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Abstract

Picard-Vessiot theory, that is, Galois theory of homogeneous linear differential equations, has been established for differential fields with algebraically closed field of constants. In this paper, we prove the existence of a Picard-Vessiot extension for a homogeneous linear differential equation defined over a real differential field K with real closed field of constants. To this aim, we use a Taylor morphism to obtain a differential embedding of the field K in a ring of formal power series. We give an adequate definition of the differential Galois group of a Picard-Vessiot extension of a real differential field with real closed field of constants, inspired in Kolchin's one for strongly normal extensions, and we prove a Galois correspondence theorem for such a Picard-Vessiot extension.

Keywords: Picard-Vessiot extension, real field, Galois correspondence
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Contents

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1 Introduction

Picard-Vessiot theory denotes Galois theory of homogeneous linear differential equations. It was established in a rigorous form by Kolchin under the hypothesis that the constant field of the base differential field is algebraically closed (see [5], [6]). The Picard-Vessiot extension associated to a given homogeneous linear differential operator is the analog of the splitting field of a given polynomial. For a homogeneous linear differential equation $\mathcal{L}(Y) = 0$ defined over a differential field K with field of constants C, a Picard-Vessiot extension is a differential field L, differentially generated over K by a fundamental system of solutions of $\mathcal{L}(Y) = 0$ and with constant field equal to C. In the case C algebraically closed, the Picard-Vessiot extension exists and is unique up to K-differential isomorphism. In [5], Kolchin quotes a remark of Baer who notes that the difficulty lies not in proving the existence of a fundamental system of solutions of the given differential equation but in proving the existence of one which brings in no new constants.

In [14], Seidenberg constructed an example of a linear differential equation defined over a differential field K with constant field the field $\mathbb{R}$ of real numbers, for which no Picard-Vessiot extension exists (see Example 2.3). At first sight, this example seems to indicate that it is not possible to obtain a general result on existence of Picard-Vessiot extension beyond the class of differential fields with algebraically closed field of constants (see [2] Section 5.3). Some misinterpretation of this example, quoted by several specialists, may explain the fact that such a general result has not been obtained since now. However, the differential field K in Seidenberg's example is not a real field (see Definition 2.4). In this paper we present an existence theorem of Picard-Vessiot extensions for real differential fields with real closed field of constants. It is worth noting that the field of rational functions $\mathbb{R}(x_1, \dots, x_n)$ and the field of real meromorphic functions are real fields hence our result will lead to applications in real analytic mechanics (see [9]).

It is well known that if L is a splitting field of a polynomial in $K[X]$, the extension $L|K$ is normal, i.e. for any $a \in L \setminus K$, there exists $\sigma \in Aut_K L$ such that $\sigma(a) \neq a$. If $L|K$ is a normal algebraic extension, then, for any field F with $K \subset F \subset L$, $L|F$ is normal as well. In his quest for a good concept of normality for differential field extensions, Kolchin observed that the direct analog of normality for differential field extensions is defective, as the property does not translate to intermediate differential fields (see Example 2.5). He defined then a differential field extension $L|K$ to be normal when for any differential field F with $K \subset F \subset L$ and any $a \in L \setminus F$, there exists a differential automorphism σ of L over F such that $\sigma(a) \neq a$. However, the Galois correspondence theorem for normal differential extensions has some failures. Kolchin finally introduced the concept of strongly normal extension (see Definition 2.1) and obtained a satisfactory Galois correspondence theorem for this class of extensions. Note that, for a strongly normal extension $L|K$, the differential Galois group is no longer the group $DAut_K L$ of K-differential automorphisms of L, rather one has to consider as well K-differential morphisms of L in larger differential fields.

It is worth noting that a Picard-Vessiot extension of a differential field with algebraically closed field of constants is normal, in Kolchin's sense, but this is no longer true for a Picard-Vessiot extension of a real differential field with real closed field of constants. However a Picard-Vessiot extension is always strongly normal. In the case of

Picard-Vessiot extensions of real fields with real closed field of constants, we can adopt a definition of the differential Galois group inspired by Kolchin's but simpler than his one. We obtain then a Galois correspondence theorem which classifies intermediate differential fields of a Picard-Vessiot extension of a real differential field with real closed field of constants in terms of its differential Galois group.

We refer the reader to [1] for topics on real field theory, to [3], [7] or [11] for topics on differential Galois theory.

2 Preliminaries

In the sequel, all fields considered will be of characteristic 0 and C_K will denote the constant field of the differential field K.

We recall now the notions of normality for differential field extensions introduced by Kolchin and the precise definition of Picard-Vessiot extension. We adopt Umemura's definition of strong morphism and strongly normal extension (see [20]).

Definition 2.1. *Let $L|K$ be an extension of differential fields.*

1. *$L|K$ is* weakly normal *if for every $a \in L \setminus K$, there exists $\sigma \in DAut_K L$ such that $\sigma(a) \neq a$.*
2. *$L|K$ is* normal *if for every differential field F, with $K \subset F \subset L$, $L|F$ is weakly normal.*
3. *If M is a differential field extension of K, $f, g : L \to M$ are differential K-morphisms, we say that f is* strong *over g if the following two conditions are satisfied.*
   (a) *$f(a) = g(a)$, for all $a \in C_L$,*
   (b) *the composite field $f(L)g(L)$ is generated by constants over $g(L)$.*
4. *$L|K$ is* strongly normal *if for any differential field extension M of L and every pair (f, g) of K-differential morphisms of L in M, f is strong over g.*

Definition 2.2. *Given a homogeneous linear differential equation*

$$\mathcal{L}(Y) := Y^{(r)} + a_{r-1}Y^{(r-1)} + \ldots + a_1 Y' + a_0 Y = 0$$

of order r over a differential field K, a differential extension $L|K$ is a Picard-Vessiot extension *for $\mathcal{L}$ if*

1. *$L = K\langle \eta_1, \ldots, \eta_r \rangle$, where $\eta_1, \ldots, \eta_r$ is a fundamental set of solutions of $\mathcal{L}(Y) = 0$ in L.*
2. *Every constant of L lies in K, i.e. $C_K = C_L$.*

As mentioned in the introduction, in the case when the constant field C_K of the differential field K is algebraically closed, it is known that there exists a Picard-Vessiot extension for a given homogeneous linear ordinary differential equation defined over K which is unique, up to K-differential isomorphism. The following example due to Seidenberg ([14]) proves that one cannot expect a Picard-Vessiot extension to exist for any linear differential equation over an arbitrary differential field.

Example 2.3. *We consider the field of real numbers $\mathbb{R}$ with trivial derivation and the differential field K obtained by adjoining to $\mathbb{R}$ a solution of the differential equation $4a^2 + a'^2 = -1$, such that $a' \neq 0$. Let us look at the homogeneous linear differential equation $Y'' + Y = 0$ defined over K. Seidenberg proved that for any differential field extension L of K containing a solution of this last equation, the inclusion of $\mathbb{R}$ in the constant field of L is strict. In other words, there is no Picard-Vessiot extension of K for this equation.*

In this paper we shall deal with linear differential equations defined over real differential fields with real closed field of constants. We recall now the meaning of real and real closed field and some of their properties.

Definition 2.4. *An* ordered field *is a field endowed with an ordering compatible with the field operations. A field K is called a* real field *if K can be ordered or equivalently if -1 is not a sum of squares in K. A real field K which has no nontrivial real algebraic extensions is called a* real closed field. *An algebraic extension L of an ordered field K is called a* real closure *of K if L is real closed and the inclusion $K \hookrightarrow L$ preserves the ordering of K.*

A real field always has characteristic zero. If K is a real field, the ring $K[i] := K[X]/(X^2+1)$ is a field which is a quadratic extension of K. If K is a real field, the field of rational functions $K(X)$ is as well real.

A field K is a real closed field if and only if the ring $K[i]$ is an algebraically closed field.

Every ordered field K has a real closure which is unique up to K-isomorphism. The fields $\mathbb{Q}$ and $\mathbb{R}$ with their natural orderings are clearly real fields. Moreover $\mathbb{R}$ is a real closed field.

Let us note that the field K in example 2.3 is not real since, by construction, -1 is a sum of squares in K. However the class of real differential fields with real closed field of constants will be a good setting to establish the existence of Picard-Vessiot extensions. We note that a partial result in this direction has been obtained in [18].

Regarding Galois correspondence theorem, it is worth noting that a Picard-Vessiot extension of a differential field with algebraically closed field of constants is normal, in Kolchin's sense (see Definition 2.1). This is not longer true for Picard-Vessiot extensions of real fields with real closed field of constants, though, as it can be seen in the following example.

Example 2.5. *Let us consider the real differential field $K := \mathbb{R}(t)$, with derivation $\frac{d}{dt}$. Its constant field is clearly $\mathbb{R}$. We consider the differential field extension $L := K(e^t)|K$. It is a Picard-Vessiot extension for the equation $Y' = Y$, defined over K. Any K-differential automorphism of L sends e^t to λe^t, with $\lambda \in \mathbb{R}^*$, so $L|K$ is weakly normal. Now consider the intermediate field $F = K(e^{3t})$. The only F-automorphism of L is identity, hence $L|F$ is not weakly normal, so $L|K$ is not normal.*

3 Existence of Picard-Vessiot extension

3.1 The Seidenberg-Singer embedding theorem

In this section we will state and prove an embedding theorem which will be crucial in the proof of the existence of a Picard-Vessiot extension for a differential equation defined over a real field K differentially finitely generated over its constant field C which is assumed to be real closed.

Definition 3.1. *Let A be a differential ring and let B be a Ritt algebra (i.e. a differential ring which is a $\mathbb{Q}$-algebra). Let $\sigma : A \to B$ be a ring homomorphism. The map*

$$T_\sigma : A \to B[[X]], \qquad a \mapsto \sum_{n \geq 0} \frac{\sigma(a^{(n)})}{n!} X^n,$$

is called the Taylor morphism *associated to σ.*

The properties of the Taylor morphism given in the following proposition are easy to prove. If I is an ideal of a differential ring A, we denote by $I^\sharp$ the largest differential ideal of A contained in I.

Proposition 3.2. *Let A, B, σ and T_σ be as in 3.1. Then:*

1. *If $B[[X]]$ is endowed with the derivation d/dX, the map T_σ is a differential homomorphism and $ker(T_\sigma) = (ker(\sigma))^\sharp$,*
2. *If A is a field, then $T_\sigma(A)$ is a field.*

In the following lemma, we prove a stronger version of results of A. Seidenberg presented in [13], which were also explained by M. F. Singer in [17].

Lemma 3.3. *Let F be an arbitrary field of characteristic zero considered as a differential field with trivial derivation and let $K = F\langle y_1, \dots, y_n\rangle$ be a differential extension of F, with derivation D. Let L be an arbitrary field of characteristic zero. Let the ring $L[[X]]$ be endowed with the derivation d/dX. Let $\sigma : K \to L$ be a field isomorphism. We consider F as a subfield of L via σ.*

Then the Taylor morphism $T_\sigma : K \to L[[X]]$ associated to σ gives a differential isomorphism $\varphi : K \to F\langle \bar{y}_1, \dots \bar{y}_n\rangle$, where $\bar{y}_i := T_\sigma(y_i)$, $1 \leq i \leq n$.

Proof. By 1. in proposition 3.2, T_σ is a differential monomorphism. So by 2. in proposition 3.2, we obtain the differential field isomorphism $\varphi : K \to Im(T_\sigma)$. □

Let us recall the *Tarski-Seidenberg principle*, which states that semialgebraic sets in C^n, where C is a real closed field, are stable under projection. We can formulate Tarski-Seidenberg Principle in the language of model theory. The principle states that the language of real closed fields admits *elimination of quantifiers*, i.e. every first-order formula in the language of real closed fields is equivalent to a quantifier-free formula (see for example [8]).

A consequence of this result is a theorem analogous to the *Principle of Lefschetz*. Namely, any formula of first-order language which holds true in one real closed field is also true in all real closed fields. So the problem always reduces to answering the question whether the statement of our result can be described in first-order language and if it holds true for some real closed field (for more details see [8] and [12]). For our purposes we can formulate the following proposition based on this argumentation.

Proposition 3.4. *Let F be a real field and let $f_i, g_j \in F[X_1, \dots, X_n]$, for $i = 1, \dots, k$ and $j = 1, \dots, l$. The polynomial system*

$$\begin{cases} f_i(X_1, \dots, X_n) & = 0, \quad i = 1, \dots, k \\ g_j(X_1, \dots, X_n) & > 0, \quad j = 1, \dots, l, \end{cases}$$

has a solution in some real extension of F if and only if it has a solution in all real closed fields containing F.

We will use the result above to prove the following lemma.

Lemma 3.5. *Let F be a real field and let K be a real extension of F of the form $K = F(\{x_\lambda\}_{\lambda\in\Lambda}, y)$, where x_λ are algebraically independent over F, $card(\Lambda) \le \kappa$ and y is algebraic over $G = F(\{x_\lambda\}_{\lambda\in\Lambda})$. Let M be a real closed field extension of F such that $trdeg(M|F) > \kappa$. Then K is isomorphic to a subfield of M.*

Proof. We consider the real field $G = F(\{x_\lambda\}_{\lambda\in\Lambda})$. Since $trdeg(M|F) > \kappa$, there exist a subfield S of M and an F-isomorphism of fields $\varphi : G \to S$. We can now extend φ to an isomorphism of the polynomial rings $\bar\varphi : G[X] \to S[X]$. Let $f \in G[X]$ be the minimal polynomial of y. We have $K \cong G[X]/(f) \cong S[X]/(\bar\varphi f)$, so $\widetilde{S} := S[X]/(\bar\varphi f)$ is a real field and $\bar\varphi f$ has a root in $\widetilde{S}$. By proposition 3.4 it has a root, say y^*, in M. Then we obtain an isomorphism of fields $\psi : K = G(y) \to S(y^*) \subset M$. □

The existence of the field M postulated in the lemma above is guaranteed by Skolem-Löwenheim theorem (see [10]). Let us recall that as a consequence of this theorem we obtain that, *if M is an infinite model of the complete theory T in the language L, then for every cardinal number κ not less than $card(M)$ and not less than $card(L)$, M has an elementary extension of cardinality κ.* The theory of real closed fields is *complete*, so this kind of choice of the real closed field is possible. In other words one can choose a real closed field extension of arbitrary large cardinality.

The following result is a generalization of the embedding theorem proved by A. Seidenberg (see [13]) and later in the real case by M. F. Singer (see [17]).

Theorem 3.6 (Seidenberg-Singer Embedding Theorem)**.** *Let F be a real closed field considered as a differential field with trivial derivation. Let (K, D) be a real differential field extension of F, differentially finitely generated over F with field of constants F. Let M be a real closed extension of F such that $trdeg(M|F) \geq \mathfrak{c}$. Then K is differentially isomorphic to a subfield $K_1 = F\langle \bar{y}_1, \ldots, \bar{y}_n \rangle$ of the ring $M[[X]]$, with derivation d/dX.*

Proof. By using the differential primitive element theorem (see [15]), we may assume that $K = F\langle y_1, \ldots, y_n \rangle$, where $y_1, \ldots, y_{n-1}$ are differentially algebraically independent over F and y_n is differentially algebraic over $G = F\langle y_1, \ldots, y_{n-1} \rangle$. We consider $G = F(D^j y_i)$, where $1 \leq i \leq n-1$ and $j \in \mathbb{N}$. By lemma 3.5, there exists a field isomorphism $\sigma : G \to S \subset M$. The element y_n is differentially algebraic over G. Let $f \in G\{Y_n\}$ be the minimal differential polynomial of y_n. Let $ord(f) = r$. Then $K = G\langle y_n \rangle = G(y_n, Dy_n, \ldots, D^r y_n)$, $y_n, Dy_n, \ldots, D^{r-1} y_n$ are algebraically independent over G and $D^r y_n$ is algebraic over the field $G_1 := G(y_n, Dy_n, \ldots, D^{r-1} y_n)$. So $y_n, Dy_n, \ldots, D^{r-1} y_n$ are algebraically independent over the real closed field F. By lemma 3.5, we can embed G_1 and also $K = G_1(D^r y_n)$ into M. We can now apply lemma 3.3 and obtain a differential field isomorphism $\varphi : K \to F\langle \overline{y}_1, \ldots, \overline{y}_n \rangle$, where $F\langle \overline{y}_1, \ldots, \overline{y}_n \rangle$ is a differential subfield of the differential ring $(M[[X]], \frac{d}{dX})$ and $\overline{y}_i$ is the image of y_i in M, $1 \leq i \leq n$. ☐

Remark 3.7. *We observe that lemma 3.5 and theorem 3.6 are also true without assuming the fields F and K to be real if we take them to be of characteristic zero and M to be an algebraically closed extension of F.*

3.2 Existence theorem

Let K be a real differential field with real closed field of constants C. We consider a homogeneous linear ordinary differential equation of order r of the form

$$\mathcal{L}(Y) := Y^{(r)} + a_{r-1} Y^{(r-1)} + \ldots + a_1 Y' + a_0 Y = 0, \tag{1}$$

where $a_i \in K$ for $i \in \{0, 1, \ldots, r-1\}$. In this section we shall prove that there exists a Picard-Vessiot extension for this equation, which moreover is a real field (see [19]). In the sequel, for a real field K, we shall denote by $\widehat{K}$ the field $K(i)$.

Theorem 3.8. *Let K be a real differential field with real closed field of constants C. For a homogeneous linear differential equation defined over K, there exists a Picard-Vessiot extension, which moreover is a real field.*

Proof. We shall consider the linear differential equation (1) defined over K. We shall prove the existence of a real Picard-Vessiot extension for such an equation first in the case K differentially finitely generated over its field of constants C, then obtain the general result by applying the Kuratowski-Zorn lemma. We shall consider the differential field $\widehat{K}$ whose constant field is the algebraically closed field $\widehat{C}$.

Case 1. We assume $K = C\langle y_1, \ldots, y_n\rangle$ for some $y_1, \ldots, y_n \in K$ and some $n \in \mathbb{N}$. We have then $\widehat{K} = \widehat{C}\langle y_1, \ldots, y_n\rangle$. Let M be a real closed extension of C such that $trdeg(M|C) \geq \mathfrak{c}$. By lemma 3.5 we may embed the real field $K = C\langle y_1, \ldots, y_n\rangle$ into the real closed field M. Let us denote this embedding by σ. We define then an embedding $\widehat{\sigma} : \widehat{K} = \widehat{C}\langle y_1, \ldots, y_n\rangle \to \widehat{M}$ extending σ.

By theorem 3.6 we have the following differential isomorphisms:

- $T_\sigma : K = C\langle y_1, \ldots, y_n\rangle \to K_1 := C\langle \bar{y}_1, \ldots, \bar{y}_n\rangle \subset M[[X]]$,
- $T_{\widehat{\sigma}} : \widehat{K} = \widehat{C}\langle y_1, \ldots, y_n\rangle \to \widehat{K}_1 := \widehat{C}\langle \bar{y}_1, \ldots, \bar{y}_n\rangle \subset \widehat{M}[[X]]$.

Now we consider equation (1) over $\widehat{K}$. The field of constants $\widehat{C}$ of $\widehat{K}$ is algebraically closed, so there exists a unique Picard-Vessiot extension of $\widehat{K}$ for this equation. Let us denote it by $\widehat{L} := \widehat{K}\langle \eta_1, \ldots, \eta_r\rangle$, where $\eta_1, \ldots, \eta_r$ is a fundamental set of solutions of equation (1) in $\widehat{L}$.

By lemma 3.3, we obtain a differential isomorphism

$$\varphi : \widehat{L} = \widehat{K}\langle \eta_1, \ldots, \eta_r\rangle \to \widehat{L}_1 := \widehat{K}_1\langle \bar{\eta}_1, \ldots, \bar{\eta}_r\rangle \subset \widehat{M}[[X]].$$

So $\bar{\eta}_1, \ldots, \bar{\eta}_r$ is a fundamental set of solutions of equation

$$Y^{(r)} + b_{r-1}Y^{(r-1)} + \ldots + b_1 Y' + b_0 Y = 0, \tag{2}$$

considered over $\widehat{K}_1$, where the coefficients $b_i := T_\sigma(a_i)$ belong to the real field K_1, for $i = 0, \ldots, r-1$.

The involution c of $\widehat{M}$ given by $i \mapsto -i$ and $c_{|M} = Id_M$ extends to $\widehat{M}[[X]]$ in a natural way. The vector space of solutions $V := \widehat{C}\bar{\eta}_1 \oplus \cdots \oplus \widehat{C}\bar{\eta}_r$ is c-stable. Let V^c be the C-subspace of V fixed by the involution c and let L_1 be the differential subfield of $\widehat{L}_1$ generated by K_1 and V^c. By definition, it is differentially generated over K_1 by a fundamental system of solutions of (2). As it is a differential subfield of $\widehat{L}_1$, $C_{L_1} \subset C_{\widehat{L}_1} = \widehat{C}$. But L_1 is contained in $M[[X]]$, so it is a real field, hence $C_{L_1} = C$. We have then proved that L_1 is a Picard-Vessiot extension for equation (2) over K_1. Hence $L := \varphi^{-1}(L_1)$ is a Picard-Vessiot extension for equation (1) over K, and L is a real field.

Case 2. Now the real differential field K has arbitrary differential degree over its constant field C. We will prove that we can embed K into a real closed extension M of C, which is large enough. Then the proof of the existence theorem of Picard-Vessiot extension for K follows the same steps as in Case 1.

By lemma 3.5, we can embed into M all these subfields of K, which are compositions of differentially transcendental extensions of arbitrary large differential transcendence degree and a differentially algebraic extension of finite differential degree. Let us denote the family of all embeddable subfields of K by $\mathcal{S}$. Let us consider the space $\mathcal{E}$ of embeddings of subfields from $\mathcal{S}$. Clearly $\mathcal{E} \neq \varnothing$. We introduce a partial order relation $\preceq$ in $\mathcal{E}$ by

$$\varphi_1 \preceq \varphi_2 \Leftrightarrow \varphi_1 \subset \varphi_2,$$

for $\varphi_1, \varphi_2 \in \mathcal{E}$. We observe that every totally ordered subset $\mathcal{G}$ of $\mathcal{E}$ has an upper bound, i.e. $\bigcup_{\varphi_i \in \mathcal{G}} \varphi_i$. Hence by *Kuratowski-Zorn lemma* there exists a maximal element φ_{max} in $\mathcal{E}$. We claim that φ_{max} is an embedding of K. Indeed, if not then there exist a proper subfield $S \in \mathcal{S}$ such that φ_{max} is an embedding of S. So there exists an element $a \in K \setminus S$ which is not embedded by φ_{max}. We consider the differential field $S\langle a \rangle$. If a is differentially transcendental over S, then there exists clearly an embedding ψ of $S\langle a \rangle$ into M. If a is differentially algebraic over S, by lemma 3.5, there exists an embedding ψ of $S\langle a \rangle$ into M. Hence $\varphi_{max} \prec \psi$. We have a contradiction with the maximality of φ_{max}. □

Remark 3.9. *In the case of a linear differential equation defined over a differential field $\widehat{K}$ with algebraically closed constant field, the Picard-Vessiot extension is proved to be unique, up to $\widehat{K}$-differential isomorphism. Hence, if we consider a linear differential equation $\mathcal{L}(Y) = 0$, defined over a real differential field K with real closed constant field C and L is a Picard-Vessiot extension for it, the set of K-isomorphism classes of Picard-Vessiot extensions of K for $\mathcal{L}(Y) = 0$ is in bijection with $H^1(\mathrm{Gal}(\widehat{K}|K), DAut_{\widehat{K}}(\widehat{L}))$. Hence, the Picard-Vessiot extension is in general not unique. For example, $H^1(C_2, SO(n, \mathbb{R}))$ is not trivial, as it can be identified with the set of equivalence classes of quadratic forms of rank n with positive discriminant (see [16] III 3.2). However, if we want to restrict to real Picard-Vessiot extensions, the problem of uniqueness is more subtle (see example 3.10 below). It is connected to the problem of determining the isomorphism classes of real fields K having isomorphic extensions $K(i)$ which is, as far as we know, not solved.*

Example 3.10. *Let us consider the differential equation $Y'' + Y = 0$ defined over the field $\widehat{K} = \mathbb{C}(t)$, with derivation d/dt. Its Picard-Vessiot extension is $L = \widehat{K}(\sin t, \cos t)$ and its differential Galois group is $\mathrm{SO}(2, \mathbb{C})$.*

We consider now the same equation over the field $K = \mathbb{R}(t)$. We have two Picard-Vessiot extensions of K for this equation which are not K-isomorphic, namely $L_1 = K(\sin t, \cos t)$ and $L_2 = K(i \sin t, i \cos t)$ corresponding to the two elements in $H^1(\mathrm{Gal}(\mathbb{C}|\mathbb{R}), \mathrm{SO}(2, \mathbb{C}))$. We observe that L_1 is a real field, while L_2 is not as $(i \sin t)^2 + (i \cos t)^2 = -1$.

4 Galois correspondence

4.1 Galois group

Let K be a real field with real closed field of constants C. For a real Picard-Vessiot extension $L|K$, we shall consider the set $DHom_K(L, \widehat{L})$ of K-differential morphisms from L into $\widehat{L}$. We shall see that we can define a group structure on this set and we shall take it as the Galois group $G(L|K)$ of the Picard-Vessiot extension $L|K$. We shall prove that it is a C-defined (Zariski) closed subgroup of some $\widehat{C}$-linear algebraic group.

We observe that we can define mutually inverse bijections

$$\begin{array}{ccc} DHom_K(L, \widehat{L}) & \rightarrow & DAut_{\widehat{K}}\widehat{L} \\ \sigma & \mapsto & \widehat{\sigma} \end{array}, \quad \begin{array}{ccc} DAut_{\widehat{K}}\widehat{L} & \rightarrow & DHom_K(L, \widehat{L}) \\ \tau & \mapsto & \tau_{|L} \end{array},$$

where $\widehat{\sigma}$ is the extension of σ to $\widehat{L}$ defined by $\widehat{\sigma}(a+i\,b)=\sigma(a)+i\sigma(b)$, for $a,b\in L$. We may then transfer the group structure from $DAut_{\widehat{K}}\widehat{L}$ to $DHom_K(L,\widehat{L})$.

Let now $\eta_1,\dots,\eta_r$ be C-linearly independent elements in L such that $L=K\langle\eta_1,\dots,\eta_r\rangle$ and $\sigma\in DHom_K(L,\widehat{L})$. We have then $\sigma(\eta_j)=\sum_{i=1}^r c_{ij}\eta_i, 1\leq j\leq r$, with $c_{ij}\in\widehat{C}$. We may then associate to σ the matrix (c_{ij}) in $\mathrm{GL}(r,\widehat{C})$.

Proposition 4.1. *Let K be a real differential field with real closed field of constants C, $L=K\langle\eta_1,\dots,\eta_r\rangle$ a real Picard-Vessiot extension of K. There exists a set S of polynomials $F(X_{ij}), 1\leq i,j\leq r$, with coefficients in C such that*

1) If $\sigma\in DHom_K(L,\widehat{L})$ and $\sigma(\eta_j)=\sum_{i=1}^r c_{ij}\eta_i$, then $F(c_{ij})=0,\forall F\in S$.

2) Given a matrix $(c_{ij})\in\mathrm{GL}(r,\widehat{C})$ with $F(c_{ij})=0,\forall F\in S$, there exists a differential K-morphism σ from L to $\widehat{L}$ such that $\sigma(\eta_j)=\sum_{i=1}^r c_{ij}\eta_i$.

Proof. The proof follows the steps of prop. 6.2.1 in [3]. Let $K\{Z_1,\dots,Z_r\}$ be the ring of differential polynomials in r indeterminates over K. We define a differential K-morphism φ from $K\{Z_1,\dots,Z_r\}$ in L by $Z_j\mapsto\eta_j$. Then $\Gamma:=\operatorname{Ker}\varphi$ is a prime differential ideal of $K\{Z_1,\dots,Z_r\}$. Let $\widehat{L}[X_{ij}], 1\leq i,j\leq r$ be the ring of polynomials in the indeterminates X_{ij} with the derivation defined by $X'_{ij}=0$. We define a differential K-morphism ψ from $K\{Z_1,\dots,Z_r\}$ to $\widehat{L}[X_{ij}]$ such that $Z_j\mapsto\sum_{i=1}^r X_{ij}\eta_i$. Let $\Delta:=\psi(\Gamma)$. Let $\{w_k\}$ be a basis of the C-vector space $\widehat{L}$. We write each polynomial in Δ as a linear combination of the w_k with coefficients polynomials in $C[X_{ij}]$. We take S to be the collection of all these coefficients.
1. Let σ be a differential K-morphism from L to $\widehat{L}$ and $\sigma(\eta_j)=\sum_{i=1}^r c_{ij}\eta_i$. We consider the diagram

$$\begin{array}{ccccc}
Z_j & \longmapsto & & & \eta_j \\
\downarrow & K\{Z_1,\dots,Z_r\} & & \xrightarrow{\varphi} & L \\
\downarrow & & \downarrow\psi & & \downarrow\sigma \\
\sum X_{ij}\eta_i & & \widehat{L}[X_{ij}] & \xrightarrow{v} & \widehat{L} \\
 & & X_{ij} & \longmapsto & c_{ij}
\end{array}$$

It is clearly commutative. The image of Γ by $\sigma\circ\varphi$ is 0. Its image by $v\circ\psi$ is Δ evaluated in $X_{ij}=c_{ij}$. Therefore all polynomials of Δ vanish at c_{ij}. Writing this down in the basis $\{w_k\}$, we see that all polynomials of S vanish at c_{ij}.
2. Let us now be given a matrix $(c_{ij})\in\mathrm{GL}(r,\widehat{C})$ such that $F(c_{ij})=0$ for every F in S. We consider the differential morphism

$$\begin{array}{ccc} K\{Z_1,\dots,Z_r\} & \to & \widehat{L} \\ Z_j & \mapsto & \sum_i c_{ij}\eta_i \end{array}.$$

By the hypothesis on (c_{ij}), and the definition of the set S, we see that the kernel of this morphism contains Γ and so, we have a differential K-morphism

$$\begin{array}{rccc} \sigma : & K\{\eta_1,\dots,\eta_r\} & \rightarrow & \widehat{L} \\ & \eta_j & \mapsto & \sum_i c_{ij}\eta_i \end{array} .$$

Taking into account that the elements $\sigma(\eta_j), 1 \leq j \leq r$, are $\widehat{C}$-linearly independent, we obtain that σ is injective and so extends to a K-differential morphism from L to $\widehat{L}$. □

If $L|K$ is a real Picard -Vessiot extension for a homogeneous linear differential equation of order r defined over K, the preceding proposition gives that $G(L|K)$ is a C-defined closed subgroup of $\mathrm{GL}(r, \widehat{C})$.

Real Picard-Vessiot extensions satisfy the following normality property.

Proposition 4.2. *Let K be a real differential field with real closed field of constants C, $L|K$ a real differential Picard-Vessiot extension. For $a \in L \setminus K$, there exists a K-differential morphism $\sigma : L \rightarrow \widehat{L}$ such that $\sigma(a) \neq a$.*

Proof. As $\widehat{L}|\widehat{K}$ is a Picard-Vessiot extension and the constant field $\widehat{C}$ of $\widehat{K}$ is algebraically closed, we know ([3] prop. 6.1.2) that there exists a $\widehat{K}$-differential automorphism $\widehat{\sigma}$ of $\widehat{L}$ such that $\widehat{\sigma}(a) \neq a$. We can then take $\sigma = \widehat{\sigma}_{|L}$. □

For a subset S of $G(L|K)$, we set $L^S := \{a \in L : \sigma(a) = a, \forall \sigma \in S\}$.

Corollary 4.3. *Let K be a real differential field with real closed field of constants C, $L|K$ a real differential Picard-Vessiot extension. We have $L^{G(L|K)} = K$.*

4.2 Fundamental theorem

Let K be a real differential field with real closed field of constants C and $L|K$ a real differential Picard-Vessiot extension. For a closed subgroup H of $G(L|K)$, L^H is a differential subfield of L containing K. If F is an intermediate differential field, i.e. $K \subset F \subset L$, then $L|F$ is a real Picard-Vessiot extension and $G(L|F)$ is a C-defined closed subgroup of $G(L|K)$.

Theorem 4.4. *Let $L|K$ be a real Picard-Vessiot extension, $G(L|K)$ its differential Galois group.*

1. *The correspondences*

$$H \mapsto L^H \quad , \quad F \mapsto G(L|F)$$

define inclusion inverting mutually inverse bijective maps between the set of C-defined closed subgroups H of $G(L|K)$ and the set of differential fields F with $K \subset F \subset L$.

2. The intermediate differential field F is a Picard-Vessiot extension of K if and only if the subgroup $G(L|F)$ is normal in $G(L|K)$. In this case, the restriction morphism

$$\begin{array}{ccc} G(L|K) & \to & G(F|K) \\ \sigma & \mapsto & \sigma_{|F} \end{array}$$

induces an isomorphism

$$G(L|K)/G(L|F) \simeq G(F|K).$$

Proof. 1. It is clear that both maps invert inclusion. If F is an intermediate differential field of $L|K$, we have $L^{G(L|F)} = F$, taking into account that $L|F$ is Picard-Vessiot and corollary 4.3. For H a C-defined closed subgroup of $G(L|K)$, the equality $H = G(L|L^H)$ follows from the correspondent equality in Picard-Vessiot theory for differential fields with algebraically closed field of constants ([3] theorem 6.3.8).
2. If F is a Picard-Vessiot extension of K, then $\widehat{F}$ is a Picard-Vessiot extension of $\widehat{K}$ and so $G(L|F)$ is normal in $G(L|K)$. Reciprocally, if $G(L|F)$ is normal in $G(L|K)$, then the subfield of $\widehat{L}$ fixed by $G(L|F)$ is a Picard-Vessiot extension of $\widehat{K}$. Now, this field is $\widehat{F}$. So, if $\widehat{F}$ is differentially generated over $\widehat{K}$ by a $\widehat{C}$-vector space of finite dimension V, then F is differentially generated over K by the C-vector space $V^c = \{y \in V : c(y) = y\}$, where c is the F-automorphism of $\widehat{F}$ determined by $c(i) = -i$. Hence $F|K$ is a real Picard-Vessiot extension. The last statement of the theorem follows from the fundamental theorem of Picard-Vessiot theory in the case of algebraically closed fields of constants ([3] theorem 6.3.8). □

Remark 4.5. *All results in sections 4.1 and 4.2 remain valid for K any differential field with real closed field of constants C and $L|K$ a Picard-Vessiot extension. Just observe that, as -1 is not a square in C, $\widehat{K} = K(i)$ is also in this case a quadratic extension of K. However, without assuming K real, we cannot assure that a Picard-Vessiot extension exists for a given linear differential equation defined over K.*

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