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Abstract

This work aims to prove the convergence in the space L^2 of an iterative algorithmic method proposed by Kozlov *et al.* [Comput. Maths. Math. Phys. 31 (1991) 45] for the reconstruction of solutions to a particular type of ill-posed Cauchy problem associated with the Helmholtz equation on an infinite strip $\mathbb{R}^3$.

1 Introduction

Kozlov and Maz'ya [4] proposed an alternating iterative method to solve Cauchy problems for general strongly elliptic and formally self-adjoint systems. From the theoretical aspect, the method is based on solving successive well-posed mixed boundary value problems by using the Cauchy data as part of the boundary data. In [3] a convergence of the alternating method was proved for problems described by elliptic, symmetric and coercive operators. In this work, we consider the following inverse Cauchy problem for the Helmholtz equation in an infinite "strip", $\Omega \subset \mathbb{R}^3$.

$$\begin{cases} \Delta u + k^2 u = 0 \ , & \text{in } \Omega, \\ u(\rho, z) = g(\rho) & \text{on } \Gamma, \\ \partial_z u(\rho, z) = h(\rho) & \text{on } \Gamma, \\ u(\cdot, z) \in L^2(\mathbb{R}) & \text{z} \in (0, d). \end{cases} \tag{1}$$

Recently, the Cauchy problem associated with the Helmholtz equation in a strip has been considered in [9]. Here $k > 0$ is wave number [9], and the exact data g, h in (1) are from the class L^2 such that there exists unique solution $u \in H^2(\Omega)$ to the problem (1). In this

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problem, the Cauchy data are given at the subset $\Gamma \subset \partial\Omega$ (figure 1). The primary purpose of this alternating method is the reconstruction of appropriate solution to the problem (1) at the subset $\Gamma_0 \subset \partial\Omega$ (figure 1).

It has been shown in [4] that the convergence of the method holds for any initial approximation data $h^0(\rho) \in L^2(\Gamma_0)$. However, it is possible to give more particular case of this requirement as is seen in this study. Here, we present the solution of this Cauchy problem explicitly and show the ill-posedness of the problem by this solution. Additionally, we present a new convergence proof for the iterative algorithms. The key of the proof is simply based on convergence of the compounds $g^n(\rho)$ and $h^n(\rho)$ forming the iteration data $u^n(\rho, z)$. To this end, besides of the solution, we also obtain the necessary estimations arising from the alternating method.

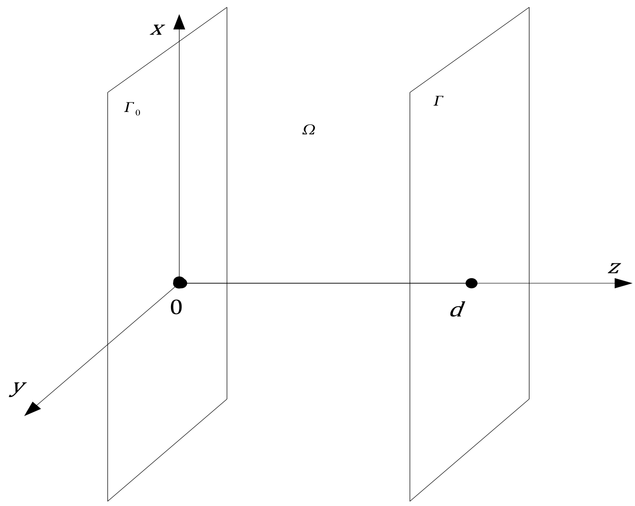

Figure 1: Geometry of the problem

The work is organized as follows. In section 2 we present the spaces and notations, and the ill-posedness of the problem (1). The alternating method is applied to the problem (1) in section 3. Moreover, still in section 3, convergence of the method is discussed.

2 Notations and definitions

We have $\Omega = \mathbb{R}^2 \times (0, d) \subset \mathbb{R}^3$, $d > 0$. First two spatial variables will be denoted by $\rho = (x, y)$. According to this notation the boundaries are $\Gamma_0 = \{(\rho\ ,\ 0) : \rho \in \mathbb{R}^2\ \} \subset \partial\Omega$ and $\Gamma = \{(\rho\ ,\ d) : \rho \in \mathbb{R}^2\ \} \subset \partial\Omega$.

As usual, $H^1(\Omega)$ denotes the first order Sobolev space of real-valued functions in Ω. The space of traces of functions from $H^1(\Omega)$ on Γ_0 is denoted by $H^{1/2}(\Gamma_0)$. This space is equipped with the norm

$$\|u\|_{H^{1/2}(\Gamma_0)} = \left(\int_{\Gamma_0} (1 + |\xi|^2)|\widehat{u}(\xi, 0)|^2 d\xi\right)^{1/2}$$

where $|\xi|^2 = {\xi_1}^2 + {\xi_2}^2$, and for $\xi\rho = \xi_1 x + \xi_2 y$, the Fourier transformation of the function $u(\rho, z)$ with respect to the variable ρ is $\widehat{u}(\xi, z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} u(\rho, z)e^{-i\xi\cdot\rho}\mathrm{d}\rho$.

In order to show the ill-posedness of the problem (1), let us find the solution of the problem. Since $u(\cdot, z) \in H^1(\mathbb{R}^2)$ for z $\in (0, d)$ and $g, h \in L^2$ we can apply to them the Fourier transform with respect to variables $\rho \in \mathbb{R}^2$. Thus

$$\begin{cases} \widehat{u}_{zz}(\xi, z) = (|\xi|^2 - k^2)\widehat{v}(\xi, z), & \xi \in \mathbb{R}^2, \mathrm{z} \in (0, d) \\ \widehat{u}(\xi,\mathrm{d}) = \widehat{g}(\xi), & \xi \in \mathbb{R}^2 \\ \partial_z \widehat{u}(\xi, d) = \widehat{h}(\xi), & \xi \in \mathbb{R}^2. \end{cases} \tag{2}$$

The solution of this problem is

$$\widehat{u}(\xi, z) = \widehat{g}(\xi)\cosh\left((d - z)\sqrt{\eta(\xi)}\right) - \widehat{h}(\xi)\frac{\sinh\left((d - z)\sqrt{\eta(\xi)}\right)}{\sqrt{\eta(\xi)}} \quad , \tag{3}$$

where $\eta(\xi) = |\xi|^2 - k^2$. It follows from here that

$$u(\rho, z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\xi\cdot\rho}\left[\widehat{g}(\xi)\cosh\left((d - z)\sqrt{\eta(\xi)}\right) - \widehat{h}(\xi)\frac{\sinh\left((d - z)\sqrt{\eta(\xi)}\right)}{\sqrt{\eta(\xi)}}\right] d\rho \quad . \tag{4}$$

Ill-posedness of the problem (1) can be seen from its solution (4). Boundedness of $u(\cdot, z)$ in the L^2-norm implies the rapid decay of $\widehat{h}(\xi)$ as $\xi \to \infty$. Specifically, for $\widehat{A}(z) = \frac{1}{\cosh\left(d\sqrt{\eta(\xi)}\right)}$, a case of (3) can be presented as an operator equation when $h(\rho) \equiv 0$

$$\widehat{A}(z)\widehat{u}(\xi, z) = \widehat{u}(\xi, d) \quad ,$$

$\widehat{A}(z) : \mathfrak{D}(\widehat{A}(z)) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$. However, the operator $\widehat{A}(z)$ is merely bounded for $kd < \frac{\pi}{2}$. For details see *Theorem 3.1* , [9] , and [10]. A well-posed system from this Cauchy problem (1) is obtained by an alternating method which is presented below.

3 Application of the alternating method

The Cauchy problem (1) cannot be solved numerically by using a direct approach, such as the Gauss elimination method, since this problem is ill-posed. Instead we use a numerical method proposed by Kozlov et al. [4] for Cauchy problems associated to linear, elliptic, self-adjoint and positive-definite operators. By this method, our objective is to reconstruct traces of the solution of an elliptic equation in the region $\Gamma_0 \subset \partial\Omega$. To do so, we successively solve on each step Dirichlet-Neumann or Neumann-Dirichlet boundary value problems. For $Lu = \Delta u + k^2 u$, the method substantially consists of the following two algorithmic steps:

Step-1: The approximate solution $u^{2n-1}(\rho, z) \in H^1(\Omega)$ for $n \geq 1$ solves the following mixed boundary value problem,

$$\begin{cases} Lu^{2n-1} = 0, & \text{in } \Omega, \\ u^{2n-1}(\rho, z) = g(\rho) & \text{on } \Gamma, \\ h^{2n-1}(\rho) \equiv \partial_z u^{2n-1}(\rho, z) = \partial_z u^{2n-2}(\rho, z) & \text{on } \Gamma_0. \end{cases} \tag{5}$$

in order to determine $u^{2n-1}(\rho, z)$ in Ω and $u^{2n-1}(\rho, z)$ on Γ_0.

Step-2: The approximate solution $u^{2n}(\rho, z) \in H^1(\Omega)$ for $n \geq 1$ is constructed, and the following mixed boundary value problem

$$\begin{cases} Lu^{2n} = 0 & \text{in } \Omega, \\ u^{2n}(\rho, z) = u^{2n-1}(\rho, z) & \text{on } \Gamma_0, \\ h^{2n}(\rho) \equiv \partial_z u^{2n}(\rho, z) = h(\rho, z) & \text{on } \Gamma, \end{cases} \tag{6}$$

is solved in order to determine $u^{2n}(\rho, z)$ in Ω and $\partial_z u^{2n}(\rho, z)$ on the boundary Γ_0.

The mixed boundary value problems (5) and (6) are solvable in $H^1(\Omega)$ provided that k^2 is not an eigenvalue of the Laplacian operator Δ, see [5].

In order for initiation of the procedure, let us consider the case of $n = 1$. Thus for a specified initial approximation $h^0(\rho)$ on Γ_0, $u^1(\rho, z) \in H^1(\Omega)$ solves the following mixed boundary value problem,

$$\begin{cases} Lu^1 = 0 & \text{in } \Omega, \\ u^1(\rho, z) = g(\rho) & \text{on } \Gamma, \\ h^1(\rho) \equiv \partial_z u^{(1)}(\rho, z) = h^0(\rho) & \text{on } \Gamma_0. \end{cases} \tag{7}$$

By this problem we obtain $u^1(\rho, z)$ in Ω, and $u^1(\rho, z)$ on the boundary Γ_0.

Theorem 3.1. *For $dk < \frac{\pi}{2}$; If the given data $(g, h^0) \in (L^2(\Gamma) \times L^2(\Gamma_0))$, then solution of the problem (7) is*

$$u^1(\rho, z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\xi\cdot\rho} \left[\widehat{g}(\xi) \frac{\cosh\left(z\sqrt{\eta(\xi)}\right)}{\cosh\left(d\sqrt{\eta(\xi)}\right)} - \widehat{h}^0(\xi) \frac{\sinh\left((d-z)\sqrt{\eta(\xi)}\right)}{\sqrt{\eta(\xi)}\cosh\left(d\sqrt{\eta(\xi)}\right)} \right] d\rho, \tag{8}$$

and this solution is unique and continuously dependent in L^2-norm on the given data g and h^0, as follows:

$$\|u^1\|^2_{L^2(\Omega)} \leq 2dK \left(\|g\|^2_{L^2(\Gamma)} + \|h^0\|^2_{L^2(\Gamma_0)}\right)^2.$$

Proof: After applying the Fourier transformation to the problem (7), its solution reads

$$\widehat{u^1}(\xi, z) = \widehat{g}(\xi)\frac{\cosh\left(z\sqrt{\eta(\xi)}\right)}{\cosh\left(d\sqrt{\eta(\xi)}\right)} - \widehat{h^0}(\xi)\frac{\sinh\left((d-z)\sqrt{\eta(\xi)}\right)}{\sqrt{\eta(\xi)}\cosh\left(d\sqrt{\eta(\xi)}\right)} \quad , \tag{9}$$

where $\eta(\xi) = |\xi|^2 - k^2$. In order to obtain the continuous dependence of the solution with respect to L^2-norm on the given data, we have to consider

$$\|u^1\|^2_{L^2(\Omega)} = \|\widehat{u^1}\|^2_{\mathrm{L}^2(\Omega)} = \int_0^d \|\widehat{u^1}(\cdot, z)\|^2_{\mathrm{L}^2(\mathbb{R}^2)}\mathrm{d}z,$$

where

$$\|\widehat{u^1}(\cdot, z)\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\widehat{u^1}(\xi, z)|^2 d\xi = \int_{|\xi|<k} |\widehat{u^1}(\xi, z)|^2 d\xi + \int_{|\xi|>k} |\widehat{u^1}(\xi, z)|^2 d\xi = I_1(k) + I_2(k).$$

Considering (9), the first integral

$$I_1(k) = \int_{|\xi|<k} \left|\widehat{g}(\xi)\frac{\cos\left(z\sqrt{-\eta(\xi)}\right)}{\cos\left(d\sqrt{-\eta(\xi)}\right)} - i\widehat{h^0}(\xi)\frac{\sin\left((d-z)\sqrt{-\eta(\xi)}\right)}{\sqrt{-\eta(\xi)}\cos\left(d\sqrt{-\eta(\xi)}\right)}\right|^2 \mathrm{d}\xi$$

$$\leq 2\int_{|\xi|<k} \left|\widehat{g}(\xi)\frac{\cos\left(z\sqrt{-\eta(\xi)}\right)}{\cos\left(d\sqrt{-\eta(\xi)}\right)}\right|^2 \mathrm{d}\xi + 2\int_{|\xi|<k} \left|\widehat{h^0}(\xi)\frac{\sin\left((d-z)\sqrt{-\eta(\xi)}\right)}{\sqrt{-\eta(\xi)}\cos\left(d\sqrt{-\eta(\xi)}\right)}\right|^2 \mathrm{d}\xi.$$

In the interval $(0, \pi/2)$, the function $\cos(x)$ is decreasing while $\sin(x)$ is increasing. So that, for $|\xi| < k$ and $0 < (d-z)\sqrt{\eta(\cdot)} < d\sqrt{\eta(\cdot)}$, $\left|\sin\left((d-z)\sqrt{-\eta(\xi)}\right)\right| \leq |\sin(dk)|$ and $\left|\cos\left(d\sqrt{-\eta(\xi)}\right)\right| \geq |\cos(dk)|$. Also, for $x \in (0, \pi/2)$, $|\cos(x)| \leq 1$, then we have

$$I_1(k) \leq 2\frac{1}{|\cos^2(dk)|}\int_{|\xi|<k} |\widehat{g}(\xi)|^2 \mathrm{d}\xi + 2\frac{|\tan^2(dk)|}{k^2}\int_{|\xi|<k} |\widehat{h^0}(\xi)|^2 \mathrm{d}\xi.$$

On the other hand,

$$I_2(k) = \int_{|\xi|>k} \left| \widehat{g}(\xi) \frac{\cosh\left(z\sqrt{\eta(\xi)}\right)}{\cosh\left(d\sqrt{\eta(\xi)}\right)} - \widehat{h^0}(\xi) \frac{\sinh\left((d-z)\sqrt{\eta(\xi)}\right)}{\sqrt{\eta(\xi)}\cosh\left(d\sqrt{\eta(\xi)}\right)} \right|^2 \mathrm{d}\xi$$

$$\leq 2\int_{|\xi|>k} \left| \widehat{g}(\xi) \frac{\cosh\left(z\sqrt{\eta(\xi)}\right)}{\cosh\left(d\sqrt{\eta(\xi)}\right)} \right|^2 \mathrm{d}\xi + 2\int_{|\xi|>k} \left| \widehat{h^0}(\xi) \frac{\sinh\left((d-z)\sqrt{\eta(\xi)}\right)}{\sqrt{\eta(\xi)}\cosh\left(d\sqrt{\eta(\xi)}\right)} \right|^2 \mathrm{d}\xi.$$

For $|\xi| > k$ and $z \in (0, d)$, $\sinh\left((d-z)\sqrt{\eta(\xi)}\right) \leq \sinh\left(d\sqrt{\eta(\xi)}\right)$ and $\cosh\left(z\sqrt{\eta(\xi)}\right) \leq \cosh\left(d\sqrt{\eta(\xi)}\right)$. So that,

$$I_2(k) \leq 2\int_{|\xi|>k} |\widehat{g}(\xi)|^2 \mathrm{d}\xi + 2\int_{|\xi|>k} |\widehat{h^0}(\xi)|^2 \left| \frac{\tanh\left(d\sqrt{\eta(\xi)}\right)}{\sqrt{\eta(\xi)}} \right|^2 \mathrm{d}\xi,$$

for $|\xi| > k$.
Since for $d, x > 0$, $\tanh(dx) \leq dx$, then

$$I_2(k) \leq 2\int_{|\xi|>k} |\widehat{g}(\xi)|^2 \mathrm{d}\xi + 2d^2 \int_{|\xi|>k} |\widehat{h^0}(\xi)|^2 \mathrm{d}\xi$$

For $C_1 = \max\{1, \frac{1}{|cos^2(dk)|}\}$ and $C_2 = \max\{d^2, \frac{|\tan^2(dk)|}{k^2}\}$, we obtain

$$\|u^1\|^2_{L^2(\mathbb{R}^2)} \leq 2\left(C_1\|g\|^2_{L^2(\Gamma)} + C_2\|h^0\|^2_{L^2(\Gamma_0)}\right),$$

it follows that for $K = \max\{C_1, C_2\}$,

$$\|u^1\|^2_{L^2(\mathbb{R}^2)} \leq 2K\left(\|g\|^2_{L^2(\Gamma)} + \|h^0\|^2_{L^2(\Gamma_0)}\right).$$

Hence,

$$\|u^1\|^2_{L^2(\Omega)} = \int_0^d \|u^1(\cdot, z)\|^2_{L^2(\mathbb{R}^2)} \mathrm{d}z \leq 2dK\left(\|g\|^2_{L^2(\Gamma)} + \|h^0\|^2_{L^2(\Gamma_0)}\right). \tag{10}$$

which is the desired conclusion.

■

We have analyzed the existence and continuous dependency of the data for the boundary value problem (7). The same estimation can also be obtained by replacing Γ with Γ_0.

Now, let us analyze the boundedness of $u^{(1)}(\rho, 0)$. To do so, we inherently address (9). So,

$$\widehat{u^1}(\xi,0)=\widehat{g}(\xi)\frac{1}{\cos\left(d\sqrt{-\eta(\xi)}\right)}-i\widehat{h^0}(\xi)\frac{\tan\left(d\sqrt{-\eta(\xi)}\right)}{\sqrt{-\eta(\xi)}}\quad,\quad \text{for}|\xi|<k \tag{11}$$

$$\widehat{u^1}(\xi,0)=\widehat{g}(\xi)\frac{1}{\cosh\left(d\sqrt{\eta(\xi)}\right)}-\widehat{h^0}(\xi)\frac{\tanh\left(d\sqrt{\eta(\xi)}\right)}{\sqrt{\eta(\xi)}}\quad,\quad \text{for}|\xi|>k. \tag{12}$$

If we multiply both sides of (11) and (12) by $\sqrt{-\eta(\xi)}$ and $\sqrt{\eta(\xi)}$ respectively, then we have

$$\widehat{u^1}(\xi,0)\sqrt{-\eta(\xi)}=\widehat{g}(\xi)\sqrt{-\eta(\xi)}\frac{1}{\cos\left(d\sqrt{-\eta(\xi)}\right)}-i\widehat{h^0}(\xi)\tan\left(d\sqrt{-\eta(\xi)}\right), \tag{13}$$

and

$$\widehat{u^1}(\xi,0)\sqrt{\eta(\xi)}=\widehat{g}(\xi)\sqrt{\eta(\xi)}\frac{1}{\cosh\left(d\sqrt{\eta(\xi)}\right)}-\widehat{h^0}(\xi)\tanh\left(d\sqrt{\eta(\xi)}\right). \tag{14}$$

By this construction of $u^1(\rho,z)$ on the boundary Γ_0, one can see that the boundedness of the specified data $h^0(\rho)$ in L^2-norm is provided by the restriction of $dk\in(0,\pi/4)$. In the following Lemma, for smooth enough functions $g(\rho)$, we show that the norm $\|u^1(\rho,0)\|_{H^{1/2}}$ is bounded under this restriction.

Lemma 3.2. *For $0<dk<\pi/4$; If the given data $(g,h^0)\in(\mathrm{H}^{1/2}(\Gamma)\times\mathrm{L}^2(\Gamma_0))$, then $u^1(\cdot,0)\in\mathrm{L}^2(\mathbb{R}^2)$, and the following estimation holds for $C_1=max\{1,\frac{1}{|cos^2(dk)|}\}$,*

$$\|u^1(\cdot,0)\|^2_{\mathrm{H}^{1/2}(\mathbb{R}^2)}\le 2\left(C_1\|g\|^2_{L^2(\Gamma)}+\|h^0\|^2_{L^2(\Gamma_0)}\right). \tag{15}$$

Proof: In the same manner with the Theorem (3.1), one can easily find the constant $C_1=\max\{1,\frac{1}{|cos^2(dk)|}\}$, such that (15) holds.

■

So far, we have analyzed the solution $u^1(\rho,z)$ over Ω and more particularly trace of it on the boundary Γ_0. However, we are interested in this analysis for $u^n(\rho,z)$, for $n\ge1$. To this end, the convergence of $h^n(\rho)$ on the boundary Γ_0 is necessary. We can define the sequence of Neumann condition emerging from the boundary value problem (6).

$$\partial_z\widehat{u}^{2n}(\xi,0)=\widehat{h}(\xi)\frac{1}{\cosh\left(d\sqrt{\eta(\xi)}\right)}-\widehat{g}^{2n-1}(\xi)\sqrt{\eta(\xi)}\tanh\left(d\sqrt{\eta(\xi)}\right)\equiv\widehat{h}^{2n}(\xi). \tag{16}$$

Here, Dirichlet condition $\widehat{g}^{2n-1}(\xi):=\widehat{u}^{2n-1}(\xi,0)$ can easily be obtained solving the former boundary value problem (6). Considering the Fourier transform to the problem, we obtain

$$\widehat{u}^{2n-1}(\xi,z)=\widehat{g}(\xi)\frac{\cosh\left(z\sqrt{\eta(\xi)}\right)}{\cosh\left(d\sqrt{\eta(\xi)}\right)}-\widehat{h}^{2n-2}(\xi)\frac{\sinh\left((d-z)\sqrt{\eta(\xi)}\right)}{\sqrt{\eta(\xi)}\cosh\left(d\sqrt{\eta(\xi)}\right)},$$

and hence

$$\widehat{u}^{2n-1}(\xi,0)=\widehat{g}(\xi)\frac{1}{\cosh\left(d\sqrt{\eta(\xi)}\right)}-\widehat{h}^{2n-2}(\xi)\frac{\tanh\left(d\sqrt{\eta(\xi)}\right)}{\sqrt{\eta(\xi)}}\equiv\widehat{g}^{2n-1}(\xi). \tag{17}$$

If we plug the formula (17) into (16), then one gets

$$\widehat{h}^{2n}(\xi)=\widehat{h}(\xi)\frac{1}{\cosh\left(d\sqrt{\eta(\xi)}\right)}+\widehat{h}^{2n-2}(\xi)\tanh^2\left(d\sqrt{\eta(\xi)}\right)-\widehat{g}(\xi)\sqrt{\eta(\xi)}\frac{\tanh\left(d\sqrt{\eta(\xi)}\right)}{\cosh\left(d\sqrt{\eta(\xi)}\right)}. \tag{18}$$

As an immediate consequence of this tiny analysis, we can give the following Lemma.

Lemma 3.3. *For the specified initial data $h^0(\rho)\in L^2(\mathbb{R}^2)$, if the formula (18) is valid for any $n=1,2,3\cdots$, then the following formula also holds for any $n=1,2,3\cdots$,*

$$\widehat{h}^{2n}(\xi)=\widehat{h}^0(\xi)\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right)+\frac{\widehat{h}(\xi)}{\cosh\left(d\sqrt{\eta(\xi)}\right)}\sum_{j=1}^{n}\tanh^{2j-2}\left(d\sqrt{\eta(\xi)}\right)-$$

$$-\frac{\widehat{g}(\xi)\sqrt{\eta(\xi)}}{\cosh\left(d\sqrt{\eta(\xi)}\right)}\sum_{j=1}^{n}\tanh^{2j-1}\left(d\sqrt{\eta(\xi)}\right). \tag{19}$$

Proof: We prove the formula (19) by the method of induction.
For $n=1$, equation (18)

$$\widehat{h}^2(\xi)=\widehat{h}^0(\xi)\tanh^2\left(d\sqrt{\eta(\xi)}\right)+\frac{\widehat{h}(\xi)}{\cosh\left(d\sqrt{\eta(\xi)}\right)}-\frac{\widehat{g}(\xi)\sqrt{\eta(\xi)}}{\cosh\left(d\sqrt{\eta(\xi)}\right)}\tanh\left(d\sqrt{\eta(\xi)}\right).$$

Thus formula (19) is true for $n=1$. Now, let us assume that the formula (19) is valid for $n-1$. That is,

$$\widehat{h}^{2n-2}(\xi)=\widehat{h}^0(\xi)\tanh^{2n-2}\left(d\sqrt{\eta(\xi)}\right)+\frac{\widehat{h}(\xi)}{\cosh\left(d\sqrt{\eta(\xi)}\right)}\sum_{j=1}^{n-1}\tanh^{2j-2}\left(d\sqrt{\eta(\xi)}\right)-$$

$$-\frac{\widehat{g}(\xi)\sqrt{\eta(\xi)}}{\cosh\left(d\sqrt{\eta(\xi)}\right)}\sum_{j=1}^{n-1}\tanh^{2j-1}\left(d\sqrt{\eta(\xi)}\right). \tag{20}$$

Since (18) is true for any n, including (20) inside (18), we get

$$\widehat{h}^{2n}(\xi) = \widehat{h}^{0}(\xi)\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right) + \frac{\widehat{h}(\xi)}{\cosh\left(d\sqrt{\eta(\xi)}\right)}\sum_{j=1}^{n}\tanh^{2j-2}\left(d\sqrt{\eta(\xi)}\right) -$$

$$- \frac{\widehat{g}(\xi)\sqrt{\eta(\xi)}}{\cosh\left(d\sqrt{\eta(\xi)}\right)}\sum_{j=1}^{n}\tanh^{2j-1}\left(d\sqrt{\eta(\xi)}\right). \tag{21}$$

This shows that formula (19) is valid for any $n = 1, 2, 3....$

■

Of course, the convergence of $u^n(\rho, z)$ depends on the reconstructed data $h^n(\rho)$. Thus, it should be expected that the data $h^n(\rho)$ is convergent. Since we are interested in the reconstruction of the approximate solution on the boundary Γ_0, h^n should converge to the exact data defined on the boundary Γ_0. Let us recall (3) to investigate this convergence. It follows that for $\partial_z\widehat{u}(\xi, z)$ where $\xi \in \Gamma_0$,

$$\partial_z\widehat{u}(\xi, z)|_{\xi\in\Gamma_0} \equiv \widehat{\widehat{h}}(\xi) = -\widehat{g}(\xi)\sqrt{\eta(\xi)}\sinh\left(d\sqrt{\eta(\xi)}\right) + \widehat{h}(\xi)\cosh\left(d\sqrt{\eta(\xi)}\right). \tag{22}$$

For $|\xi| < k$;

$$\widehat{h}^{2n}(\xi) = (-1)^n\widehat{h}^{0}(\xi)\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right) + \frac{\widehat{h}(\xi)}{\cos\left(d\sqrt{-\eta(\xi)}\right)}\sum_{j=1}^{n}\left(i\tan\left(d\sqrt{-\eta(\xi)}\right)\right)^{2j-2} -$$

$$- \frac{\widehat{g}(\xi)\sqrt{-\eta(\xi)}}{\cos\left(d\sqrt{-\eta(\xi)}\right)}\sum_{j=1}^{n}\left(i\tan\left(d\sqrt{-\eta(\xi)}\right)\right)^{2j-1},$$

by considering the value of two finite sums above, we obtain

$$\widehat{h}^{2n}(\xi) = (-1)^n\widehat{h}^{0}(\xi)\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right) + \widehat{h}(\xi)\cos\left(d\sqrt{-\eta(\xi)}\right)\left[1 + (-1)^{n+1}\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right)\right] -$$

$$-i\widehat{g}(\xi)\sqrt{-\eta(\xi)}\sin\left(d\sqrt{-\eta(\xi)}\right)\left[1 + (-1)^{n+1}\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right)\right],$$

which is in other words

$$\widehat{h}^{2n}(\xi) = (-1)^n \widehat{h}^0(\xi) \tan^{2n}\left(d\sqrt{-\eta(\xi)}\right) + \{\widehat{h}(\xi)\cos\left(d\sqrt{-\eta(\xi)}\right) -$$

$$-i\widehat{g}(\xi)\sqrt{-\eta(\xi)}\sin\left(d\sqrt{-\eta(\xi)}\right)\}\left[1 + (-1)^{n+1}\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right)\right]. \quad (23)$$

On the other hand, for $|\xi| > k$;

$$\widehat{h}^{2n}(\xi) = \widehat{h}^0(\xi)\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right) + \frac{\widehat{h}(\xi)}{\cosh\left(d\sqrt{\eta(\xi)}\right)}\sum_{j=1}^{n}\tanh^{2j-2}\left(d\sqrt{\eta(\xi)}\right) -$$

$$-\frac{\widehat{g}(\xi)\sqrt{\eta(\xi)}}{\cosh\left(d\sqrt{\eta(\xi)}\right)}\sum_{j=1}^{n}\tanh^{2j-1}\left(d\sqrt{\eta(\xi)}\right),$$

again, in the same manner of above one can easily get

$$\widehat{h}^{2n}(\xi) = \widehat{h}^0(\xi)\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right) + \{\widehat{h}(\xi)\cosh\left(d\sqrt{\eta(\xi)}\right)$$

$$-\widehat{g}(\xi)\sqrt{\eta(\xi)}\sinh\left(d\sqrt{\eta(\xi)}\right)\}\left[1 - \tanh^{2n}\left(d\sqrt{\eta(\xi)}\right)\right]. \quad (24)$$

For $|\xi| < k$,

$$\widehat{h}^{2n}(\xi) - \widehat{\widehat{h}}(\xi) = (-1)^n \widehat{h}^0(\xi)\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right) -$$

$$-(-1)^n\{\widehat{h}(\xi) - i\widehat{g}(\xi)\sqrt{-\eta(\xi)}\sin\left(d\sqrt{-\eta(\xi)}\right)\}\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right),$$

thus

$$\widehat{h}^{2n}(\xi) - \widehat{\widehat{h}}(\xi) = (-1)^n \widehat{h}^0(\xi)\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right) - (-1)^n\widehat{\widehat{h}}(\xi)\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right), \quad (25)$$

where $\widehat{\widehat{h}}(\xi) = -i\widehat{g}(\xi)\sqrt{-\eta(\xi)}\sin\left(d\sqrt{-\eta(\xi)}\right) + \widehat{h}(\xi)\cos\left(d\sqrt{-\eta(\xi)}\right)$, for $|\xi| < k$. On the other hand, for $|\xi| > k$, we have

$$\widehat{h}^{2n}(\xi) - \widehat{\widehat{h}}(\xi) = \widehat{h}^0(\xi)\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right) -$$

$$-\{\widehat{h}(\xi)\cosh\left(d\sqrt{\eta(\xi)}\right) - \widehat{g}(\xi)\sqrt{\eta(\xi)}\sinh\left(d\sqrt{\eta(\xi)}\right)\}\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right),$$

and thus

$$\widehat{h}^{2n}(\xi) - \widehat{\widetilde{h}}(\xi) = \widehat{h}^0(\xi)\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right) - \widehat{\widetilde{h}}(\xi)\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right), \tag{26}$$

where $\widehat{\widetilde{h}}(\xi) = -\widehat{g}(\xi)\sqrt{\eta(\xi)}\sinh\left(d\sqrt{\eta(\xi)}\right) + \widehat{h}(\xi)\cosh\left(d\sqrt{\eta(\xi)}\right)$, for $|\xi| > k$.

The main result of this work can be formulated as follows;

Theorem 3.4. *For $dk \in (0, \frac{\pi}{4})$. If $\widetilde{h}$, $h^0 \in H^{1/2}(\Gamma_0)$, then $\|h^{2n} - \widetilde{h}\|_{\mathrm{L}^2} \to 0$ as $n \to \infty$.*

Proof: Again we have to consider

$$\begin{aligned}
\|h^{2n}(\rho) - \widetilde{h}(\rho)\|^2_{L^2} &= \int_{|\xi|<k} |\widehat{h}^{2n}(\xi) - \widehat{\widetilde{h}}(\xi)|^2 \mathrm{d}\xi + \int_{|\xi|>k} |\widehat{h}^{2n}(\xi) - \widehat{\widetilde{h}}(\xi)|^2 \mathrm{d}\xi \\
&\le \int_{|\xi|<k}\left[\left|\widehat{h}^0(\xi)\right|\left|\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right)\right| + \left|\widehat{\widetilde{h}}(\xi)\right|\left|\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right)\right|\right]^2 \mathrm{d}\xi + \\
&+ \int_{|\xi|>k}\left[\left|\widehat{h}^0(\xi)\right|\left|\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right)\right| + \left|\widehat{\widetilde{h}}(\xi)\right|\left|\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right)\right|\right]^2 \mathrm{d}\xi
\end{aligned}$$

In the name of simplicity let us set

$$I_1(k) = \int_{|\xi|<k}\left[\left|\widehat{h}^0(\xi)\right|\left|\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right)\right)| + \left|\widehat{\widetilde{h}}(\xi)\right|\left|\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right)\right|\right]^2 \mathrm{d}\xi$$

and

$$I_2(k) = \int_{|\xi|>k}\left[\left|\widehat{h}^0(\xi)\right|\left|\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right)\right| + \left|\widehat{\widetilde{h}}(\xi)\right|\left|\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right)\right|\right]^2 \mathrm{d}\xi.$$

Since $sup_{(0,\pi/4)}|\tan(x)| = 1$, thus

$$I_1(k) \le \int_{|\xi|<k}\left[\left|\widehat{h}^0(\xi)\right|\left|\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right)\right| + \left|\widehat{\widetilde{h}}(\xi)\right|\left|\tan^{2n}\left(d\sqrt{-\eta(\xi)}\right)\right|\right]^2 \mathrm{d}\xi.$$

The function $\tan(x)$ is decreasing in this interval. So that, this last inequality can also be written as follows;

$$I_1(k) \le |\tan^{2n}(dk)|^2 \int_{|\xi|<k}\left(|\widehat{h}^0(\xi)| + |\widehat{\widetilde{h}}(\xi)|\right)^2 \mathrm{d}\xi. \tag{27}$$

For a $\delta \in (0, \pi/4)$, let us set $t_n = \sup_{0<dk\le\pi/4-\delta}|\tan^{2n}(dk)|^2$. By this setting one can see that as $n \to \infty$, $t_n \to 0$. In the same manner, the following inequality also holds for $|\xi| > k$;

$$\begin{aligned}
I_2(k) &\le \int_{|\xi|>k}\left[\left|\widehat{h}^0(\xi)\right|\left|\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right)\right| + \left|\widehat{\widetilde{h}}(\xi)\right|\left|\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right)\right|\right]^2 \mathrm{d}\xi \\
&\le 2\int_{|\xi|>k}\left[\left|\widehat{h}^0(\xi)\right|^2\left|\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right)\right|^2 + \left|\widehat{\widetilde{h}}(\xi)\right|^2\left|\tanh^{2n}\left(d\sqrt{\eta(\xi)}\right)\right|^2\right] \mathrm{d}\xi
\end{aligned}$$

$$\leq 2\int_{|\xi|>k}\left[\left|\widehat{h}^0(\xi)\right|^2\left|\tanh^{2n}\left(d(1+|\xi|^2)\right)\right|^2+\left|\widehat{\widetilde{h}}(\xi)\right|^2\left|\tanh^{2n}\left(d(1+|\xi|^2)\right)\right|^2\right]\mathrm{d}\xi$$

$$=2\int_{|\xi|>k}\left[|\widehat{h}^0(\xi)|^2(1+|\xi|^2)\frac{|\tanh^{2n}\left(d(1+|\xi|^2)\right)|^2}{(1+|\xi|^2)}+|\widehat{\widetilde{h}}(\xi)|^2(1+|\xi|^2)\frac{|\tanh^{2n}\left(d(1+|\xi|^2)\right)|^2}{(1+|\xi|^2)}\right]\mathrm{d}\xi.$$

In order to estimate $I_2(k)$ we are going to estimate the maximum value of the function $F(\xi)=\frac{|\tanh^{4n}\left(d(1+|\xi|^2)\right)|}{(1+|\xi|^2)}$ on the set $\{\xi\in\mathbb{R}^2:|\xi|\in[k,\infty)\}$. Let us rewrite this function; $F(\xi)=d\frac{|\tanh^{4n}\left(d(1+|\xi|^2)\right)|}{d(1+|\xi|^2)}$. Maximum value of the function $F(\xi)$ can also be obtained studying on the function $\widetilde{F}(x)=\frac{|\tanh^n(x)|}{x}$. If we evaluate the derivative of the last function and set it to zero, then we get

$$n=\frac{\sinh(2x_n)}{2x_n}, \tag{28}$$

here $n\to\infty$ as $x_n\to\infty$. There exists unique solution x_n for each $n\in\mathbb{N}$. The equation (28) can not be solved explicitly. So that, it is necessary to reform the function $\widetilde{F}(x)$ with respect to x_n for estimating $\widetilde{F}(x_n)$. It follows from the identity $\tanh(2\varphi)=\frac{2\tanh(\varphi)}{1+\tanh^2(\varphi)}$ that

$$\begin{aligned}\tanh(\varphi)&=\frac{\cosh(2\varphi)-1}{\sinh(2\varphi)}\\&=\frac{\sqrt{1+\sinh^2(2\varphi)}-1}{\sinh(2\varphi)}.\end{aligned}$$

This identity can also be defined with respect to (28) as follows

$$\tanh(x_n)=\frac{\sqrt{1+4x_n^2n^2}-1}{2x_nn}.$$

Thus, the maximum value of $\widetilde{F}(x)$ for each $n\in\mathbb{N}$ is

$$\widetilde{F}(x_n)=\frac{(\sqrt{1+4x_n^2n^2}-1)^n}{x_n(2x_nn)^n}=\frac{\left(\sqrt{\frac{1}{x_n^2n^2}+4}-\frac{1}{x_nn}\right)^n}{2^nx_n} \tag{29}$$

$$\lim_{n\to\infty}F(x_n)=\lim_{n\to\infty}\frac{1}{x_n}\left(\frac{\sqrt{\frac{1}{x_n^2n^2}+4}-\frac{1}{x_nn}}{2}\right)^n=\lim_{n\to\infty}\frac{1}{x_n}\left(\sqrt{\frac{1}{4x_n^2n^2}+1}-\frac{1}{2x_nn}\right)^n$$

[1]

$$\leq\lim_{n\to\infty}\frac{1}{x_n}\left(1+\frac{1}{8x_n^2n^2}-\frac{1}{2x_nn}\right)^n=\lim_{n\to\infty}\frac{1}{x_n}\left(1+\frac{1-4x_nn}{8x_n^2n^2}\right)^n$$

[1] Here, we have used a basic result of the Taylor expansion which is $\sqrt{1+x^2}\leq1+\frac{x^2}{2}$.

$$= \lim_{n\to\infty} \frac{1}{x_n} \left[\left(1 + \frac{1}{\frac{8x_n^2 n}{1-4x_n n}} \right)^{\frac{8x_n^2 n^2}{1-4x_n n}} \right]^{\frac{n-4x_n^2 n^2}{8x_n n}} = 0, \tag{30}$$

which together with the sequence emerged from (27) gives the desired result.

■

4 Conclusion

In this work, we have used an alternating method proposed by Kozlov & Maz'ya (1989) for solving a Cauchy problem for the Helmholtz operator which is defined on an infinite "strip" $\Omega \subset \mathbb{R}^3$. Besides of where our problems is defined, another challenging side of this work is that non-coercivity of the Helmholtz operator. In Marin *et al.* (2003) [6], it was shown that if the operator is coercive, then the method is convergent. In this paper we show that the method may also be convergent for certain problems with non-coercive operators.

It is known that Cauchy problems for elliptic equations are ill-posed [2], i.e. the solution does not depend continuously on the boundary data. By the alternating method applied to this problem, we have solved successive well-posed boundary value problems.

Note that, for solvability of the problems (5) and (6), it is sufficient that the specified data $h^0(\rho)$ is the class of $L^2(\Gamma_0)$. However, for the convergence of the method, it is necessary to take $h^0(\rho)$ from the class $H^{1/2}(\Gamma_0)$.

A number of numerical methods have been proposed to solve the problem. Marin et al. [6, 7, 8] have solved the Cauchy problem for the Helmholtz equation by employing the boundary element method (BEM) in conjunction with iterative algorithm.

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