R}^n)$ will be called the *rapidly decreasing distributions on* $\mathbb{R}^n$ [1]. For every $T\in\mathcal{O}'_C(\mathbb{R}^n)$ the convolution operator $T*$ is a continuous linear operator from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$, and a continuous linear operator from $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$. Furthermore, $\mathcal{O}'_C(\mathbb{R}^n)$ is a convolution

2010 *Mathematics Subject Classification*: Primary 35E15, 46E10, 47D06; Secondary 42B99.

Key words and phrases: systems of PDE with constant coefficients, Petrovskiĭ condition, rapidly decreasing distributions, slowly increasing functions.

[1]Our definition of $\mathcal{O}'_C(\mathbb{R}^n)$ is equivalent to one given in Sec. VI.5 of [S], p. 244. See Theorem 2.2.

algebra [2]. The topology in $\mathcal{O}'_C(\mathbb{R}^n)$ is induced by the mapping $\mathcal{O}'_C \ni T \mapsto T* \in L_\beta(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$ [3].

Let $m \in \mathbb{N}$, and let M_m be the set of $m\times m$ matrices with complex entries. The above facts referring to classes of scalar functions and distributions remain valid for analogous classes of M_m-valued functions and distributions like $\mathcal{S}(\mathbb{R}^n; M_m)$, $\mathcal{S}'(\mathbb{R}^n; M_m)$, $\mathcal{O}'_C(\mathbb{R}^n; M_m)$.

By a one-parameter *infinitely differentiable convolution semigroup in* $\mathcal{O}'_C(\mathbb{R}^n; M_m)$, briefly *i.d.c.s.*, we mean a mapping

$$(1.1) \qquad [0,\infty[\, \ni t \mapsto S_t \in \mathcal{O}'_C(\mathbb{R}^n; M_m)$$

such that

(1.2) $S_{t+s} = S_t * S_s$ for every $s, t \in [0,\infty[$,

(1.3) $S_0 = \mathbb{1} \otimes \delta$ where $\mathbb{1}$ is the unit $m \times m$ matrix and δ is the Dirac distribution,

(1.4) the map (1.1) is infinitely differentiable.

In (1.4) it is understood that the derivatives at zero are right-side derivatives, and that the topology in $\mathcal{O}'_C(\mathbb{R}^n; M_m)$ is that defined above.

The *infinitesimal generator* of the i.d.c.s. $(S_t)_{t\ge0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_m)$ is defined as the distribution

$$G := \frac{d}{dt} S_t \bigg|_{t=0} \in \mathcal{O}'_C(\mathbb{R}^n; M_m).$$

It follows that

$$\frac{d}{dt} S_t = G * S_t = S_t * G \qquad \text{for every } t \in [0,\infty[.$$

Furthermore, *any i.d.c.s. in* $\mathcal{O}'_C(\mathbb{R}^n; M_m)$ *is uniquely determined by its infinitesimal generator*. Indeed, suppose that a distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_m)$ is the infinitesimal generator of two i.d.c.s. $(S_t)_{t\ge0}, (T_t)_{t\ge0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_m)$. Fix any $t \in\,]0,\infty[$. Using the Banach–Steinhaus theorem, one concludes that the function

$$[0,t] \ni \tau \mapsto S_\tau * T_{t-\tau} \in \mathcal{O}'_C(\mathbb{R}^n; M_m)$$

is infinitely differentiable and $\frac{d}{d\tau}(S_\tau * T_{t-\tau}) = (\frac{d}{d\tau} S_\tau) * T_{t-\tau} + S_\tau * (\frac{d}{d\tau} T_{t-\tau})$. Consequently, $\frac{d}{d\tau}(S_\tau * T_{t-\tau}) = (S_\tau * G) * T_{t-\tau} - S_\tau * (G * T_{t-\tau}) = 0$ for

[2] Due to our definition of $\mathcal{O}'_C(\mathbb{R}^n)$ it is convenient to define convolution in $\mathcal{O}'_C(\mathbb{R}^n)$ imitating Sec. VI.3 of [Y], pp. 158–159.

[3] The subscript β means that $L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$ is equipped with the topology of uniform convergence on bounded subsets of $\mathcal{S}(\mathbb{R}^n)$. By Theorem 2.2 below, the above topology in $\mathcal{O}'_C(\mathbb{R}^n)$ coincides on bounded subsets of $\mathcal{O}'_C(\mathbb{R}^n)$ with the topology defined in Sec. VII.5 of [S], p. 244.

every $\tau \in [0,t]$, so that $S_\tau * T_{t-\tau}$ is independent of τ for $\tau \in [0,t]$, and $S_t = S_\tau * T_{t-\tau}|_{\tau=t} = S_\tau * T_{t-\tau}|_{\tau=0} = T_t$.

Let now $d \in \mathbb{N}$, and suppose that for every multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ of length $|\alpha| = \alpha_1 + \cdots + \alpha_n \le d$ we are given a matrix $A_\alpha \in M_m$. Consider the *matricial differential operator with constant coefficients*

$$P\left(\frac{\partial}{\partial x}\right) := \sum_{|\alpha|\le d} A_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha \tag{1.5}$$

where $(\partial/\partial x)^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, and its *symbol*

$$P(i\xi) := \sum_{|\alpha|\le d} i^{|\alpha|}\xi^\alpha A_\alpha \in M_m \tag{1.6}$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. Thus the symbol is an $m \times m$ matrix whose entries are polynomials on $\mathbb{R}^n$ with complex coefficients. The *Petrovskiĭ index* $\omega_0(P)$ of the differential operator $P(\partial/\partial x)$ is defined to be

$$\begin{aligned} \omega_0(P) &= \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(P(i\xi)),\ \xi \in \mathbb{R}^n\} \\ &= \sup\{\operatorname{Re}\lambda : \lambda \in \mathbb{C},\ \xi \in \mathbb{R}^n, \det(\lambda\mathbb{1} - P(i\xi)) = 0\} \end{aligned} \tag{1.7}$$

where $\sigma(B)$ denotes the spectrum of the matrix $B \in M_m$.

Our aim is to prove

Theorem 1.1. *For every matricial differential operator with constant coefficients of the form* (1.5) *the following two conditions are equivalent:*

(1.8) $\omega_0(P) < \infty$,

(1.9) *the M_m-valued distribution* $P(\partial/\partial x)\delta := \sum_{|\alpha|\le d} A_\alpha \otimes (\partial/\partial x)^\alpha \delta$ *is the infinitesimal generator of an i.d.c.s.* $(S_t)_{t\ge 0} \subset \mathcal{O}_C'(\mathbb{R}^n; M_m)$.

Furthermore, if these equivalent conditions are satisfied, then

(1.10) $\omega_0(P) = \inf\{\omega \in \mathbb{R} : (e^{-\omega t} S_t\, *)_{t\ge 0}$ *is an equicontinuous semigroup of operators on* $\mathcal{S}(\mathbb{R}^n; M_m)\}$.

In Theorem 1.1 the Petrovskiĭ condition (1.8) plays an independent role. But most frequently (1.8) occurs as part of the Gårding assumptions of hyperbolicity for the non-characteristic Cauchy problem. The relation between Theorem 1.1 and the hyperbolic situation may by elucidated by the following result whose proof is omitted in the present paper [4].

[4] The proof of Theorem 1.2 is based on Lemma 2.8 from [G], the Paley–Wiener–Schwartz theorem about Fourier transforms of compactly supported distributions, and the non-uniqueness theorem for the characteristic Cauchy problem.

Theorem 1.2. *Suppose that the condition* (1.8) *is satisfied, let* $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_m)$ *be the i.d.c.s. generated by the* M_m*-valued distribution* $P(\partial/\partial x)\delta$*, and let*

$$\det(\lambda \mathbb{1} - P(\xi_1,\dots,\xi_n)) = \lambda^m + q_{m-1}(\xi_1,\dots,\xi_m)\lambda^{m-1} + \dots + q_1(\xi_1,\dots,\xi_n)\lambda + q_0(\xi_1,\dots,\xi_n).$$

Then

(1.11) *there is* $r \in]0,\infty[$ *such that* $\max\{|x| : x \in \operatorname{supp} S_t\} \leq rt$ *for every* $t \in [0,\infty[$

if and only if

(1.12) *for every* $k = 0,\dots,m-1$ *the degree of the polynomial* $q_k(\zeta_1,\dots,\zeta_n)$ *is no greater than* $m-k$.

The condition (1.12) may be equivalently expressed by saying that

(1.12)′ the vector $(1,0,\dots,0) \in \mathbb{R}^{n+1}$ is not characteristic for the polynomial $q(\lambda,\zeta_1,\dots,\zeta_n)$,

i.e. $p(1,0,\dots,0) \neq 0$ where $p(\lambda,\zeta_1,\dots,\zeta_n)$ is the main homogeneous part of $q(\lambda,\zeta_1,\dots,\zeta_n)$. In the terminology of [G], conditions (1.8) and (1.12)′ together mean that the polynomial $q(\lambda,\zeta_1,\dots,\zeta_n)$ is hyperbolic with respect to the vector $(1,0,\dots,0) \in \mathbb{R}^{n+1}$. The hyperbolicity of $q(\lambda,\zeta_1,\dots,\zeta_n)$ with respect to $(1,0,\dots,0)$ implies its hyperbolicity with respect to $(-1,0,\dots,0)$. Therefore if the M_m-valued distribution $P(\partial/\partial x)\delta$ is the generator of an i.d.c.s. satisfying (1.12), then so also is $-P(\partial/\partial x)\delta$, and hence the i.d.c.s. generated by $P(\partial/\partial x)\delta$ extends to an infinitely differentiable one-parameter convolution group of distributions with compact support.

2 Rapidly decreasing distributions on $\mathbb{R}^n$

Sections 2 and 3 are devoted to a self-contained presentation of some results identical or similar to those stated, in part without proofs, in the book of L. Schwartz [S]. These results constitute a basis for our subsequent arguments, and for this reason we give complete proofs.

Let $\mathcal{D}_{L^1}(\mathbb{R}^n)$ be the space of infinitely differentiable complex functions φ on $\mathbb{R}^n$ such that $(\partial/\partial x)^\alpha\varphi \in L^1(\mathbb{R}^n)$ for every $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$. The topology in $\mathcal{D}_{L^1}(\mathbb{R}^n)$ is determined by the system of seminorms $p_\alpha(\varphi) = \int_{\mathbb{R}^n} |(\partial/\partial x)^\alpha \varphi(x)|\,dx$, $\alpha \in \mathbb{N}_0^n$, $\varphi \in \mathcal{D}_{L^1}(\mathbb{R}^n)$. $\mathcal{D}_{L^1}(\mathbb{R}^n)$ is a Fréchet space, and $\mathcal{D}(\mathbb{R}^n)$ is densely and continuously imbedded in $\mathcal{D}_{L^1}(\mathbb{R}^n)$. We say that a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is *bounded on* $\mathbb{R}^n$ if it extends to a linear functional continuous on $\mathcal{D}_{L^1}(\mathbb{R}^n)$. The set of bounded distributions on $\mathbb{R}^n$ is denoted by $\mathcal{B}'(\mathbb{R}^n)$.

Let $C_b(\mathbb{R}^n)$ be the Banach space of complex continuous bounded functions on $\mathbb{R}^n$. In the present section we will base on the following result contained in [S].

Theorem 2.1. *For any family $\mathcal{B}' \subset \mathcal{D}'(\mathbb{R}^n)$ the following three conditions are equivalent:*

(2.1) *there are $m \in \mathbb{N}_0$ and a bounded subset $\{f_{T,\alpha} : T \in \mathcal{B}',\ \alpha \in \mathbb{N}_0^n,\ |\alpha| \le m\}$ of $C_b(\mathbb{R}^n)$ such that*

$$T = \sum_{|\alpha|\le m} \left(\frac{\partial}{\partial x}\right)^{\alpha} f_{T,\alpha} \quad \text{for every } T \in \mathcal{B}',$$

(2.2) *$\mathcal{B}' \subset \mathcal{B}'(\mathbb{R}^n)$ and the distributions belonging to $\mathcal{B}'$ are equicontinuous with respect to the topology of $\mathcal{D}_{L^1}(\mathbb{R}^n)$,*

(2.3) *whenever $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then $\{T * \varphi : T \in \mathcal{B}'\}$ is a bounded subset of $C_b(\mathbb{R}^n)$.*

Proof. The implications (2.1)⇒(2.2)⇒(2.3) follow at once from two facts:

(i) whenever (2.1) holds, then

$$T(\varphi) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} \int_{\mathbb{R}^n} f_{T,\alpha}(x) \left(\frac{\partial}{\partial x}\right)^{\alpha} \varphi(x)\, dx$$

for every $T \in \mathcal{B}'$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

(ii) $(T * \varphi)(x) = \langle T, \varphi(x - \cdot)\rangle$ for every $T \in \mathcal{B}'$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

The implication (2.3)⇒(2.1) is proved by a more refined argument similar to one on p. 196 of [S]. Let the subscript x denote translation by x, and superscript $\vee$ the reflection. Suppose that (2.3) holds. Since $(T * \varphi)(x) = \langle (T_x)^{\vee}, \varphi\rangle$, (2.3) implies that $\{(T_x)^{\vee} : T \in \mathcal{B}',\ x \in \mathbb{R}^n\}$ is a pointwise bounded family of continuous linear functionals on $\mathcal{D}(\mathbb{R}^n)$. Since $\mathcal{D}(\mathbb{R}^n)$ is a barrelled space, the Banach–Steinhaus theorem implies that this family is equicontinuous. Let $K = \{y \in \mathbb{R}^n : |y| \le 1\}$. Equicontinuity of $\{(T_x)^{\vee} : T \in \mathcal{B}',\ x \in \mathbb{R}^n\}$ implies that there are $k \in \mathbb{N}_0$ and $C \in\ ]0, \infty[$ such that whenever $\varphi \in C_K^{\infty}(\mathbb{R}^n)$, $T \in \mathcal{B}'$ and $x \in \mathbb{R}^n$, then

$$|(T * \varphi)(x)| = |\langle (T_x)^{\vee}, \varphi\rangle| \le C \sup\left\{ \left| \left(\frac{\partial}{\partial y}\right)^{\alpha} \varphi(y) \right| : |\alpha| \le k,\ y \in K \right\}.$$

This estimate implies that whenever $\phi \in C^k_K(\mathbb{R}^n) \subset \mathcal{E}'(\mathbb{R}^n)$, then for every $T \in \mathcal{B}'$ the distribution $\phi * T$ is a function belonging to $C_b(\mathbb{R}^n)$, and

(2.4) $\{\phi * T : T \in \mathcal{B}'\}$ is a bounded subset of $C_b(\mathbb{R}^n)$.

If $l \in \mathbb{N}$ is sufficiently large and E is the fundamental solution for Δ^l depending only on $|x|$, then $E \in C^k(\mathbb{R}^n)$ and $E|_{\mathbb{R}^n\setminus\{0\}} \in C^\infty(\mathbb{R}^n \setminus \{0\})$ [5]. Let $\gamma \in C^\infty_K(\mathbb{R}^n)$ be such that $\gamma(x) = 1$ whenever $|x| \le 1/2$. Then $\gamma E \in C^k_K(\mathbb{R}^n)$, $(1-\gamma)E \in C^\infty(\mathbb{R}^n)$, and $\Delta^l[(1-\gamma)E] \in C^\infty_K(\mathbb{R}^n)$. For every $T \in \mathcal{B}'$ one has

$$T = \Delta^l \delta * E * T = \Delta^l[(\gamma E) * T] + [\Delta^l((1-\gamma)E)] * T = \Delta^l f_T + g_T$$

where

$$f_T = (\gamma E) * T \quad \text{and} \quad g_T = \Delta^l((1-\gamma)E) * T.$$

Furthermore, $\{f_T : T \in \mathcal{B}'\}$ and $\{g_T : T \in \mathcal{B}'\}$ are bounded subsets of $C_b(\mathbb{R}^n)$, by (2.4) and (2.3) respectively. Hence (2.3) implies (2.1).

Theorem 2.2. *For every family of distributions $\mathcal{F}' \subset \mathcal{D}'(\mathbb{R}^n)$ the following three conditions are equivalent:*

(2.5) *for every polynomial $P(x_1, \ldots, x_n)$ the family of distributions $\{P \cdot T : T \in \mathcal{F}'\}$ is a subset of $\mathcal{B}'(\mathbb{R}^n)$ equicontinuous with respect to the topology of $\mathcal{D}_{L^1}(\mathbb{R}^n)$,*

(2.6) *there is a sequence $(m_k)_{k\in\mathbb{N}_0} \subset \mathbb{N}_0$ and a mapping*

$$\mathbb{N}_0 \times \mathcal{F}' \ni (k, T) \mapsto \{f_{T,k,\alpha} : \alpha \in \mathbb{N}_0^n,\ |\alpha| \le m_k\} \subset C_b(\mathbb{R}^n)$$

such that

$$T = \sum_{|\alpha|\le m_k} \left(\frac{\partial}{\partial x}\right)^\alpha f_{T,k,\alpha} \quad \textit{whenever } (k,T) \in \mathbb{N}_0 \times \mathcal{F}'$$

and

$$\sup\{(1+|x|)^k |f_{T,k,\alpha}| : T \in \mathcal{F}',\ |\alpha| \le m_k,\ x \in \mathbb{R}^n\} < \infty$$

for every $k \in \mathbb{N}_0$,

(2.7) *$\mathcal{F}' \subset \mathcal{O}_C(\mathbb{R}^n)$ and the set $\{T* : T \in \mathcal{F}'\} \subset L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ is equicontinuous.*

[5] See Sec. VII.10 of [S], Example 2, p. 288.

If $\mathcal{F}'$ contains only one distribution T, then, in accordance with the definifion of $\mathcal{O}'_C(\mathbb{R}^n)$ given in Section 1, each of the conditions (2.5)–(2.7) means that T rapidly decreases at infinity. Such a definition is equivalent to one in [S], Sec. VII.5, p. 244 [6]. The equivalence (2.6)⇔(2.7) is fundamental for the proof of Theorem 1.1.

Proof of (2.5)⇒(2.6). Let $r^2 \in C^\infty(\mathbb{R}^n)$ be the function such that $r^2(x) = |x|^2 = x_1^2 + \cdots + x_n^2$ for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then

$$(2.8) \qquad (1+r^2)^{\frac{1}{2}|\alpha|+a}\left(\frac{\partial}{\partial x}\right)^{\alpha}((1+r^2)^{-a}) \in C_b(\mathbb{R}^n) \qquad \text{for all } a \in \,]0,\infty[\text{ and } \alpha \in \mathbb{N}_0^n,$$

because $(\partial/\partial x)^\alpha((1+r^2)^{-a}) = (1+r^2)^{-a-|\alpha|}P_\alpha$ where P_α is a polynomial on $\mathbb{R}^n$ of degree no greater than $|\alpha|$. Suppose that (2.5) is satisfied. Fix $k \in \mathbb{N}_0$. By the implication (2.2)⇒(2.1) from Theorem 2.1, there is $m_k \in \mathbb{N}_0$ and for every $T \in \mathcal{F}'$ and $\beta \in \mathbb{N}_0^n$ such that $|\beta| \le m_k$ there is $g_{T,k,\beta} \in C_b(\mathbb{R}^n)$ such that

$$T = (1+r^2)^{-k}\sum_{|\beta|\le m_k}\left(\frac{\partial}{\partial x}\right)^{\beta} g_{T,k,\beta}$$

and

$$(2.9) \qquad \sup\{|g_{T,k,\beta}(x)| : T \in \mathcal{F}',\ |\beta| \le m_k,\ x \in \mathbb{R}^n\} < \infty.$$

It follows that

$$T = \sum_{|\alpha|\le m_k}\left(\frac{\partial}{\partial x}\right)^{\alpha} f_{T,k,\alpha}$$

where

$$f_{T,k,\alpha} = \sum_{\alpha\le\beta,\,|\beta|\le m_k}(-1)^{|\beta-\alpha|}\binom{\beta}{\alpha} g_{T,k,\beta}\left(\frac{\partial}{\partial x}\right)^{\beta-\alpha}(1+r^2)^{-k}.$$

By (2.8) and (2.9), one has

$$\sup\{(1+|x|)^{2k}|f_{T,k,\alpha}(x)| : T \in \mathcal{F}',\ |\alpha| \le m_k,\ x \in \mathbb{R}^n\} < \infty.$$

[6] For $m = 1$ and $\mathcal{F}'$ consisting of a single T the equivalence (2.5)⇔(2.6) follows from Theorem IX in Sec. VII.5 of [S], p. 244, stated there with just an indication of the method of proof.

Proof of (2.6)⇒(2.7). Suppose that (2.6) holds. By the Banach–Steinhaus theorem, (2.7) will follow once it is proved that whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\{T * \varphi : T \in \mathcal{F}'\}$ is a bounded subset of $\mathcal{S}(\mathbb{R}^n)$. Since $(\partial/\partial x)^\alpha (T * \varphi) = T * ((\partial/\partial x)^\alpha \varphi)$, it is sufficient to show that

$$\sup\left\{\left(1+\frac{1}{2}|x|\right)^k |(T*\varphi)(x)| : T \in \mathcal{F}',\, x \in \mathbb{R}^n\right\} < \infty$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $k = n+1, n+2, \ldots$. So, fix any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $k = n+1, n+2, \ldots$. Then, by (2.6), for every $T \in \mathcal{F}'$ and $x \in \mathbb{R}^n$ one has

$$\begin{aligned}
&|(T*\varphi)(x)| \\
&\quad \le \sum_{|\alpha|\le m_k} \left(\int_{|y|\ge\frac{1}{2}|x|} + \int_{|x-y|\ge\frac{1}{2}|x|}\right) |f_{T,k,\alpha}(y)| \cdot \left|\left(\frac{\partial}{\partial x}\right)^\alpha \varphi(x-y)\right| dy \\
&\quad \le \left(1+\frac{1}{2}|x|\right)^{-k} \sum_{|\alpha|\le m_k} \left(C_k \int_{\mathbb{R}^n} \left|\left(\frac{\partial}{\partial y}\right)^\alpha \varphi(y)\right| dy + D_k \int_{\mathbb{R}^n} |f_{T,k,\alpha}(y)|\, dy\right) \\
&\quad \le \left(1+\frac{1}{2}|x|\right)^{-k} (\#\{\alpha \in \mathbb{N}_0^n : |\alpha| \le m_k\}) 2 C_k D_k \int_{\mathbb{R}^n} (1+|y|)^{-k}\, dy,
\end{aligned}$$

where

$$\begin{aligned}
C_k &= \sup\{(1+|y|)^k |f_{T,k,\alpha}(y)| : T \in \mathcal{F}',\, |\alpha| \le m_k,\, y \in \mathbb{R}^n\} < \infty, \\
D_k &= \sup\left\{(1+|y|)^k \left|\left(\frac{\partial}{\partial y}\right)^\alpha \varphi(y)\right| : |\alpha| \le m_k,\, y \in \mathbb{R}^n\right\} < \infty.
\end{aligned}$$

Proof of (2.7)⇒(2.5). Suppose that (2.7) holds. It is sufficient to prove (2.5) for the monomials $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. We will prove that whenever $\alpha \in \mathbb{N}_0^n$ and $\varphi \in \mathcal{D}(\mathbb{R})$, then

(2.10) $\{(x^\alpha T) * \varphi : T \in \mathcal{F}'\}$ is a bounded subset of $\mathcal{S}(\mathbb{R}^n)$.

From (2.10) it follows that whenever $\alpha \in \mathbb{N}_0^n$ and $\varphi \in \mathcal{D}(\mathbb{R})$, then $\{(x^\alpha T) * \varphi : T \in \mathcal{F}'\}$ is a bounded subset of $C_b(\mathbb{R}^n)$, whence, by the implication (2.3)⇒(2.2) of Theorem 2.1, condition (2.5) holds for the monomials x^α.

We will prove (2.10) by induction on $|\alpha|$. If $|\alpha| = 0$, then $x^\alpha \equiv 1$ and (2.10) is a direct consequence of (2.7). Furthermore, the condition

$(2.10)_m$ *the statement* (2.10) *holds for every* $\varphi \in \mathcal{D}(\mathbb{R}^n)$ *and* $\alpha \in \mathbb{N}_0^n$ *such that* $|\alpha| = m$

implies $(2.10)_{m+1}$. Indeed, if $\alpha = (\alpha_1, \dots, \alpha_n)$ and $|\alpha| = m+1$, then $\alpha_i \geq 1$ for some $i = 1, \dots, n$, so that $\alpha = \beta + \gamma$ where $\beta \in \mathbb{N}_0^n$, $|\beta| = m$ and $\gamma = (\delta_{i,1}, \dots, \delta_{i,n})$. Consequently, $x^\alpha = x_i x^\beta$ and

$$(2.11) \qquad (x^\alpha T) * \varphi = (x_i x^\beta T) * \varphi = x_i((x^\beta T) * \varphi) - (x^\beta T) * (x_i \varphi)$$

for every $T \in \mathcal{F}'$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is fixed then, by $(2.10)_m$, $\{(x^\beta T) * \varphi : T \in \mathcal{F}'\}$ and $\{(x^\beta T) * (x_i \varphi) : T \in \mathcal{F}'\}$ are bounded subsets of $\mathcal{S}(\mathbb{R}^n)$, whence, by (2.11), so is $\{(x^\alpha T) * \varphi : T \in \mathcal{F}'\}$.

3 Infinitely differentiable slowly increasing functions on $\mathbb{R}^n$

A function $\phi \in C(\mathbb{R}^n)$ is called *continuous slowly increasing* if

$$\sup\{(1+|\xi|)^{-m}|\phi(\xi)| : \xi \in \mathbb{R}^n\} < \infty$$

for some $m \in \mathbb{N}_0$. A function $\phi \in C^\infty(\mathbb{R}^n)$ is called *infinitely differentiable slowly increasing* [7] if for every $k \in \mathbb{N}_0$ there is $m_k \in \mathbb{N}_0$ such that

$$\sup\left\{(1+|\xi|)^{-m_k}\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha \phi(\xi)\right| : \alpha \in \mathbb{N}_0^n,\ |\alpha| \leq k,\ \xi \in \mathbb{R}^n\right\} < \infty.$$

The set of infinitely differentiable slowly increasing functions on $\mathbb{R}$ is denoted by $\mathcal{O}_M(\mathbb{R}^n)$. One has $\mathcal{O}_M(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. We will say that the functions belonging to a subset Φ of $\mathcal{O}_M(\mathbb{R}^n)$ are *uniformly slowly increasing* if for every $k \in \mathbb{N}_0$ there is $m_k \in \mathbb{N}_0$ such that

$$\sup\left\{(1+|\xi|)^{-m_k}\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha \phi(\xi)\right| : \phi \in \Phi,\ \alpha \in \mathbb{N}_0^n,\ |\alpha| \leq k,\ \xi \in \mathbb{R}^n\right\} < \infty.$$

Let $\mathfrak{F}$ denote the Fourier transformation defined by

$$\hat{\varphi}(\xi) = (\mathfrak{F}\varphi)(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi}\varphi(x)\,dx$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. Then $\mathfrak{F}$ is a continuous automorphism of $\mathcal{S}(\mathbb{R}^n)$, and it extends uniquely to a continuous automorphism of $\mathcal{S}'_b(\mathbb{R}^n)$.

Theorem 3.1. [8] *$\mathfrak{F}\mathcal{O}'_C(\mathbb{R}^n) = \mathcal{O}_M(\mathbb{R}^n)$.*

[7] Infinitely differentiable slowly increasing functions play a fundamental role in Petrovskiĭ's paper [P] devoted to the Cauchy problem for systems of PDE whose coefficients are either constant or depend only on time. See [P], Bedingung A, p. 3, and Lemmas 1 and 2, pp. 7–8.

[8] Theorem 3.1 is contained in Theorem XV of Sec. VII.8 of [S], p. 268. We present a proof based directly on Theorem 2.2. The second part of our proof differs from that in [S].

Proof of $\mathcal{F}\mathcal{O}'_C(\mathbb{R}^n) \subset \mathcal{O}_M(\mathbb{R}^n)$. Suppose that $T \in \mathcal{O}'_C(\mathbb{R}^n)$. Fix $k \in \mathbb{N}$ such that $k > n$. Then, by (2.6), one has

$$\mathcal{F}T = \sum_{|\alpha| \le m_k} (i\xi)^\alpha \mathcal{F} f_{T,k,\alpha}$$

where $f_{T,k,\alpha} \in L^1(\mathbb{R}^n)$ and so $\mathcal{F} f_{T,k,\alpha} \in C_0(\mathbb{R}^n)$. Consequently $\mathcal{F}T$ is a continuous slowly increasing function on $\mathbb{R}^n$. Furthermore,

$$\left(\frac{\partial}{\partial \xi}\right)^\alpha \mathcal{F}T = \mathcal{F}((-ix)^\alpha T) \quad \text{for every } \alpha \in \mathbb{N}_0^n. \tag{3.1}$$

By (2.5) one has $(ix)^\alpha T \in \mathcal{O}'_C(\mathbb{R}^n)$, so that, by what we have already proved, $\mathcal{F}((ix)^\alpha T)$ is a continuous slowly increasing function. Since $\alpha \in \mathbb{N}_0^n$ in (3.1) is arbitrary, it follows that $\mathcal{F}T \in \mathcal{O}_M(\mathbb{R}^n)$.

Proof of $\mathcal{O}_M(\mathbb{R}^n) \subset \mathcal{F}\mathcal{O}'_C(\mathbb{R}^n)$. Pick $\phi \in \mathcal{O}_M(\mathbb{R}^n)$ and set $T = \mathcal{F}^{-1}\phi$. Then $T \in \mathcal{S}'(\mathbb{R}^n)$. Furthermore, whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\mathcal{F}(T * \varphi) = (\mathcal{F}T) \cdot \hat{\varphi} = \phi \cdot \hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$, and hence $T * \varphi \in \mathcal{S}(\mathbb{R}^n)$. It follows that $T \in \mathcal{O}'_C(\mathbb{R}^n)$, and so $\phi = \mathcal{F}T \in \mathcal{F}\mathcal{O}'_C(\mathbb{R}^n)$.

Theorem 3.2. *For any subset Φ of $\mathcal{O}_M(\mathbb{R}^n)$ the following three conditions are equivalent:*

(3.2) *the functions belonging to Φ increase uniformly slowly,*

(3.3) *the set $\{\phi \cdot : \phi \in \Phi\} \subset L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ is equicontinuous,*

(3.4) *the set $\{(\mathcal{F}^{-1}\phi) * : \phi \in \Phi\} \subset L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ is equicontinuous.*

Proof. The implication (3.2)⇒(3.3) is straightforward. If $\phi \in \mathcal{O}_M(\mathbb{R}^n)$, then $\phi \cdot \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$, $(\mathcal{F}^{-1}\phi) * \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ and $(\mathcal{F}^{-1}\phi) * = \mathcal{F}^{-1} \circ (\phi \cdot)$, so that (3.3)⇔(3.4). The implication (3.4)⇒(3.2) may be proved by an argument based on (2.7)⇒(2.6), similar to one used in the proof of the inclusion $\mathcal{F}\mathcal{O}'_C(\mathbb{R}^n) \subset \mathcal{O}_M(\mathbb{R}^n)$.

Remark 3.1. It is stressed by L. Schwartz that the condition (2.6) is strictly weaker than the statement that

$$T = \sum_{|\alpha| \le m} \left(\frac{\partial}{\partial x}\right)^\alpha f_\alpha \tag{3.5}$$

for some $m \in \mathbb{N}$ where all f_α are continuous rapidly decreasing functions on $\mathbb{R}^n$. Indeed, it is easy to see that

$$\text{if } T \text{ satisfies (3.5) and } \phi \in \mathcal{O}_M(\mathbb{R}^n), \text{ then } T * \phi \in \mathcal{O}_M(\mathbb{R}^n). \tag{3.6}$$

However, the example in Sec. VII.5 of [S], p. 245, shows that if $n = 1$ and $\phi_0(x) = e^{ix^2/2}$ for $x \in \mathbb{R}$, then $\phi_0 \in \mathcal{O}_M(\mathbb{R})$, and if T_0 is a distribution equal to the function ϕ_0, then T_0 satisfies (2.6) with $\mathcal{F}' = \{T_0\}$, so that $T_0 \in \mathcal{O}'_C(\mathbb{R})$. If T_0 were to satisfy (3.5), then, by (3.6), one would have $T_0 * \overline{\phi}_0 \in \mathcal{O}_M(\mathbb{R})$. But this does not hold because $\mathfrak{F}T_0 = \mathfrak{F}\phi_0 = c\overline{T}_0 = c\overline{\phi}_0$ where $c \in \mathbb{C} \setminus \{0\}$ is a constant, so that $\mathfrak{F}\overline{\phi}_0 = \overline{\mathfrak{F}\phi_0}^{\vee} = \overline{c}\phi_0$, $\mathfrak{F}(T_0 * \overline{\phi}_0) = \mathfrak{F}T_0 \cdot \mathfrak{F}\overline{\phi}_0 = |c|^2$, and $T_0 * \overline{\phi}_0 = |c|^2\delta$. Hence T_0 cannot be represented in the form (3.5).

Note that $T_0 = \phi_0 \in \mathcal{O}_M(\mathbb{R}) \cap \mathcal{O}'_C(\mathbb{R})$ differs only by a multiplicative constant from a member of the infinitely differentiable convolution group in $\mathcal{O}'_C(\mathbb{R})$ whose infinitesimal generator is equal to $i\delta''$. This group is related to the Schrödinger equation. See [R], Sec. 3.2–3.4 and 4.4.

Remark 3.2. Whenever $\phi \in C^\infty(\mathbb{R}^n)$, then

$$\begin{aligned}\phi \in \mathcal{O}_M(\mathbb{R}^n) \;&\Rightarrow\; \phi\cdot \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))\\ &\Rightarrow\; \phi \text{ is a multiplier for } \mathcal{S}'(\mathbb{R}^n) \;\Rightarrow\; \phi \in \mathcal{O}_M(\mathbb{R}^n).\end{aligned}$$

Here the only non-trivial implication is the last one, resulting from Theorem VI of Sec. VII.4 of [S], p. 239, and stated in Sec. VII.5 of [S], after Theorem X, p. 246. However, *an element of* $\mathcal{O}_M(\mathbb{R}^n)$ *may not be a multiplier for* $\mathcal{O}'_C(\mathbb{R}^n)$. Indeed, if $n = 1$ and, as in Remark 3.1, $\phi_0(x) = e^{ix^2/2}$ for $x \in \mathbb{R}$, and T_0 is the same function treated as a distribution on $\mathbb{R}$, then $\overline{\phi}_0 \in \mathcal{O}_M(\mathbb{R})$, $T_0 \in \mathcal{O}'_C(\mathbb{R})$, and $\overline{\phi}_0 \cdot T_0 \in \mathcal{S}'(\mathbb{R})$ is a function identically equal to one. Therefore $\overline{\phi}_0 \cdot T_0 \notin \mathcal{O}'_C(\mathbb{R})$ and $\overline{\phi}_0$ is not a multiplier for $\mathcal{O}'_C(\mathbb{R})$.

In the following we will consider functions and distributions on $\mathbb{R}^n$ with values in the space M_m of complex $m \times m$ matrices. In this setting the theorems proved earlier for the scalar case remain valid.

Consider the matricial differential operator $P(\partial/\partial x)$ defined by (1.5), and its symbol $P(i\xi)$ defined by (1.6). As $\frac{d}{dt}\exp(tP(i\xi)) = P(i\xi)\exp(tP(i\xi))$, the theorem about differentiation of solutions of ODE with respect to parameters implies that the mapping $\mathbb{R}^{1+n} \ni (t,\xi) \mapsto \exp(tP(i\xi)) \in M_m$ is infinitely differentiable. Therefore, for any $t \in \mathbb{R}$, the formula

$$\phi_t(\xi) := \exp(tP(i\xi)), \qquad \xi \in \mathbb{R}^n, \tag{3.7}$$

defines a function $\phi_t \in C^\infty(\mathbb{R}^n; M_m)$.

Theorem 3.3. *The condition* (1.9) *from Theorem* 1.1 *is satisfied if and only if*

(3.8) *$\phi_t \in \mathcal{O}_M(\mathbb{R}^n; M_m)$ for every $t \in [0,\infty[$, and the functions in $\{\phi_t : t \in [0,T]\}$ increase uniformly slowly for each $T \in]0,\infty[$.*

Furthermore, if the equivalent conditions (1.9) *and* (3.8) *are satisfied, then $\mathcal{F}S_t = \phi_t$ for all $t \in [0,\infty[$, and, for each fixed $\omega \in \mathbb{R}$,*

(3.9) *the semigroup of convolution operators $(e^{-\omega t}S_t\, *)_{t\geq 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m); \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ is equicontinuous*

if and only if

(3.10) *the functions in $\{e^{-\omega t}\phi_t : t \in [0,\infty[\}$ increase uniformly slowly.*

Proof of (1.9)⇒(3.8). If (1.9) is satisfied, then $((\mathcal{F}S_t)\,\cdot)_{t\geq 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m); \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ is a one-parameter semigroup of multiplication operators such that for every $\phi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ the trajectory $[0,\infty[\ \ni t \mapsto (\mathcal{F}S_t)\cdot\varphi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ is infinitely differentiable. The infinitesimal generator of this semigroup is multiplication by the function

$$G : \mathbb{R}^n \ni \xi \mapsto P(i\xi) \in M_m \tag{3.11}$$

which belongs to $\mathcal{O}_M(\mathbb{R}^n; M_m)$. Consequently, whenever $\varphi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ and $t \in [0,\infty[$, then $\frac{d}{dt}(\mathcal{F}S_t)\cdot\varphi = G\cdot(\mathcal{F}S_t)\cdot\varphi$, with differentiation in the topology of $\mathcal{S}(\mathbb{R}^n; M_m)$. It follows that $\frac{d}{dt}(\mathcal{F}S_t)(\xi) = P(i\xi)(\mathcal{F}S_t)(\xi)$ for every $(t,\xi) \in [0,\infty[\times\mathbb{R}^n$. Furthermore, $(\mathcal{F}S_0)(\xi) = (\mathcal{F}\delta)(\xi) = 1$ for every $\xi \in \mathbb{R}^n$. The last two properties imply that

$$(\mathcal{F}S_t)(\xi) = \exp(tP(i\xi)) \quad \text{for every } (t,\xi) \in [0,\infty[\times\mathbb{R}^n. \tag{3.12}$$

Now the implication (1.9)⇒(3.8) is an easy consequence of (3.7), (3.12), Theorem 3.1, the Banach–Steinhaus theorem, and Theorem 3.2.

Proof of (3.8)⇒(1.9). If (3.8) is satisfied, then the multiplication operators $\phi_t\,\cdot$ constitute a one-parameter semigroup

$$(\phi_t\,\cdot)_{t\geq 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m); \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)). \tag{3.13}$$

Since the mapping $\mathbb{R}^{1+n} \ni (t,\xi) \mapsto \phi_t(\xi) = \exp(tP(i\xi)) \in M_m$ is infinitely differentiable, from (3.8) it follows that for every $\varphi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ the trajectory $[0,\infty[\ \ni t \mapsto \phi_t\cdot\varphi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ of the semigroup (3.13) is infinitely differentiable. The infinitesimal generator of the semigroup (3.13) is the multiplication operator $\frac{d}{dt}(\phi_t\,\cdot)|_{t=0} = G\,\cdot$ where $G \in \mathcal{O}_M(\mathbb{R}^n; M_m)$ is defined by (3.11). Consequently, by Theorem 3.1,

$$((\mathcal{F}^{-1}\phi_t)\,*)_{t\geq 0} = (\mathcal{F}^{-1}\circ(\phi_t\,\cdot)\circ\mathcal{F})_{t\geq 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m); \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$$

is a semigroup with infinitely differentiable trajectories and infinitesimal generator $(\mathcal{F}^{-1}G)* = (P(\partial/\partial x)\delta)*$. By the Banach–Steinhaus theorem, the mapping $[0,\infty[\ \ni t \mapsto \mathcal{F}^{-1}\phi_t \in \mathcal{O}'_C(\mathbb{R}^n; M_m)$ is infinitely differentiable in the topology of $\mathcal{O}'_C(\mathbb{R}^n; M_m)$, i.e. the topology induced from $L_\beta(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m); \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$.

Proof of (3.9)⇔(3.10). We have already proved that (1.9) implies (3.12). Therefore the relation (1.9)⇒[(3.9)⇔(3.10)] is a consequence of Theorem 3.2.

4 Proof of Theorem 1.1

A. Necessity of the Petrovskiĭ condition

Proof of (1.9)⇒(1.8). By Theorem 3.3, instead of showing that (1.9) implies the Petrovskiĭ condition (1.8), it is sufficient to prove that (3.8) implies (1.8). Thus assume that (3.8) holds. Then the mapping $\mathbb{R}^n \ni \xi \mapsto \phi_1(\xi) = \exp(P(i\xi)) \in M_m$ belongs to $\mathcal{O}_M(\mathbb{R}^n; M_m)$. For any $\xi \in \mathbb{R}^n$,

$$\rho(\xi) := \exp(\max\{\operatorname{Re}\lambda : \lambda \in \sigma(P(i\xi))\})$$

is equal to the spectral radius of the matrix $\phi_1(\xi)$. Hence $\rho(\xi)$ is no greater than $\|\phi_1(\xi)\|_{M_m} = \max\{\|\phi_1(\xi)z\|_{\mathbb{C}^m} : z \in \mathbb{C}^m, \|z\|_{\mathbb{C}^m} \le 1\}$. Since $\phi_1 \in \mathcal{O}_M(\mathbb{R}^n; M_m)$, it follows that there are $C \in\ ]0,\infty[$ and $k \in \mathbb{R}$ such that

$$\rho(\xi) \le C(1+|\xi|)^k \quad \text{for every } \xi \in \mathbb{R}^n,$$

or, what is the same,

$$\text{(4.1)}\quad \max\{\operatorname{Re}\lambda : \lambda \in \sigma(P(i\xi))\} \le \log C + k\log(1+|\xi|) \quad \text{for every } \xi \in \mathbb{R}^n.$$

By the Lemma of L. Gårding from [G], p. 11, the inequality (4.1) implies that

$$\text{(4.2)}\quad \text{the function } \mathbb{R}^n \ni \xi \mapsto \max\{\operatorname{Re}\lambda : \lambda \in \sigma(P(i\xi))\} \in \mathbb{R} \text{ is bounded,}$$

which means that (1.8) is satisfied [9].

[9]The implication (4.1)⇒(4.2) was conjectured by I. G. Petrovskiĭ in [P], footnote on p. 24. L. Hörmander observed that, replacing part of Gårding's proof by a direct argument based on the projection theorem for semi-algebraic subsets of $\mathbb{R}^n$, one can obtain still another result having important applications to PDE. See the Appendix to [H].

B. Sufficiency of the Petrovskiĭ condition

We will base on the inequality

(4.3) if $A \in M_m$, $\omega_A = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ and $t \in [0,\infty[$, then
$$\|\exp(tA)\|_{L(\mathbb{C}^m)} \le e^{\omega_A t}\Big(1 + \sum_{k=1}^{m-1} \frac{(2t)^k}{k!}\|A\|^k_{L(\mathbb{C}^m)}\Big).$$

This is proved in Sec. II.6.1 of [G-S] by means of the Genocchi–Hermite formula[10] for the divided differences, related to interpolation polynomials of Newton. The same proof of (4.3) is given in Sec. 7.2 of [F].

In the notation of Section 1, let d be the degree of $P(i\xi)$ treated as a polynomial on $\mathbb{R}^n$ with coefficients in M_m.

Theorem 4.1. *For every $\omega \in \mathbb{R}$ the following three conditions are equivalent:*

(4.4) $\omega_0(P) \le \omega$,

(4.5) $\sup\{e^{-(\omega+\varepsilon)t}(1+|\xi|)^{-(m-1)d}\|\exp(tP(i\xi))\|_{L(\mathbb{C}^m)} : t \in [0,\infty[, \xi \in \mathbb{R}^n\} < \infty$ *for every* $\varepsilon > 0$,

(4.6) $\sup\Big\{e^{-(\omega+\varepsilon)t}(1+|\xi|)^{-(md-1)(|\alpha|+1)}\Big\|\Big(\frac{\partial}{\partial x}\Big)^\alpha \exp(tP(i\xi))\Big\|_{L(\mathbb{C}^m)} : t \in [0,\infty[, \xi \in \mathbb{R}^n\Big\} < 0$ *for every* $\alpha \in \mathbb{N}_0^n$ *and every* $\varepsilon > 0$.

Proof of (4.4)⇒(4.5). There is $C \in [1,\infty[$ such that $\|P(i\xi)\|_{L(\mathbb{C}^m)} \le C(1+|\xi|)^d$ for every $\xi \in \mathbb{R}^n$. If (4.4) holds, then by (4.3) for every $t \in [0,\infty[$ and $\xi \in \mathbb{R}^n$ one has

$$\begin{aligned}\|\exp(tP(i\xi))\|_{L(\mathbb{C}^m)} &\le e^{\omega_0(P)t}\Big(1 + \sum_{k=1}^{m-1}\frac{(2t)^k}{k!}C^k(1+|\xi|)^{kd}\Big)\\ &\le e^{\omega_0(P)t} m[(1+2t)C(1+|\xi|)^d]^{m-1},\end{aligned}$$

whence (4.5) follows.

Proof of (4.5)⇒(4.6). Suppose that (4.5) holds. Then (4.6) is satisfied for $|\alpha| = 0$. By induction on $|\alpha|$ we will prove that (4.6) is satisfied for every $|\alpha| \in \mathbb{N}_0^n$. So suppose that (4.6) is satisfied whenever $|\alpha| \le l$, and take any

[10] The formula is attributed to Genocchi and Hermite in Sec. 16 of the Appendix B to [Hig], p. 333.

$\alpha_0 \in \mathbb{N}_0^n$ such that $|\alpha_0| = l+1$. Let

$$U_\alpha(t,\xi) = \left(\frac{\partial}{\partial\xi}\right)^\alpha e^{tP(i\xi)},$$
$$V(t,\xi) = \sum_{\alpha\le\alpha_0,\, |\alpha|\le l} \binom{\alpha_0}{\alpha}\left[\left(\frac{\partial}{\partial\xi}\right)^{\alpha_0-\alpha} P(i\xi)\right] U_\alpha(t,\xi).$$

Then

$$\sup\{e^{-(\omega+\varepsilon)t}(1+|\xi|)^{-(m-1)d}|U_0(t,\xi)| : t\in[0,\infty[,\ \xi\in\mathbb{R}^n\} < \infty \tag{4.7}$$

and

$$\sup\{e^{-(\omega+\varepsilon)t}(1+|\xi|)^{-(md-1)(l+2)+(m-1)d}|V(t,\xi)| : t\in[0,\infty[,\ \xi\in\mathbb{R}^n\} < \infty \tag{4.8}$$

for every $\varepsilon > 0$, because whenever $|\alpha| \le l$, then

$$\begin{aligned}(d-l-1+|\alpha|) + (md-1)(|\alpha|+1) &\le (d-1) + (md-1)(l+1)\\ &= (md-1)(l+2) - (m-1)d.\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial}{\partial t}U_{\alpha_0}(t,\xi) &= \left(\frac{\partial}{\partial\xi}\right)^{\alpha_0}\frac{\partial}{\partial t}U_0(t,\xi) = \left(\frac{\partial}{\partial\xi}\right)^{\alpha_0}[P(i\xi)U_0(t)]\\ &= P(i\xi)U_{\alpha_0}(t,\xi) + V(t,\xi)\end{aligned}$$

and $U_{\alpha_0}(0,\xi) = 0$, it follows that

$$U_{\alpha_0}(t,\xi) = \int_0^t U_0(t-\tau,\xi)V(\tau,\xi)\,d\tau. \tag{4.9}$$

From (4.7)–(4.9) it follows that

$$\sup\{e^{-(\omega+\varepsilon)t}(1+|\xi|)^{-(md-1)(l+2)}|U_{\alpha_0}(t,\xi)| : t\in[0,\infty[,\ \xi\in\mathbb{R}^n\} < \infty$$

for every $\varepsilon > 0$, so that (4.6) holds whenever $|\alpha| \le l+1$.

Proof of (4.6)⇒(4.4). If (4.6) holds, then, taking $\alpha = 0$, one concludes that

$$\max\{\operatorname{Re}\lambda : \lambda\in\sigma(P(i\xi))\} = \lim_{t\to\infty}\frac{1}{t}\log\|\exp(tP(i\xi))\|_{L(\mathbb{C}^m)} \le \omega$$

for every $\xi\in\mathbb{R}^n$, whence $\omega_0(P)\le\omega$. See [E-N], Sec. IV.2, Corollary 2.4, p. 252.

Now we are in a position to complete the proof of Theorem 1.1, i.e. to prove the implication (1.8)⇒(1.9) and the equality (1.10). Indeed, if (1.8) holds, then, by Theorem 4.1, $\omega_0(P)$ is equal to the infimum of the numbers ω such that the functions

$$\phi_t : \mathbb{R}^n \ni \xi \mapsto e^{-\omega t} \exp(tP(i\xi)) \in M_m$$

increase uniformly slowly for t ranging over $[0, \infty[$. By Theorem 3.3, this in turn implies that the distributions $S_t := \mathcal{F}^{-1}\phi_t$ constitute an i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n;\, M_m)$ satisfying (1.10) and having the infinitesimal generator $P(\partial/\partial x)\delta$.

5 Remarks

Remark 5.1. We have

Lemma 5.1. *Suppose that the equivalent conditions* (1.8) *and* (1.9) *are satisfied, and d is the degree of $P(i\xi)$ treated as a polynomial on $\mathbb{R}^n$ with coefficients in M_m. Suppose moreover that $k_0 \in \mathbb{N}$ and $k_0 \geq d + \frac{1}{2}(md-1)(n+2) + \frac{1}{2}(n+1)$. Then there is a mapping $(t \mapsto f_t) \in C^1([0,\infty[;\, L^1(\mathbb{R}^n;\, M_m))$ such that*

$$\text{(5.1)} \quad \sup\left\{ e^{-\omega t} \left\| \left(\frac{d}{dt}\right)^l f_t \right\|_{L^1(\mathbb{R}^n;\, M_m)} : t \in [0,\infty[,\ l = 0, 1 \right\} < \infty \text{ for every } \omega > \omega_0(P)$$

and

$$\text{(5.2)} \quad S_t = (1-\Delta)^{k_0} f_t \quad \text{for every } t \in [0,\infty[,$$

the action of the differential operator $(1-\Delta)^{k_0}$ being understood in the sense of distributions [11].

Proof. Whenever $|\alpha| \leq n+1$, $l = 0, 1, 2$, $t \in [0, \infty[$ and $\omega > \omega_0(P)$, then, by (2.8) and (4.6),

$$\begin{aligned}
\int_{\mathbb{R}^n} \left| \left(\frac{\partial}{\partial \xi}\right)^\alpha \left[(1+|\xi|^2)^{-k_0} \left(\frac{d}{dt}\right)^l \phi_t(\xi) \right] \right| d\xi \\
&= \int_{\mathbb{R}^n} \left| \left(\frac{\partial}{\partial \xi}\right)^\alpha [(1+|\xi|^2)^{-k_0} (P(i\xi))^l \phi_t(\xi)] \right| d\xi \\
&\leq \text{const} \cdot e^{\omega t} \int_{\mathbb{R}^n} (1+|\xi|^2)^{-k_0 + \frac{1}{2}ld + \frac{1}{2}(md-1)(n+2)}\, d\xi = \text{const} \cdot e^{\omega t}.
\end{aligned}$$

[11] The equality (5.2) should be compared with (2.6).

This implies that whenever $|\alpha| \le n+1$, then

$$[0,\infty[\, \ni t \mapsto x^\alpha f_t = x^\alpha (1-\Delta)^{-k_0} S_t \in C_b(\mathbb{R}^n;\, M_m)$$

is a C^1-mapping such that

$$\sup\left\{ e^{-\omega t} \left\| x^\alpha \left(\frac{d}{dt}\right)^l f_t \right\|_{C_b(\mathbb{R}^n;\, M_m)} : t \in [0,\infty[,\ l = 0,1 \right\} < \infty$$

and hence the mapping $[0,\infty[\, \ni t \mapsto f_t \in L^1(\mathbb{R}^n; M_m)$ satisfies (5.1) and (5.2).

Following L. Schwartz [S], Sec. VI.8, for every $p \in [1,\infty]$ denote by $\mathcal{D}_{L^p}(\mathbb{R}^n;\, \mathbb{C}^m)$ the Fréchet space of all functions $\varphi \in C^\infty(\mathbb{R}^n;\, \mathbb{C}^m)$ such that $(\partial/\partial x)^\alpha \varphi \in L^p(\mathbb{R}^n;\, \mathbb{C}^m)$ for every $\alpha \in \mathbb{N}_0^n$. Whenever $1 \le p < q \le \infty$, then

$$\mathcal{S}(\mathbb{R}^n;\, \mathbb{C}^m) \subset \mathcal{D}_{L^p}(\mathbb{R}^n;\, \mathbb{C}^m) \subset \mathcal{D}_{L^q}(\mathbb{R}^n;\, \mathbb{C}^m) \subset \mathcal{S}'(\mathbb{R}^n;\, \mathbb{C}^m).$$

If the equivalent conditions (1.8) and (1.9) are satisfied and $\varphi \in \mathcal{D}_{L^p}(\mathbb{R}^n;\, \mathbb{C}^m)$ for some $p \in [1,\infty]$, then, by Lemma 5.1,

$$S_t * \varphi = f_t * (1-\Delta)^{k_0} \varphi \quad \text{for every } t \in [0,\infty[\tag{5.3}$$

where, for fixed t, $S_t * \varphi$ is understood as the convolution of the distribution $S_t \in \mathcal{O}'_C(\mathbb{R}^n;\, M_m)$ with the distribution $\varphi \in \mathcal{S}'(\mathbb{R}^n;\, \mathbb{C}^m)$, while, again for fixed t, the right side of (5.3) is the convolution of the function $f_t \in L^1(\mathbb{R}^n;\, M_m)$ and the function $(1-\Delta)^{k_0}\varphi \in \mathcal{D}_{L^p}(\mathbb{R}^n;\, \mathbb{C}^m)$. From (5.3) one infers easily that *for any fixed* $p \in [1,\infty]$,

(5.4) $((S_t *)|_{\mathcal{D}_{L^p}(\mathbb{R}^n;\mathbb{C}^m)})_{t\ge 0} \subset L(\mathcal{D}_{L^p}(\mathbb{R}^n;\, \mathbb{C}^m);\, \mathcal{D}_{L^p}(\mathbb{R}^n;\, \mathbb{C}^m))$ *is a one-parameter semigroup of operators with all trajectories in* $C^\infty([0,\infty[;\, \mathcal{D}_{L^p}(\mathbb{R}^n;\, \mathbb{C}^m))$ *and with infinitesimal generator* $P(\partial/\partial x)|_{\mathcal{D}_{L^p}(\mathbb{R}^n;\mathbb{C}^m)}$.

Furthermore, if $\omega_0(P) \le \omega < \infty$, *then*

(5.5) *for every* $\varepsilon > 0$ *the semigroup of operators* $(e^{-(\omega+\varepsilon)t}(S_t *)|_{\mathcal{D}_{L^p}(\mathbb{R}^n;\mathbb{C}^m)})_{t\ge 0} \subset L(\mathcal{D}_{L^p}(\mathbb{R}^n;\, \mathbb{C}^m);\, \mathcal{D}_{L^p}(\mathbb{R}^n;\, \mathbb{C}^m))$ *is equicontinuous.*

Notice that (5.4) for $p=\infty$ is equivalent to the original result of I. G. Petrovskiĭ [P] proved in 1938 by an elementary method (discussed in Sec. 12 of [K]). From Theorem 1 of [K] it follows that if $p=2$ or $p=\infty$, then (5.5) holds if and only if $\omega_0(P) \le \omega < \infty$. In connection with (5.5) let us recall that the theory of equicontinuous one-parameter semigroups in locally convex spaces is presented in Chapter IX of [Y].

Directly from Theorem 1.1 it follows that if $E = \mathcal{S}(\mathbb{R}^n;\, \mathbb{C}^m)$ or $E = \mathcal{S}'(\mathbb{R}^n;\, \mathbb{C}^m)$, then $((S_t *)|_E)_{t\ge 0} \subset L(E;\, E)$ is a one-parameter semigroup of operators with all trajectories in $C^\infty([0,\infty[;\,E)$ and with infinitesimal generator $P(\partial/\partial x)|_E$.

A prototype of the above results is Theorem 10.1 of T. Ushijima [U], p. 118, in which $E = \{u \in L^2(\mathbb{R}^n;\mathbb{C}^m) : P(\partial/\partial x)^k u \in L^2(\mathbb{R}^n;\mathbb{C}^m)$ for $k = 1,2,\ldots\}$ and the topology of E is determined by the system of seminorms $\|P(\partial/\partial x)^k u\|_{L^2(\mathbb{R}^n;\mathbb{C}^m)}$, $k = 0,1,\ldots$. In the proof of Ushijima's theorem an application of inequality (4.3) is replaced by estimations based on the interpolation polynomials of E. A. Gorin.

Remark 5.2. *A distribution $G \in \mathcal{O}'_C(\mathbb{R}^n;\,M_m)$ is the infinitesimal generator of an i.d.c.s. $(S_t)_{t\ge0} \subset \mathcal{O}'_C(\mathbb{R}^n;\,M_m)$ if and only if there are $a \in [0,\infty[$ and $b \in \mathbb{R}$ such that*

$$\text{(5.6)}\quad \max\{\operatorname{Re}\lambda : \lambda \in \sigma((\mathcal{F}G)(\xi)\} \le a\log(1+|\xi|) + b \quad \textit{for every } \xi \in \mathbb{R}^n.$$

Furthermore,

$$\text{(5.7)}\qquad \max\{\operatorname{Re}\lambda : \lambda \in \sigma((\mathcal{F}G)(\xi)),\ \xi \in \mathbb{R}^n\} \le \omega < \infty$$

if and only if

(5.8) *for every $\varepsilon > 0$ the semigroup of operators $(e^{-(\omega+\varepsilon)t}S_t\,*)_{t\ge0} \subset L(\mathcal{S}(\mathbb{R}^n;\,\mathbb{C}^m);\,\mathcal{S}(\mathbb{R}^n;\,\mathbb{C}^m))$ is equicontinuous.*

The above follows by arguments similar to those used in the proof of Theorem 1.1. This time it is not asserted that (5.6)⇔(5.7). In the pioneering paper [P] of I. G. Petrovskiĭ the case of $G = (\partial/\partial x)\delta$ was investigated, but instead of (1.8) the condition

$$\text{(5.9)}\qquad \max\{\operatorname{Re}\lambda : \lambda \in \sigma(P(i\xi))\} \le a\log(1+|\xi|) + b \qquad \text{for every } \xi \in \mathbb{R}^n$$

was used. The equivalence (1.8)⇔(5.9) was only a hypothesis at that time.

Remark 5.3. Similarly to operator semigroups (5.4) one can consider the operator semigroups related to the Cauchy problem for systems of PDE of the form

$$A\Big(\frac{\partial}{\partial x}\Big)\frac{\partial}{\partial t}\vec{u}(t,x) = B\Big(\frac{\partial}{\partial x}\Big)\vec{u}(t,x) \quad \text{ for } (t,x) \in [0,\infty[\,\times\mathbb{R}^n$$

with given $\vec{u}(0,x)$. The corresponding i.d.c.s. is defined as follows. $A(\zeta_1,\ldots,\zeta_n)$ and $B(\zeta_1,\ldots,\zeta_n)$ are $m\times m$ matrices whose entries are complex polynomials of n variables $\zeta_1,\ldots,\zeta_n$. It is assumed that

$$A(i\xi_1,\ldots,i\xi_n) \text{ is invertible for every } (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$$

and

$$\sup\{\operatorname{Re}\lambda : \lambda \in \mathbb{C},\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,\ \det(\lambda A(i\xi) - B(i\xi)) = 0\} < \infty.$$

Then there is a unique i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_m)$ whose infinitesimal generator is the distribution in $\mathcal{O}'_C(\mathbb{R}^n; M_m)$ whose Fourier transform is the function $\xi \mapsto A(i\xi)^{-1}B(i\xi)$. This last function belongs to $\mathcal{O}_M(\mathbb{R}^n; M_m)$ by the same argument as in Example A.2.7 in the Appendix to [H]. From the result obtained in this way for systems of PDE one can deduce the theorem of J. Rauch [R], p. 128, concerning a single PDE of higher order.

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