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IM PAN Preprint 724 (2010)

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Computing Gromov-Witten Invariants of some Fano Varieties

Presented by Piotr M. Hajac

Published as manuscript

Received 05 May 2010

COMPUTING GROMOV-WITTEN INVARIANTS OF SOME FANO VARIETIES.

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Abstract. We present a recursive algorithm computing all the genus-zero Gromov-Witten invariants from a finite number of initial ones, for Fano varieties with generically tame semi-simple quantum (and small quantum) (p,p)-type cohomology, whose first Chern class is a strictly positive combination of effective integral basic divisors.

1. Introduction. The recursive formula of Kontsevich for numbers N_d of rational planar curves of degree d passing through $3d-1$ points in general position

$$N_d = \sum_{d_1+d_2=d} d_1^2 d_2 \left[d_2 \begin{pmatrix} 3d-4 \\ 3d_1-2 \end{pmatrix} - d_1 \begin{pmatrix} 3d-4 \\ 3d_1-1 \end{pmatrix} \right] N_{d_1} N_{d_2}$$

for $d > 1$ ($N_1 = 1$ is obvious), as well as its generalizations for genus zero Gromov-Witten invariants of all projective spaces and Del Pezzo surfaces [20], was originally derived from the associativity relations and the homogeneity condition for quantum cohomology [20] of these varieties. (For particular Fano threefolds all Gromov-Witten invariants can be computeted from a finite number of them by other methods [26].) If the (p,p)-part of Hodge cohomology of a Fano variety is generated by $H^{1,1}$ then the First Reconstruction Theorem of [20] gives an inductive procedure reconstructing all restricted genus zero Gromov-Witten classes from a finite number of codimension zero basic classes [20]. In many papers quantum cohomology of other Fano varieties was studied and in many cases completely determined [1], [3], [6], [4], [7], [9], [10], [11], [16], [18], [19], [22], [23], [29]. But the structure of the general associativity relations is so complicated that it is not clear how to compute Gromov-Witten invariants in general, in particular no general recursive structure is apparent.

However, if the Hodge (p,p)-part of the quantum cohomology of a Fano variety V is generically semi-simple and, moreover, admits a tame semi-simple point lying in the subspace $H^{1,1}(V)/(H^{1,1}(V) \cap 2\pi\sqrt{-1}H^2(V,\mathbb{Z}))$ (parameter space of the small quantum deformation), then all genus zero Gromov-Witten invariants of $\bigoplus_p H^{p,p}(V)$ can be reconstructed from a finite number of correlators (Reconstruction Theorem of [3]).

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†The author was partially supported by KBN grants 1P03A 036 26 and 115/E-343/SPB/6.PR UE/DIE 50/2005-2008.

Mathematics Subject Classification (2000): 14N35, 53D45.

In fact it is a consequence of the two properties: 1) the quantum (and small quantum) (p,p)-type Hodge cohomology algebra is generically tame semi-simple, 2) the system of partial differential equations encoding associativity of quantum multiplication on (p,p)-type Hodge cohomology together with the quasi-homogeneity condition, is formally integrable. These two properties imply that generically there exist local orthogonal coordinates $(u_0, ...u_\sigma)$ (canonical coordinates of Dubrovin, whose basis tangent vectors form a complete system of orthogonal idempotents), in which flatness of the metric (the Darboux-Egoroff system of partial differential equations) together with homogeneity, is equivalent to the Pfaff system

$$dv_{ik} = -\sum_{j\neq i,k} v_{ij}v_{kj}\ d\log\frac{u_i-u_j}{u_k-u_j}$$

for a skew-symmetric $(\sigma+1)\times(\sigma+1)$ matrix $v = (v_{ij} = -v_{ji})_{i,j=0}^{\sigma}$. From theoretical point of view, every diagonalizable solution v reconstructs the Frobenius manifold structure encoding the quantum cohomology up to fixing a finite number of parameters [13]. The *isomonodromic deformations method* based on isomonodromicity of these equations can be found in [13], [14], [17], [27], [28]. However, already for $\sigma = 2$ this system reduces to the Painlevé VI equation and the inversion formula contains inversion of complicated transcendental functions derived from Painlevé transcendents [17]. In particular, this method computes Gromov-Witten numbers N_d of $\mathbb{P}^2$ only term by term through successive expansions inverting a complicated series [17]. In general, such a series is even not known.

In the present paper we show that the semi-simple associativity together with the quasi-homogeneity condition reduce to another Pfaff system

$$dy_{ab} = \sum_c r_{abc}(x,y)\ dx_c$$

with some rational functions $r_{abc}(x,y)$ with constant coefficients, fully symmetric in indices abc, of a symmetric $\sigma\times\sigma$ matrix $y = (y_{ab})_{a,b=1}^{\sigma}$ satisfying some algebraic constraint. Here independent variables $(x_1,...,x_\sigma)$ form a part of usual affine coordinates $(x_0,...,x_\sigma)$ on the space of the sum of (p,p)-Hodge cohomology. In this way we overpass the previous hard inversion problem from [17].

In the case of $\mathbb{P}^2$ we solve our equation explicitly re-obtaining the Kontsevich recursive formula for numbers N_d. This shows that our equation does the same job as the associativity and quasi-homogeneity equations do.

In general, using the initial data at a tame semi-simple point in the small quantum deformation and the Newton method computing the Taylor series of the solution, we get an algorithm computing all Gromov-Witten numbers from a finite number of initial ones.

Our recursive algorithm generalizes the Kontsevich formula for the projective plane and makes effective the Bayer-Manin Reconstruction Theorem [3], provided the first Chern class of a given Fano manifold is a strictly positive combination of effective integral basic divisors.

2. Gromov-Witten numbers. Let V be a connected complex Fano manifold of dimension n. Consider the complex linear space

$$H(V) = \bigoplus_{p=0}^{d} H^{p,p}(V)$$

of the (p,p)-Hodge cohomology algebra with the cup product $\cup$ and with the unit $1 \in H^{0,0}(V)$. We have the embedding of additive groups

$$\mathrm{Pic}(V) \cong H^{1,1}(V) \cap H^2(V,\mathbb{Z}) \subset H(V)$$

($\mathrm{Pic}(V) \cong H^2(V,\mathbb{Z})$ is torsion free) and in $H^{1,1}(V)$ there is the cone $H^{1,1}_{+}(V)$ of Kähler classes. The first Chern class $c_1(V)$ lies in $H^{1,1}_{+}(V) \cap H^2(V,\mathbb{Z})$.

According to [20], given a smooth connected complex Fano variety V, one chooses a basis $(h_0, \ldots, h_\sigma)$ of $H(V)$ of classes Poincaré dual to classes of integral cycles $(Z_0, \ldots, Z_\sigma)$ in general position, not necessarily algebraic, such that Z_0 is the fundamental cycle, $(Z_1, \ldots, Z_\rho)$ are integral effective divisors generating the Picard group and Z_σ is a point. Then $h_0 = 1$ and $h_1, \ldots, h_\rho \in H^{1,1}(V) \cap H^2(V,\mathbb{Z})$ span $H^{1,1}(V)$. One defines the symmetric integer valued non-degenerate symmetric matrix of the intersection form

$$g_{\alpha\beta} := \int_V h_\alpha \cup h_\beta,$$

and the integer valued inverse matrix $\bar{g}_{\alpha\beta}$. We will use also fully symmetric integer valued symbols

$$g_{\alpha\beta\gamma} := \int_V h_\alpha \cup h_\beta \cup h_\gamma,$$

integers c_α such that

$$c_1(V) = \sum_\alpha c_\alpha h_\alpha,$$

and the integer valued symmetric matrix

$$c_{\alpha\beta} := \int_V c_1(V) \cup h_\alpha \cup h_\beta = \sum_\gamma g_{\alpha\beta\gamma} c_\gamma.$$

We will consider the following additional condition on the above basis $h_1, \ldots, h_\rho$ of $\mathrm{Pic}(V)$.

Condition C. The integers c_α are strictly positive for $\alpha = 1, \dots, \rho$.

Such a basis exists at least for the following classes of Fano manifolds:

i) Generalized flag varieties $V = G/B$, where G is semisimple and B is the Borel subgroup [8]. As $h_1, \dots, h_\rho$ one can take first Chern classes of homogeneous line bundles associated with fundamental weights. Then $c_1(V)$ corresponds to a dominant weight μ_V and $c_\alpha = \langle \mu_V, \alpha \rangle = 2$, where α is a simple root of G, which can be computed using a formula from [30].

ii) Fano manifolds with $\rho = 1$. As h_1 one can take the cohomology class of the ample generator of $\mathrm{Pic}(V)$ and c_1 is equal to the index of the Fano manifold V .

iii) Fano threefolds with $\rho = 2$. Then V admits two extremal contractions $f_\alpha : V \to V_\alpha$, corresponding to extremal rays of lengths λ_α, $\alpha = 1, 2$, onto projective varieties V_α with $\mathrm{Pic}(V_\alpha) \cong \mathbb{Z}$ and $\mathrm{Pic}(V) \cong f_1^*\mathrm{Pic}(V_1) \oplus f_2^*\mathrm{Pic}(V_2)$. As h_1, h_2 one can take pull-backs of ample generators of $\mathrm{Pic}(V_\alpha)$ and then $c_1 = \lambda_2, c_2 = \lambda_1$ [25].

iv) Fano n-fold $V = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-1})$, $n \geq 2$. As a basis of $\mathrm{Pic}(V)$ one can take h_1 equal to the pull-back of $\mathcal{O}_{\mathbb{P}^1}(1)$ and h_2 equal to the Grothendieck tautological line bundle $\mathcal{O}_{V/\mathbb{P}^1}(1)$. Then $c_1 = 1, c_2 = n$.

v) For a toric Fano manifold V, corresponding to a nonsingular complete fan with edges containing primitive vectors $v_1, \dots, v_{n+\rho}$ in $\mathbb{Z}^n$, one has the split exact sequence [15]

$$0 \to \mathbb{Z}^n \to \mathbb{Z}^{n+\rho} \to \mathrm{Pic}(V) \to 0,$$

where the map $\mathbb{Z}^n \to \mathbb{Z}^{n+\rho}$ is given by the matrix $V = (v_1, \dots, v_{n+\rho})$.

Therefore there exists an integer valued matrix $W = (w_{\alpha i})$, $\alpha = 1, \dots, \rho$, $i = 1, \dots, n + \rho$, such that

$$\det \begin{pmatrix} V \\ W \end{pmatrix} = \pm 1,$$

which defines integers $(c_1, \dots, c_\rho)$ as follows

$$(1, \dots, 1) \begin{pmatrix} V \\ W \end{pmatrix}^{-1} = (*, \dots, *, c_1, \dots, c_\rho).$$

Let $D_1, \dots, D_{n+\rho}$ be irreducible effective invariant divisors corresponding to the edges containing primitive vectors $v_1, \dots, v_{n+\rho}$. Then divisors

$$h_\alpha := \sum_{i=1}^{n+\rho} w_{\alpha i} D_i,$$

for $\alpha = 1, \dots, \rho$, form a basis in $\mathrm{Pic}(X)$ and

$$-K_X = \sum_{\alpha=1}^{\rho} c_\alpha h_\alpha.$$

If W can be chosen in such a way that all $w_{\alpha i} \geq 0$ and $c_\alpha > 0$ then the above condition C is satisfied.

It is easy to check that it is so for Hirzebruch toric Del Pezzo surfaces $\mathbb{F}_k$, $k = 1, 6, 7$, where

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} -1 \\ k \end{pmatrix}, \ v_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \ v_4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Namely, for $\mathbb{F}_1$ (resp. $\mathbb{F}_6$ or $\mathbb{F}_7$) we can take

$$h_1 = D_1 + D_2, \quad h_2 = D_1 + 2D_2$$

(resp. $h_1 = D_1$, $h_2 = D_2$ or $h_1 = D_1 + 4D_2$, $h_2 = D_1 + 5D_2$), and then

$$-K_X = h_1 + h_2$$

(resp. $-K_X = 2h_1 + 8h_2$ or $-K_X = h_1 + h_2$).

Definition. One defines the number $N_{k_1 \dots k_\sigma}$ as the number of rational curves intersecting cycles $(Z_1, \dots, Z_\sigma)$ with finite multiplicities $(k_1, \dots, k_\sigma)$.

One calls $N_{k_1 \dots k_\sigma}$'s the *Gromov-Witten numbers*.

Correctness of this definition follows from the intersection theory on moduli stacks of stable maps [5], [20]. Gromov-Witten numbers are derived from the (genus zero) *Gromov-Witten invariants*. The Fano condition implies that multiplicities $(k_1, \dots, k_\rho)$ determine the homology class of a rational curve uniquely. The collection of Gromov-Witten invariants forms a quite complicated combinatorial structure. Luckily, the combinatorial identities can be encoded in properties of a generating function.

3. Generating function. Let $(x_0, ..., x_\sigma)$ be the coordinate system on $H(V)$ dual to the basis $(h_0, ..., h_\sigma)$. Then the variable point of $H(V)$ has the form $x = \sum_\alpha x_\alpha h_\alpha$ and one defines a *generating function* as the formal series of the form

$$F(x) = \frac{1}{6} \int_V x^{\cup 3} + f(x),$$

$$f(x) = \sum_{k_1, \dots, k_\sigma} N_{k_1 \dots k_\sigma} e^{k_1 x_1 + \dots + k_\rho x_\rho} \frac{x_{\rho+1}^{k_{\rho+1}}}{k_{\rho+1}!} \cdots \frac{x_\sigma^{k_\sigma}}{k_\sigma!},$$

with the following condition for non-vanishing summands

$$\sum_{\alpha=1}^{\rho} k_\alpha c_\alpha + \sum_{\alpha=\rho+1}^{\sigma} k_\alpha (1 - p_\alpha) = 3 - n.$$

4. Homogeneity. One defines an action of $\mathbb{C}^*$ on the quotient space

$$X := H(V) / (H^{1,1}(V) \cap 2\pi\sqrt{-1} H^2(V, \mathbb{Z})) \cong (\mathbb{C}^*)^\rho \times \mathbb{C}^{\sigma+1-\rho}$$

(Dubrovin's flow) as follows

$$s \cdot \sum_p x_{(p)} := (x_{(1)} + \log s \ \ c_1(V)) + \sum_{p \neq 1} s^{1-p} \ x_{(p)},$$

where $s \in \mathbb{C}^*$, $x_{(p)} \in H^{p,p}(V)$. The part $f(x)$ of the generating function is homogeneous of weight $3-n$ with respect to this action [20]. Using the Euler field generating the above flow

$$E = \sum_\alpha E_\alpha \frac{\partial}{\partial x_\alpha} := \sum_\alpha (c_\alpha + (1-p_\alpha)x_\alpha)\frac{\partial}{\partial x_\alpha} \tag{1}$$

one gets equivalently

$$E(f) = (3-n)f.$$

On the other hand, decomposing x into homogeneous components we get

$$E(\int_V x^{\cup 3}) = 3\int_V c_1(V) \cup x^{\cup 2} + (3-n)\int_V x^{\cup 3},$$

implying the following quasi-homogeneity property for the generating function F

$$E(F) = \frac{1}{2}\int_V c_1(V) \cup x^{\cup 2} + (3-n)F,$$

which in coordinates takes the form

$$\sum_\delta E_\delta \frac{\partial F}{\partial x_\delta} = \frac{1}{2}\sum_{\alpha\beta} c_{\alpha\beta} x_\alpha x_\beta + (3-n)F. \tag{2}$$

Taking second partial derivatives of this and using (1) we get the following useful formula

$$\sum_\delta E_\delta \frac{\partial^3 F}{\partial x_\alpha \partial x_\beta \partial x_\delta} = c_{\alpha\beta} + (1-n+p_\alpha+p_\beta)\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}. \tag{3}$$

Note that on the right hand side $c_{\alpha\beta}$ can be non-zero only if $1-n+p_\alpha+p_\beta = 0$.

5. Flat metric. The intersection form defines on X a flat riemannian metric

$$g(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}) = g_{\alpha\beta}.$$

The metric tensor is homogeneous of weight $(2-n)$ under the Dubrovin flow.

6. Associativity equations. One introduces the following multiplication on the (trivial) tangent bundle of the quotient space X

$$\frac{\partial}{\partial x_\alpha} \cdot \frac{\partial}{\partial x_\beta} = \sum_{\gamma,\delta} \bar{g}_{\gamma\delta} \frac{\partial^3 F}{\partial x_\alpha \partial x_\beta \partial x_\delta} \frac{\partial}{\partial x_\gamma}.$$

It is commutative by symmetry of partial derivatives and associative with the unit $\frac{\partial}{\partial x_0}$ by axioms of Gromov-Witten invariants [20]. The associativity condition reads as a system of quadratic equations on third partial derivatives of F. The multiplication, viewed as a tensor, is homogeneous of weight 1 under the Dubrovin flow, as well as the unit of this multiplication.

The metric and the multiplication of tangent vectors satisfy

$$g(\frac{\partial}{\partial x_\alpha}\cdot\frac{\partial}{\partial x_\beta},\frac{\partial}{\partial x_\gamma}) = g(\frac{\partial}{\partial x_\alpha},\frac{\partial}{\partial x_\beta}\cdot\frac{\partial}{\partial x_\gamma}).$$

7. (Tame) Semi-simplicity. A point of X is called *semi-simple* if at this point the generating function is convergent and the above algebra of tangent vectors is semi-simple.

According to [13], locally around every semi-simple point, there exist orthogonal coordinates $(u_0,\dots,u_\sigma)$ with a domain $U\subset X$ in which the above multiplication of tangent vectors and the Euler vector field take the standard form

$$\frac{\partial}{\partial u_i}\cdot\frac{\partial}{\partial u_j} = \delta_{ij}\frac{\partial}{\partial u_i},$$

$$E = \sum_{i=0}^{\sigma} u_i\frac{\partial}{\partial u_i}.$$

In particular, coordinates $(u_0,\dots,u_\sigma)$ are eigenvalues of the multiplication by the Euler field E. A semi-simple point of X is called *tame* if at this point $\prod_{i\neq j}(u_i-u_j)\neq 0$. This condition is independent of the choice of coordinates $(u_0,\dots,u_\sigma)$.

Discussion of the (tame) semi-simplicity condition, examples and non-examples can be found in [2], [3], [12], [31].

8. Derivation of the main formula.

Lemma 1. *Let the vector* $(z_{i0},\dots,z_{i\sigma})^\intercal$ *be a solution to the following eigen-problem with simple eigen-values* $(a_0,\dots,a_\sigma)$

$$\sum_{\beta,\gamma}\bar g_{\beta\gamma}(c_{\alpha\beta}+(1-n+p_\alpha+p_\beta)Y_{\alpha\beta})z_{i\gamma} = a_i z_{i\alpha}. \tag{4}$$

for a symmetric $(\sigma+1)\times(\sigma+1)$ *matrix* $Y=(Y_{\alpha\beta}=Y_{\beta\alpha})_{\alpha,\beta=0}^{\sigma}$. *Then the expression*

$$R_{\alpha\beta\gamma}(Y) = \sum_i \frac{z_{i\alpha}z_{i\beta}z_{i\gamma}}{\sum_{\delta,\epsilon}\bar g_{\delta\epsilon}z_{i\delta}z_{i\epsilon}z_{i0}}, \tag{5}$$

is a rational function of Y.

Proof: First we rewrite the eigen-problem (12) as

$$\sum_\beta (a_{\alpha\beta}-a_i\delta_{\alpha\beta})z_{i\beta} = 0.$$

The simplicity of eigen-values means that for every $i=0,\dots,\sigma$ the rank of the matrix $(a_{\alpha\beta}-a_i\delta_{\alpha\beta})$ is equal to σ. Therefore there are σ rows of this matrix (we can assume that they are the first σ rows) whose exterior product defines the one dimensional eigen-space, according to the canonical isomorphism of vector spaces

$$\bigwedge^{\sigma} V^* \otimes det(V) \xrightarrow{\cong} V,$$

where $\dim(V) = \sigma + 1$. Thus we can replace every eigen-vector $(z_{i0}, \ldots, z_{i\sigma})^{\intercal}$ in the homogeneous function (11) by the vector of alternated σ-minors of the above σ rows of the matrix $(a_{\alpha\beta} - a_i\delta_{\alpha\beta})_{\alpha,\beta=0}^{\sigma}$. It is clear that we obtain a symmetric rational function in variables $(a_0, \ldots, a_\sigma)$ with coefficients in the field of rational functions of entries $a_{\alpha\beta}$, which are polynomials in $Y_{\alpha\beta}$'s. In this way we see that the right hand side of (11) is a rational function of $Y_{\alpha\beta}$'s, because we can compute this function expressing the above symmetric rational function of eigen-values $(a_0, \ldots, a_\sigma)$ in terms of their elementary symmetric polynomials, which are polynomials in $(a_{\alpha\beta})_{\alpha,\beta=0}^{\sigma}$, hence polynomials in $Y_{\alpha\beta}$'s. $\square$

Practically, to find the rational function $R_{\alpha\beta\gamma}(Y)$ we can use here the `RootSum` function of *Mathematica*. We will do so in the example of the projective plane in paragraph 9.

Next, let us form the partial differential equation

$$\frac{\partial Y_{\alpha\beta}}{\partial x_\gamma} = R_{\alpha\beta\gamma}(Y) \tag{6}$$

and an algebraic constraint

$$\sum_\gamma E_\gamma R_{\alpha\beta\gamma}(Y) = c_{\alpha\beta} + (1 - n + p_\alpha + p_\beta) Y_{\alpha\beta}. \tag{7}$$

Since the right hand side of (6) is fully symmetric in indices $\alpha\beta\gamma$ then locally any solution Y to the equation (6) is of the form

$$Y_{\alpha\beta} = \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \tag{8}$$

for some function F. Since the matrix $(a_{\alpha\beta})_{\alpha,\beta=0}^{\sigma}$ in the proof of Lemma 1 is self-adjoint with respect to the non-degenerate symmetric form $\bar{g}$ and has simple eigen-values its eigen-vectors are mutually orthogonal. Therefore by (5)

$$R_{\alpha\beta 0}(Y) = \sum_i \frac{z_{i\alpha} z_{i\beta}}{\sum_{\delta,\epsilon} \bar{g}_{\delta\epsilon} z_{i\delta} z_{i\epsilon}} = g_{\alpha\beta}, \tag{9}$$

By (5), (6), (8) and (9) F automatically satisfies the associativity equations with the unit $\frac{\partial}{\partial x_0}$. Let us define

$$f := F - \frac{1}{6} \int_V x^{\cup 3}. \tag{10}$$

By (6), (8) and (9) we have

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}(\frac{\partial f}{\partial x_0}) = 0, \tag{11}$$

so f is independent of x_0 up to adding a polynomial of degree two, at most quadratic in x_0 and at most linear in other variables.

By (1) we have

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}(E(f) - (3-n)f) = \tag{12}$$

$$= \sum_\gamma E_\gamma \frac{\partial^3 F}{\partial x_\alpha \partial x_\beta \partial x_\gamma} - (c_{\alpha\beta} + (1 - n + p_\alpha + p_\beta)\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}).$$

Therefore every f defined in (10) by means of a solution (8) to the equation (6), satisfying the constraint (7), is homogeneous of weight $(3-n)$ up to adding a polynomial of degree one.

We can focus only on f. For this we introduce

$$y_{ab} := Y_{ab} - \int_V x \cup h_a \cup h_b,$$

$$r_{abc} := R_{abc} - g_{abc},$$

where latin indices run through $(1, \dots, \sigma)$. Then locally

$$y_{ab} = \frac{\partial^2 f}{\partial x_a \partial x_b}, \tag{13}$$

hence by (11) and (13) we have

$$\frac{\partial y_{ab}}{\partial x_0} = \frac{\partial^3 f}{\partial x_a \partial x_b \partial x_0} = 0,$$

which means that y_{ab}'s depend only on variables $x = (x_a)_{a=1}^\sigma$. Then the equation (6) reduces to

$$dy_{ab} = \sum_c r_{abc}(x, y) dx_c, \tag{14}$$

with the functions $r_{abc}(x, y)$ rational in variables $x = (x_a)_{a=1}^\sigma$, $y = (y_{ab} = y_{ba})_{a,b=1}^\sigma$, and fully symmetric in indices abc. The algebraic constraint (7) reduces to

$$\sum_c (c_c + (1 - p_c)x_c) r_{abc}(x, y) = n \sum_c g_{abc} x_c + (1 - n + p_a + p_b) y_{ab}. \tag{15}$$

On the other hand, in the context of semi-simple quantum (p,p)-cohomology, comparing the multiplication of tangent vectors, the unit, the metric and the multiplication by the Euler field in coordinates $(x_0, \ldots, x_\sigma)$ and $(u_0, \ldots, u_\sigma)$ we get

$$\sum_{\gamma,\delta} \bar{g}_{\gamma\delta} \frac{\partial^3 F}{\partial x_\alpha \partial x_\beta \partial x_\delta} \frac{\partial u_i}{\partial x_\gamma} = \frac{\partial u_i}{\partial x_\alpha} \frac{\partial u_i}{\partial x_\beta}, \tag{16}$$

$$\frac{\partial u_i}{\partial x_0} = 1, \tag{17}$$

$$\sum_{\gamma,\delta} \bar{g}_{\gamma\delta} \frac{\partial u_i}{\partial x_\gamma} \frac{\partial u_j}{\partial x_\delta} = \delta_{ij} \sum_{\gamma,\delta} \bar{g}_{\gamma\delta} \frac{\partial u_i}{\partial x_\gamma} \frac{\partial u_i}{\partial x_\delta}, \tag{18}$$

$$\sum_{\beta,\gamma,\delta} \bar{g}_{\beta\gamma} E_\delta \frac{\partial^3 F}{\partial x_\alpha \partial x_\beta \partial x_\delta} \frac{\partial u_i}{\partial x_\gamma} = u_i \frac{\partial u_i}{\partial x_\beta}. \tag{19}$$

Equations (16)-(18) imply

$$\frac{\partial^3 F}{\partial x_\alpha \partial x_\beta \partial x_\gamma} = \sum_i \frac{\frac{\partial u_i}{\partial x_\alpha} \frac{\partial u_i}{\partial x_\beta} \frac{\partial u_i}{\partial x_\gamma}}{\sum_{\delta,\epsilon} \bar{g}_{\delta\epsilon} \frac{\partial u_i}{\partial x_\delta} \frac{\partial u_i}{\partial x_\epsilon} \frac{\partial u_i}{\partial x_0}}, \tag{20}$$

Using (3) and (19) we get

$$\sum_{\beta,\gamma} \bar{g}_{\beta\gamma} (c_{\alpha\beta} + (1 - n + p_\alpha + p_\beta) \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}) \frac{\partial u_i}{\partial x_\gamma} = u_i \frac{\partial u_i}{\partial x_\alpha}, \tag{21}$$

which means that for every i the vector $(\frac{\partial u_i}{\partial x_0}, \ldots, \frac{\partial u_i}{\partial x_\sigma})^\intercal$ is an eigenvector with the eigenvalue u_i of the matrix depending rationally on second partial derivatives $\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}$ and constants $\bar{g}_{\alpha\beta}$, p_α, $c_{\alpha\beta}$. By Lemma 1 the right hand side of (20) is a rational function of $\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}$ depending on constants $\bar{g}_{\alpha\beta}$, p_α, $c_{\alpha\beta}$. Then (3) defines an algebraic constraint (7) on the matrix of second derivatives of F.

In this way we obtain the following theorem.

Theorem 1. *The semi-simple associativity condition on a function*

$$F = \frac{1}{6} \int_V x^{\cup 3} + f$$

for f which is

- *independent of x_0 up to adding a polynomial of degree two, at most quadratic in x_0 and at most linear in other variables,*
- *homogeneous of weight $(3 - n)$ up to adding a polynomial of degree one,*

is locally equivalent to the problem (14)-(15).

9. Example $V = \mathbb{P}^1$**.** The basis (h_0, h_1) in (p,p)-Hodge cohomology is defined uniquely by the above convention. We have $n = 1$, $p_0 = 0, p_1 = 1$, $g_{00} = g_{11} = 0$, $g_{01} = 1$, $g_{000} = g_{011} = g_{111} = 0$, $g_{001} = 1$, $c_0 = 0, c_1 = 2$, and $c_{00} = 2$, $c_{01} = c_{11} = 0$.

Then we obtain

$$r_{111}(x,y) = y_{11},$$

so the system (14) has the form

$$dy_{11} = y_{11} dx_1, \tag{22}$$

and the constraint (15) is satisfied automatically. For

$$f(x_1) = \sum_{k_1} N_{k_1} e^{k_1 x_1}$$

the condition on non-vanishing terms is

$$2k_1 = 2,$$

which determines f uniquely up to a constant factor, as well as the system (14) does.

10. Example $V = \mathbb{P}^2$**.** The basis (h_0, h_1, h_2) in (p,p)-Hodge cohomology as above is defined uniquely by the condition that the cycle Z_1 is a hyperplane. Now $n = 2$, $p_\alpha = \alpha$, $g_{\alpha\beta} = 1$ for $\alpha + \beta = 2$, $g_{\alpha\beta\gamma} = 1$ for $\alpha + \beta + \gamma = 2$, $c_\alpha = 3$ for $\alpha = 1$, $c_{\alpha\beta} = 3$ for $\alpha + \beta = 1$, and all symbols are zero otherwise.

Then we obtain (using *Mathematica*)

$$\begin{aligned}
r_{111}(x,y) &= \frac{9y_{11} + x_2(y_{11}^2 + 6y_{12}) + 3x_2^2 y_{22}}{27 + 3x_2 y_{11} - 2x_2^2 y_{12}}, \\
r_{112}(x,y) &= \frac{18y_{12} + x_2(2y_{11}y_{12} + 9y_{22})}{27 + 3x_2 y_{11} - 2x_2^2 y_{12}}, \\
r_{122}(x,y) &= \frac{27y_{22} + 4x_2 y_{12}^2}{27 + 3x_2 y_{11} - 2x_2^2 y_{12}}, \\
r_{222}(x,y) &= \frac{12y_{12}^2 - 9y_{11}y_{22} + 6x_2 y_{12} y_{22}}{27 + 3x_2 y_{11} - 2x_2^2 y_{12}},
\end{aligned}$$

so the Pfaff system (14) has the form

$$dy_{11} = \frac{9y_{11} + x_2(y_{11}^2 + 6y_{12}) + 3x_2^2 y_{22}}{27 + 3x_2 y_{11} - 2x_2^2 y_{12}} dx_1 + \frac{18y_{12} + x_2(2y_{11}y_{12} + 9y_{22})}{27 + 3x_2 y_{11} - 2x_2^2 y_{12}} dx_2, \tag{23}$$

$$dy_{12} = \frac{18y_{12} + x_2(2y_{11}y_{12} + 9y_{22})}{27 + 3x_2 y_{11} - 2x_2^2 y_{12}} dx_1 + \frac{27y_{22} + 4x_2 y_{12}^2}{27 + 3x_2 y_{11} - 2x_2^2 y_{12}} dx_2, \tag{24}$$

$$dy_{22} = \frac{27y_{22} + 4x_2 y_{12}^2}{27 + 3x_2 y_{11} - 2x_2^2 y_{12}} dx_1 + \frac{12y_{12}^2 - 9y_{11}y_{22} + 6x_2 y_{12} y_{22}}{27 + 3x_2 y_{11} - 2x_2^2 y_{12}} dx_2. \tag{25}$$

One can check (using *Mathematica*) that the algebraic constraint (15) is satisfied identically.

The system (23)-(25) is equivalent to the system

$$d(x_2y_{11}) = \frac{9x_2y_{11} + x_2^2(y_{11} + 6y_{12}) + 3x_2^3y_{22}}{27 + 3x_2y_{11} - 2x_2^2y_{12}} d(x_1 + 3\log x_2), \tag{26}$$

$$d(3x_2y_{11} - x_2^2y_{12}) = x_2y_{11}d(x_1 + 3\log x_2), \tag{27}$$

$$d(18x_2y_{11} - 9x_2^2y_{12} + x_2^3y_{22}) = 2(3x_2y_{11} - x_2^2y_{12})d(x_1 + 3\log x_2). \tag{28}$$

If we introduce the expression ϕ such that

$$2\phi := 18x_2y_{11} - 9x_2^2y_{12} + x_2^3y_{22}, \tag{29}$$

then the system (26)-(28) means that

$$\phi = \phi(x_1 + 3\log x_2), \tag{30}$$

$$\phi''' = \frac{6\phi - 33\phi' + 54\phi'' + \phi''^2}{27 + 2\phi' - 3\phi''}, \tag{31}$$

which can be written equivalently as

$$27\phi''' - 54\phi'' + 33\phi' - 6\phi = 3\phi''\phi''' - 2\phi'\phi''' + \phi''^2. \tag{32}$$

For

$$f(x_1, x_2) = \sum_{k_1,k_2} N_{k_1k_2}e^{k_1x_1}\frac{x_2^{k_2}}{k_2!}$$

the condition on non-vanishing terms is

$$3k_1 - k_2 = 1.$$

Since Z_1 is a hyperplane k_1 equals to the degree d of a rational curve, hence $k_2 = 3d - 1$, $N_{k_1k_2} = N_d$ and

$$f = \sum_d N_d e^{dx_1}\frac{x_2^{3d-1}}{(3d-1)!}.$$

Now we compute ϕ substituting f into (13) and next the obtained result to (29). We get

$$\phi = \sum_d N_d e^{dx_1}\frac{x_2^{3d}}{(3d-1)!} = \sum_d \frac{N_d}{(3d-1)!}e^{d(x_1+3\log x_2)}. \tag{33}$$

Inserting this to (32) we get finally

$$N_d = \sum_{d_1+d_2=d} d_1^2d_2\left[d_2\begin{pmatrix} 3d-4 \\ 3d_1-2 \end{pmatrix} - d_1\begin{pmatrix} 3d-4 \\ 3d_1-1 \end{pmatrix}\right]N_{d_1}N_{d_2}.$$

This shows that our system provides the same answer as the original use of associativity and homogeneity conditions.

11. Generalization. The algorithm. In general, we can use the Newton method to compute the Taylor series of the solution. To apply this we have to know that the denominator of the rational function on the right hand side of our equation (15) do not vanish. It is so at a tame semi-simple point because all denominators on the right hand side of (20) don't vanish by (17), (18) and non-degeneracy of the matrix $\bar{g}$. Computing derivatives of the solution y which are derivatives of the generating function f, at a tame semi-simple point $(x_1, \ldots, x_\rho, 0, \ldots, 0)$ in the parameter space of the small quantum deformation

$$\frac{\partial^{m_1+\ldots+m_\sigma} f}{\partial x_1^{m_1} \ldots \partial x_\sigma^{m_\sigma}} |_{x_{\rho+1}=\ldots=x_\sigma=0} = \sum_{k_1,\ldots,k_\rho} k_1^{m_1} \ldots k_\rho^{m_\rho} e^{k_1 x_1+\ldots+k_\rho x_\rho} N_{k_1\ldots k_\rho m_{\rho+1}\ldots m_\sigma} \tag{34}$$

with the following condition on non-vanishing summands

$$\sum_{a=1}^{\rho} k_a c_a = \sum_{a=\rho+1}^{\sigma} m_a(p_a - 1) + 3 - n, \tag{35}$$

we see, provided the condition C is satisfied, that on the right hand side of (34) we have a finite sum, because there is only a finite number of non-negative vectors $(k_1, \ldots, k_\rho)$ which are solutions to (35).

By the Lagrange interpolation we find polynomials

$$P_{l_1\ldots l_\rho}(t_1, \ldots, t_\rho) \in \mathbb{Q}[t_1, \ldots, t_\rho]$$

such that for all (a finite number of) non-negative solutions $(k_1, \ldots, k_\rho)$ to (35)

$$P_{l_1\ldots l_\rho}(k_1, \ldots, k_\rho) = \delta_{k_1 l_1} \ldots \delta_{k_\rho l_\rho}. \tag{36}$$

If we expand these polynomials as follows

$$P_{l_1\ldots l_\rho}(t_1, \ldots, t_\rho) = \sum_{m_1,\ldots m_\rho} a_{l_1\ldots l_\rho, m_1\ldots m_\rho} t_1^{m_1} \ldots t_\rho^{m_\rho} \tag{37}$$

and apply this expansion to monomials $k_1^{m_1} \ldots k_\rho^{m_\rho}$ on the right hand side of (34) substituted instead of $t_1^{m_1} \ldots t_\rho^{m_\rho}$, we get by (36) the following expression for the Gromov-Witten numbers

$$N_{k_1\ldots k_\sigma} = \tag{38}$$

$$= e^{-(k_1 x_1+\ldots+k_\rho x_\rho)} \sum_{m_1,\ldots,m_\rho} a_{k_1\ldots k_\rho, m_1\ldots m_\rho} \frac{\partial^{m_1+\ldots+m_\rho+k_{\rho+1}+\ldots+k_\sigma} f}{\partial x_1^{m_1} \ldots \partial x_\rho^{m_\rho} \partial x_{\rho+1}^{k_{\rho+1}} \ldots \partial x_\sigma^{k_\sigma}} |_{x_{\rho+1}=\ldots=x_\sigma=0} .$$

Note that the result is independent of the choice of a semi-simple point $(x_1, \ldots, x_\rho, 0, \ldots, 0)$.

Corollary 1. *The process of consecutive differentiation of (14) and substitution of already computed derivatives using (13) is a recursive algorithm computing all Gromov-Witten numbers from a finite number of initial ones by evaluating (38) at values of recursively computed derivatives.*

By (34) and (35) the set of initial Gromov-Witten numbers consists of

$$N_{k_1\ldots k_\rho 0\ldots 0} \qquad \text{where} \quad \sum_{c=1}^{\rho} k_c c_c = 3 - n, \tag{39}$$

$$N_{k_1\ldots k_\rho 0\ldots 1_a\ldots 0} \qquad \text{where} \quad \sum_{c=1}^{\rho} k_c c_c = p_a + 2 - n, \tag{40}$$

$$N_{k_1\ldots k_\rho 0\ldots 1_a\ldots 1_b\ldots 0} \qquad \text{where} \quad \sum_{c=1}^{\rho} k_c c_c = p_a + p_b + 1 - n, \tag{41}$$

$$N_{k_1\ldots k_\rho 0\ldots 2_a\ldots 0} \qquad \text{where} \quad \sum_{c=1}^{\rho} k_c c_c = 2p_a + 1 - n. \tag{42}$$

In view of paragraph 10, this can be regarded as a generalization of the Kontsevich recursive formula for the projective plane.

Remark. For a Fano n-fold of index r, among the initial Gromov-Witten numbers the numbers $N_{k_1\ldots k_\rho 0\ldots 0}$ (resp. $N_{k_1\ldots k_\rho 0\ldots 1_a\ldots 0}$, $N_{k_1\ldots k_\rho 0\ldots 1_a\ldots 1_b\ldots 0}$, $N_{k_1\ldots k_\rho 0\ldots 2_a\ldots 0}$) can appear only if $n+r \leq 3$ (resp. $n+r-2 \leq p_a \leq n$, $n+r-1 \leq p_a+p_b \leq 2n-1$, $(n+r-1)/2 \leq p_a \leq n$).

In particular, for $\mathbb{P}^n$, $n \geq 2$, the only initial Gromov-Witten number with respect to the basic classes $H, H^2, \ldots, H^n$ is the number $N_{10\ldots 02} = 1$ of lines passing through two points in general position. The only computational difficulty here consists in computing the rational function $R_{\alpha\beta\gamma}(Y)$. However, for $n > 2$ the result obtained with use of `RootSum` function of *Mathematica* according to Lemma 1, is hard to display. Since applying our algorithm needs further preparations by hand, it seems that it requires better computational tools than *Mathematica*, regretfully not known to the author.

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