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On congruences for the sums $\sum_{i=1}^{[n/r]} \frac{\chi_n(i)}{n-ri}$ of E.Lehmer's type

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On congruences for the sums $\sum_{i=1}^{[n/r]} \frac{\chi_n(i)}{n-ri}$ of E. Lehmer's type*

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Abstract

The present paper is an appendix to the paper [9]. We obtain some new congruences for the sums $U_r(n) = \sum_{i=1}^{[n/r]} \frac{\chi_n(i)}{n-ri} \pmod{n^{s+1}}$ for $s \in \{0, 1, 2\}$ and $r \mid 24$. These congruences are consequences of those proved in [9] by using an identity from [15]. Our congruences for $s = 1$ extend those obtained in [2] and [3] for $r \in \{2, 3, 4, 6\}$ and $2, 3 \nmid n$. These four congruences have the same form as those proved by E. Lehmer [11] in the case when $n = p$ is an odd prime. They are rational linear combinations of Euler's quotients. In the case when $r \in \{8, 12, 24\}$, omitted in [11], [2] and [3], the congruences are linear combinations of the Euler quotients and three generalized Bernoulli numbers $\frac{1}{n\phi(n)} B_{n\phi(n),\chi} \prod_{p|n} \left(1 - p^{n\phi(n)-1}\right)$ attached to even quadratic characters χ of conductor dividing 24. Also some new congruences for $s = 2$ with one additional summand $-\frac{n^2}{2r^3} B_{n^2\phi(n)-2} \prod_{p|n} \left(1 - p^{n^2\phi(n)-3}\right)$ for all $r \mid 24$ are obtained.

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1 Notation and introduction

Following [9], let $n \in \mathbb{N}$ be odd and let χ_n be the trivial Dirichlet character modulo n. For $r \geq 2$ coprime to n, $q_r(n)$ denotes the Euler quotient, i.e.,

$$q_r(n) = \frac{r^{\phi(n)} - 1}{n}$$

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where ϕ is the Euler *phi*-function. Let $B_{i,\chi}$ denote the i-th generalized Bernoulli number attached to a Dirichlet character χ; as usual B_i are the ordinary Bernoulli numbers. For definitions see [17], [7] or [16].

Given the discriminant d of a quadratic field, let χ_d denote its quadratic character (Kronecker symbol). It follows from [4] that the quotients $B_{i,\chi_d}/i$ are rational integers unless $d = -4$ or $d = \pm p$, where p is an odd prime of a special form. We shall consider such numbers (with $d = \pm p$) only if $d = -3$; then we have the so-called D-numbers defined in [10] and [5] by $D_{i-1} = -3B_{i,\chi_{-3}}/i$ for i odd.[(1)]

If $d = -4$ and i is odd, then the numbers $E_{i-1} = -2B_{i,\chi_{-4}}/i$ are odd integers, called the Euler numbers. Following [9], we also consider the rational integers $A_{i-1} = B_{i,\chi_8}/i$, $F_{i-1} = B_{i,\chi_{-3}\chi_{-4}}/i$ and $G_{i-1} = B_{i,\chi_{-3}\chi_{-8}}/i$, if $i \geq 2$ even, and $C_{i-1} = -B_{i,\chi_{-8}}/i$ and $H_{i-1} = -B_{i,\chi_{-3}\chi_8}/i$ if $i \geq 1$ odd.

In the present paper we find congruences for the sums

$$U_r(n) = \sum_{0<i<n/r} \frac{\chi_n(i)}{n - ri}$$

modulo n^{s+1} for $s \in \{0, 1, 2\}$ and $r \mid 24$ $(1 < r < n)$ coprime to n. To obtain such congruences it suffices to use appropriate congruences for the sums

$$T_{r,k}(n) = \sum_{0<i<n/r} \frac{\chi_n(i)}{i^k}$$

or, equivalently, for the sums

$$S_{r,k,s}(n) = \sum_{0<i<n/r} \chi_n(i) i^{n^s\phi(n)-k}$$

for $k \in \{1, 2, 3\}$. Such congruences were shown in [9]. In [9] the congruence

$$T_{r,k}(n) \equiv S_{r,k,s}(n) \pmod{n^{s+1}}\,. \tag{1}$$

was proved by using an identity from [15][(2)] and a well-known congruence

$$i^{n^s\phi(n)} \equiv 1 \pmod{n^{s+1}}, \tag{2}$$

which holds for $(i, n) = 1$ and is implied by Euler's congruence $i^{\phi(n)} \equiv 1 \pmod{n}$. Here we assume that $n^s\phi(n) - k \geq 0$.

(1) The denominators of the numbers D_i are powers of 3. For details, see [5].

(2) This identity was already successfully exploited in [13], [6], [1], [8] and [9]. Using it, T. Cai [1] generalized, in an elegant way, E. Lehmer's congruence for $T_{2,1}(n) \pmod{n^2}$.

The congruences obtained for the sum $U_r(n)$ extend those proved in [11], [2] and [3].(3) For other related papers, see also [12] and [14]. Throughout the paper, following [9], we set

$$\begin{aligned}
\widetilde{B}_i &= B_i \prod_{p|n} \big(1 - p^{i-1}\big),\ \ \widehat{B}_i = \frac{\widetilde{B}_i}{i},\\
\widetilde{A}_i &= (-1)^{\frac{n^2-1}{8}} A_i \prod_{p|n} \big(1 - (-1)^{\frac{p^2-1}{8}} p^i\big),\\
\widetilde{C}_i &= (-1)^{\frac{(n-1)(n+5)}{8}} C_i \prod_{p|n} \big(1 - (-1)^{\frac{(p-1)(p+5)}{8}} p^i\big),\\
\widetilde{D}_i &= (-1)^{\nu(n)} D_i \prod_{p|n} \big(1 - (-1)^{\nu(p)} p^i\big),\\
\widetilde{E}_i &= (-1)^{\frac{n-1}{2}} E_i \prod_{p|n} \big(1 - (-1)^{\frac{p-1}{2}} p^i\big),\\
\widetilde{F}_i &= (-1)^{\frac{n-1}{2}+\nu(n)} F_i \prod_{p|n} \big(1 - (-1)^{\frac{p-1}{2}+\nu(p)} p^i\big),\\
\widetilde{G}_i &= (-1)^{\frac{(n-1)(n+5)}{8}+\nu(n)} G_i \prod_{p|n} \big(1 - (-1)^{\frac{(p-1)(p+5)}{8}+\nu(p)} p^i\big),\\
\widetilde{H}_i &= (-1)^{\frac{n^2-1}{8}+\nu(n)} H_i \prod_{p|n} \big(1 - (-1)^{\frac{p^2-1}{8}+\nu(p)} p^i\big),
\end{aligned}$$

where $\chi_{-3}(n) = (-1)^{\nu(n)}$, $\nu(n) \in \{0,1\}$ for $3 \nmid n$.

2 The main results

In the Theorem we find some congruences for $U_r(n)$ modulo n^{s+1} for $s \in \{0,1,2\}$ in each of the seven cases $r = 2$, 3, 4, 6, 8, 12 or 24. Some of these congruences for $s \in \{0,1\}$ and $r \in \{2,3,4,6\}$ were proved in [2] and [3]. The remaining ones are new. Three of them for $s = 1$ and $r \in \{8,12,24\}$ were omitted both in [11] and in [2], [3].

Write $\rho_i(r) = 1 - \delta_{\mathrm{ord}_i(r),0}$ $(i = 2,3)$ where, as usual, $\delta_{X,Y}$ denotes the Kronecker delta function. Given odd $n > r$, we set

$$\begin{aligned}
EQ_r(n) &= \alpha_2(r)q_2(r) + \alpha_3(r)q_3(r) + \beta_2(r)nq_2^2(n) + \beta_3(r)nq_3^2(n)\\
&\quad + \gamma_2(r)n^2q_2^3(n) + \gamma_3(r)n^2q_3^3(n)\,,
\end{aligned} \tag{3}$$

(3)E. Lehmer proved her congruences in the case when $n = p$ is an odd prime. The congruences proved in [2] and [3] are for n odd and not divisible by 3.

where

$$\alpha_2(r)=\rho_2(r)\Big(\frac{\operatorname{ord}_2(r)}{r}+\frac{1}{2\phi(r)}-\frac{\rho_3(r)}{6\phi(r)}\Big),$$
$$\alpha_3(r)=\rho_3(r)\Big(\frac{\operatorname{ord}_3(r)}{r}+\frac{1}{3\phi(r)}-\frac{\rho_2(r)}{6\phi(r)}\Big),$$
$$\beta_2(r)=\rho_2(r)\Big(-\frac{\operatorname{ord}_2(r)}{2r}-\frac{1}{4\phi(r)}+\frac{\rho_3(r)}{12\phi(r)}\Big),$$
$$\beta_3(r)=\rho_3(r)\Big(-\frac{\operatorname{ord}_3(r)}{2r}-\frac{1}{6\phi(r)}+\frac{\rho_3(r)}{12\phi(r)}\Big),$$
$$\gamma_2(r)=\rho_2(r)\Big(\frac{\operatorname{ord}_2(r)}{3r}+\frac{1}{6\phi(r)}-\frac{\rho_3(r)}{18\phi(r)}\Big),$$
$$\gamma_3(r)=\rho_3(r)\Big(\frac{\operatorname{ord}_3(r)}{3r}+\frac{1}{9\phi(r)}-\frac{\rho_2(r)}{18\phi(r)}\Big)$$

and

$$B_r(n)=-\frac{n^2}{2r^3}\widetilde{B}_{n^2\phi(n)-2}\,.$$

Set $EQ'_r(n)=\alpha_2(r)q_2(r)+\alpha_3(r)q_3(r)$ and $EQ''_r=\alpha_2(r)q_2(r)+\alpha_3(r)q_3(r)+\beta_2(r)nq_2^2(n)+\beta_3(r)nq_3^2(n)$. Obviously, we have $EQ_r(n)\equiv EQ'_r(n)\pmod n$ and $EQ_r(n)\equiv EQ''_r(n)\pmod{n^2}$. Note that $B_r(n)\equiv 0\pmod n$, and $B_r(n)\equiv 0\pmod{n^2}$ if n is not divisible by 3.

Following [9], set

$$Q_2(n)=-2q_2(n)+nq_2^2(n)-\frac{2}{3}n^2q_2^3(n),\ Q_3(n)=-\frac{3}{2}q_3(n)+\frac{3}{4}nq_3^2(n)-\frac{1}{2}n^2q_3^3(n)\,.$$

The sums $T_{r,1}(n)$ presented in the lemmas below are congruent to linear combinations of Euler's quotients $\widehat{EQ}_r(n)$ plus some generalized Bernoulli numbers. It was shown in [9] that $\widehat{EQ}_2(n)=Q_2(n)$, $\widehat{EQ}_3(n)=Q_3(n)$, $\widehat{EQ}_4(n)=\frac{3}{2}Q_2(n)$, $\widehat{EQ}_6(n)=Q_2(n)+Q_3(n)$, $\widehat{EQ}_8(n)=2Q_2(n)$, $\widehat{EQ}_{12}(n)=\frac{3}{2}Q_2(n)+Q_3(n)$ and $\widehat{EQ}_{24}(n)=2Q_2(n)+Q_3(n)$. In view of Proposition 1 below we have $EQ_r(n)=-\frac{1}{r}\widehat{EQ}_r(n)$.

Theorem. *Assume that $s\in\{0,1,2\}$ and $r\mid 24$. Let $n>r$ be odd and not divisible by 3 if $s=1$ or $3\mid r$. Then, in the above notation:*
(i)

$$U_r(n)\equiv EQ_r(n)+B_r(n)\pmod{n^{s+1}}$$

if $r\le 6$;
(ii)

$$U_r(n)\equiv EQ_r(n)+B_r(n)-\frac{1}{4}\widetilde{A}_{n^s\phi(n)-1}\pmod{n^{s+1}}$$

if $r = 8$*;*
(iii)

$$U_r(n) \equiv EQ_r(n) + B_r(n) - \frac{1}{4}\widetilde{F}_{n^s\phi(n)-1} \pmod{n^{s+1}}$$

if $r = 12$*;*
(iv)

$$U_r(n) \equiv EQ_r(n) + B_r(n) - \frac{1}{6}\widetilde{A}_{n^s\phi(n)-1} - \frac{1}{8}\widetilde{F}_{n^s\phi(n)-1} - \frac{1}{8}\widetilde{G}_{n^s\phi(n)-1} \pmod{n^{s+1}}$$

if $r = 24$*. Here* $EQ_r(n) \equiv EQ'_r(n) \pmod{n}$*,* $EQ_r(n) \equiv EQ''_r(n) \pmod{n^2}$*,* $B_r(n) \equiv 0 \pmod{n^2}$ *if* n *is not divisible by* 3 *and* $B_r(n) \equiv 0 \pmod{n}$*.*

3 Some useful observations

We deduce the main theorem of the paper from Propositions 1, 2 and Lemmas 1–21 below on congruences for the sums $T_{r,k}(n)$. For proofs of the lemmas, we refer the reader to [9]. First we find some useful congruences between the sums $U_r(n)$ and some linear combinations of $T_{r,1}(n)$, $T_{r,2}(n)$ and $T_{r,3}(n)$ modulo powers of n.

Proposition 1. *Assume that* $n > 1$ *is odd and* r $(1 < r < n)$ *is coprime to* n*. Then:*

$$U_r(n) \equiv \begin{cases} -\frac{1}{r}T_{r,1}(n) - \frac{n}{r^2}T_{r,2}(n) - \frac{n^2}{r^3}T_{r,3}(n) \pmod{n^3} \\ -\frac{1}{r}T_{r,1}(n) - \frac{n}{r^2}T_{r,2}(n) \pmod{n^2} \\ -\frac{1}{r}T_{r,1}(n) \pmod{n} \end{cases}$$

Proof. Obviously, $(n, i) = 1$ if and only if $(n - ri, n) = 1$. Consequently, by (2),

$$\begin{aligned} U_r(n) &\equiv \sum_{0<i<n/r} \chi_n(i)\big(n - ri\big)^{n^s\phi(n)-1} \\ &= \sum_{0<i<n/r} \chi_n(i) \sum_{j=0}^{n^s\phi(n)-1} \binom{n^s\phi(n)-1}{j} n^j(-ri)^{n^s\phi(n)-1-j} \pmod{n^{s+1}}, \end{aligned}$$

and hence, since $r^{n^s\phi(n)-j} \equiv r^{-j} \pmod{n^{s+1}}$ and $\binom{n^2\phi(n)-1}{2}n^2 \equiv n^2 \pmod{n^3}$,

$$U_r(n) \equiv \begin{cases} -\frac{1}{r}S_{r,1,2}(n) - \frac{n}{r^2}S_{r,2,2}(n) - \frac{n^2}{r^3}S_{r,3,2}(n) \pmod{n^3}, \\ -\frac{1}{r}S_{r,1,1}(n) - \frac{n}{r^2}S_{r,2,2}(n) \pmod{n^2}, \\ -\frac{1}{r}S_{r,1,0}(n) \pmod{n}. \end{cases}$$

Now Proposition 1 follows from (1) at once. □

In [9] some formulae for $\widehat{EQ}_r(n)$ are determined. Since, by Proposition 1, we have $EQ_r(n) = -\frac{1}{r}\widehat{EQ}_r(n)$, the formulae imply corresponding formulae for $EQ_r(n)$. In the next proposition, we present the formulae in a slightly different form.

Proposition 2. (cf. [9]) *In the above notation, if* $r \mid 24$, *then* (3) *holds.*

Proof. Following [9, (15)] and Proposition 1 we know that

$$EQ_r(n) = -\frac{1}{r}\widehat{EQ}_r(n) \equiv -\frac{1}{r}\widehat{B}_{m+1}\Big(-1+\frac{1}{\phi(r)r^m}\prod_{q|r}\big(1-q^m\big)\Big) \pmod{n^{s+1}}$$

where $m = n^s\phi(n)-1$. Consequently,

$$EQ_r(n) \equiv \frac{X\widetilde{B}_{m+1}}{r^{m+1}(m+1)} \pmod{n^{s+1}} \tag{4}$$

where

$$X = r^m - \frac{1}{\phi(r)}\prod_{q|r}\big(1-q^m\big)\,.$$

Thus, in view of (5) and the congruence

$$\frac{n\widetilde{B}_{n^s\phi(n)}}{\phi(n)} \equiv 1 \pmod{n^{s+1}}$$

(see [9, (20)]) to obtain (3) it suffices to determine $X \pmod{n^{s+4}}$.

Indeed, we have

$$\begin{aligned}X &= \frac{1}{r}\big(r^{\phi(n)}\big)^{n^s} - \frac{1}{\phi(r)}\Big(1-\frac{\rho_2(r)}{2}\big(2^{\phi(n)}\big)^{n^s}\Big)\Big(1-\frac{\rho_3(r)}{3}\big(3^{\phi(n)}\big)^{n^s}\Big)\\ &= \frac{1}{r}\big(2^{\phi(n)}\big)^{\mathrm{ord}_2(r)n^s}\big(3^{\phi(n)}\big)^{\mathrm{ord}_3(r)n^s} - \frac{1}{\phi(r)} + \frac{\rho_2(r)}{2\phi(r)}\big(2^{\phi(n)}\big)^{n^s} + \frac{\rho_3(r)}{3\phi(r)}\big(3^{\phi(n)}\big)^{n^s}\\ &\quad - \frac{\rho_2(r)\rho_3(r)}{6\phi(r)}\big(2^{\phi(n)}\big)^{n^s}\big(3^{\phi(n)}\big)^{n^s},\end{aligned}$$

and by virtue of $i^{\phi(n)} = 1 + nq_i(n)$ $(i = 2, 3)$

$$\begin{aligned}X &= \frac{1}{r}\big(1+nq_2(n)\big)^{\mathrm{ord}_2(r)n^s}\big(1+nq_3(n)\big)^{\mathrm{ord}_3(r)n^s} - \frac{1}{\phi(r)} + \frac{\rho_2(r)}{2\phi(r)}\big(1+nq_2(n)\big)^{n^s}\\ &\quad + \frac{\rho_3(r)}{3\phi(r)}\big(1+nq_3(n)\big)^{n^s} - \frac{\rho_2(r)\rho_3(r)}{6\phi(r)}\big(1+nq_2(n)\big)^{n^s}\big(1+nq_3(n)\big)^{n^s}.\end{aligned}$$

Thus,

$$\begin{aligned}X\equiv\ &\frac{1}{r}\Big(1+n^{s+1}\mathrm{ord}_2(r)q_2(n)+n^{s+1}\mathrm{ord}_3(r)q_3(n)-\frac{1}{2}n^{s+1}\mathrm{ord}_2(r)nq_2^2(n)\\&-\frac{1}{2}n^{s+1}\mathrm{ord}_3(r)nq_3^2(n)+\frac{1}{3}n^{s+1}\mathrm{ord}_2(r)n^2q_2^3(n)+\frac{1}{3}n^{s+1}\mathrm{ord}_3(r)n^2q_3^3(n)\Big)\\&-\frac{1}{\phi(r)}+\frac{\rho_2(r)}{2\phi(r)}\Big(1+n^{s+1}q_2(n)-\frac{1}{2}n^{s+1}nq_2^2(n)+\frac{1}{3}n^{s+1}n^2q_2^3(n)\Big)\\&+\frac{\rho_3(r)}{3\phi(r)}\Big(1+n^{s+1}q_3(n)-\frac{1}{2}n^{s+1}nq_3^2(n)+\frac{1}{3}n^{s+1}n^2q_3^3(n)\Big)\\&-\frac{\rho_2(r)\rho_3(r)}{6\phi(r)}\Big(1+n^{s+1}q_2(n)+n^{s+1}q_3(n)-\frac{1}{2}n^{s+1}nq_2^2(n)\\&-\frac{1}{2}n^{s+1}nq_3^2(n)+\frac{1}{3}n^{s+1}n^2q_2^3(n)+\frac{1}{3}n^{s+1}n^2q_3^3(n)\Big)\pmod{n^{s+4}},\end{aligned}$$

and so,

$$\begin{aligned}X\equiv\ &Y+\frac{1}{r}n^{s+1}\Big(\mathrm{ord}_2(r)q_2(n)+\mathrm{ord}_3(r)q_3(n)-\frac{1}{2}\mathrm{ord}_2(r)nq_2^2(n)\\&-\frac{1}{2}\mathrm{ord}_3(r)nq_3^2(n)+\frac{1}{3}\mathrm{ord}_2(r)n^2q_2^3(n)+\frac{1}{3}\mathrm{ord}_3(r)n^2q_3^3(n)\Big)\\&+\frac{\rho_2(r)}{2\phi(r)}n^{s+1}\Big(q_2(n)-\frac{1}{2}nq_2^2(n)+\frac{1}{3}n^2q_2^3(n)\Big)\\&+\frac{\rho_3(r)}{3\phi(r)}n^{s+1}\Big(q_3(n)-\frac{1}{2}nq_3^2(n)+\frac{1}{3}n^2q_3^3(n)\Big)\\&-\frac{\rho_2(r)\rho_3(r)}{6\phi(r)}n^{s+1}\Big(q_2(n)+q_3(n)-\frac{1}{2}nq_2^2(n)\\&-\frac{1}{2}nq_3^2(n)+\frac{1}{3}n^2q_2^3(n)+\frac{1}{3}n^2q_3^3(n)\Big)\pmod{n^{s+4}},\end{aligned}$$

where

$$Y=\frac{1}{r}-\frac{1}{\phi(r)}+\frac{\rho_2(r)}{2\phi(r)}+\frac{\rho_3(r)}{3\phi(r)}-\frac{\rho_2(r)\rho_3(r)}{6\phi(r)}.$$

An easy verification shows that $Y=0$. To check it we consider the cases. If $\rho_2(r)=0$ and $\rho_3(r)=1$; then $r=3$ and obviously $Y=0$. If $\rho_2(r)=1$ and $\rho_3(r)=0$; then $r=2,\,4,\,8$ and we have $Y=\frac{1}{r}-\frac{1}{2\phi(r)}=0$ since $r=2\phi(r)$ for these r. Finally, if $\rho_2(r)=\rho_3(r)=1$; then $r=6,\,12,\,24$ and $Y=\frac{1}{r}-\frac{1}{3\phi(r)}=0$ since $r=3\phi(r)$ in these cases. This completes the proof of Proposition 2. □

4 Proof of the Theorem

The proof of the Theorem falls naturally into seven cases $r = 2, 3, 4, 6, 8, 12$ or 24. In view of Proposition 1, in each of the cases, it suffices to determine:

(i) the sums $T_{r,1}(n) \pmod{n^{s+1}}$ for $s \in \{0, 1, 2\}$, which are determined in Theorems $4, 9, 14, 19, 24, 29$ or 34 of [9];

(ii) the congruences for $nT_{r,2}(n) \pmod{n^{s+1}}$ for $s \in \{1, 2\}$, which follow immediately from parts (i) and (ii) of Theorems $5, 10, 15, 20, 25, 30$ or 35 of [9];

(iii) the congruences for $n^2T_{r,3}(n) \pmod{n^3}$, which follow easily from parts (ii) of Theorems $1, 6, 11, 16, 21, 26$ or 31 of [9] for $k = 3$.(4)

Set $Q_i'(n) \equiv Q_i(n) \pmod{n}$ and $Q_i''(n) \equiv Q_i(n) \pmod{n^2}$ $(i = 2, 3)$. We consider the cases:

1. If $r = 2$, then part (i) of the Theorem for $s = 2$ is a consequence of Proposition 1, Theorems 4(i), 5(i) and Theorem 1(ii) of [9]; then for $n > 1$ odd we have

$$T_{2,1}(n) \equiv Q_2(n) - \frac{7}{8}n^2\widetilde{B}_{n^2\phi(n)-2} \pmod{n^3},\ nT_{2,2}(n) \equiv \frac{7}{2}n^2\widetilde{B}_{n^2\phi(n)-2} \pmod{n^3}$$

and

$$n^2T_{2,3}(n) \equiv -3n^2\widetilde{B}_{n^2\phi(n)-2} \pmod{n^3}.$$

The first of these congruences is the same as that in [9] and the second one is an immediate consequence of that in [9]. The third congruence follows immediately from Theorem 1(ii) [9] for $k = 3$; then $n^2T_{2,3}(n) \equiv 6n^2\widehat{B}_{n^2\phi(n)-2} \pmod{n^3}$. On the other hand,

$$n^2\widehat{B}_{n^2\phi(n)-2} \equiv -\frac{1}{2}n^2\widetilde{B}_{n^2\phi(n)-2} \pmod{n^3}, \tag{5}$$

which completes the proof in this case.

The part (i) of the Theorem for $s = 1$ follows immediately from Proposition 1 and Theorems 4(ii), 5(ii) of [9]; then

$$T_{2,1}(n) \equiv Q_2''(n) - \frac{7}{8}n^2\widetilde{B}_{n\phi(n)-2} \pmod{n^2},\ nT_{2,2}(n) \equiv \frac{7}{2}n^2\widetilde{B}_{n\phi(n)-2} \pmod{n^2}.$$

(4)More precisely, we need to determine $T_{r,1}(n)$, $nT_{r,2}(n)$, $n^2T_{r,3}(n) \pmod{n^3}$ if $s = 2$, $T_{r,1}(n)$, $nT_{r,2}(n) \pmod{n^2}$ if $s = 1$ and $T_{r,1}(n) \pmod{n}$ if $s = 0$.

If we assume that $3 \nmid n$, then $\widetilde{B}_{n\phi(n)-2}$ is p-integral for any $p|n$ and so

$$T_{2,1}(n) \equiv Q_2''(n) \pmod{n^2}, \ nT_{2,2}(n) \equiv 0 \pmod{n^2},$$

as claimed. The part (i) of the Theorem for $s = 0$ follows at once from Theorem 4(iii) of [9]; then $T_{2,1}(n) \equiv Q_2'(n) \pmod{n}$.

2. If $r = 3$, then part (i) of the Theorem for $s = 2$ is an immediate consequence of Proposition 1, Theorems 9(i), 10(i) and Theorem 6(ii) of [9]; then for $n > 1$, $3 \nmid n$ we have

$$T_{3,1}(n) \equiv Q_3(n) - \frac{1}{2} n \widetilde{D}_{n^2\phi(n)-2} - \frac{13}{18} n^2 \widetilde{B}_{n^2\phi(n)-2} \pmod{n^3},$$

$$nT_{3,2}(n) \equiv \frac{3}{2} n \widetilde{D}_{n^2\phi(n)-2} + \frac{13}{3} n^2 \widetilde{B}_{n^2\phi(n)-2} \pmod{n^3}$$

and

$$n^2 T_{3,3}(n) \equiv -6n^2 \widetilde{B}_{n^2\phi(n)-2} \pmod{n^3}.$$

Again the first congruence is the same as that in [9] and the second one is an easy consequence of that in [9]. The third congruence follows from Theorem 6(ii) of [9] for $k = 3$ and (5); then $n^2 T_{3,3}(n) \equiv 12n^2 \widehat{B}_{n^2\phi(n)-2} \pmod{n^3}$.

The part (i) of the Theorem for $s = 1$ follows immediately from Proposition 1 and Theorems 9(ii), 10(ii) of [9]; then

$$T_{3,1}(n) \equiv Q_3''(n) - \frac{1}{2} n \widetilde{D}_{n\phi(n)-2} \pmod{n^2}, \ nT_{3,2}(n) \equiv \frac{3}{2} n \widetilde{D}_{n\phi(n)-2} \pmod{n^2}.$$

Likewise, part (i) of the Theorem for $s = 0$ is an obvious consequence of Proposition 1 and Theorem 9(iii) of [9]; then $T_{3,1}(n) \equiv Q_3'(n) \pmod{n}$.

3. If $r = 4$, then part (i) of the Theorem for $s = 2$ follows from Proposition 1 and Theorems 14(i), 15(i) and 11(ii) of [9]; then for $n > 3$ odd we have

$$T_{4,1}(n) \equiv \frac{3}{2} Q_2(n) - n \widetilde{E}_{n^2\phi(n)-2} - \frac{7}{8} n^2 \widetilde{B}_{n^2\phi(n)-2} \pmod{n^3},$$

$$nT_{4,2}(n) \equiv 4n \widetilde{E}_{n^2\phi(n)-2} + 7n^2 \widetilde{B}_{n^2\phi(n)-2} \pmod{n^3}$$

and

$$n^2 T_{4,3}(n) \equiv -\frac{27}{2} n^2 \widetilde{B}_{n^2\phi(n)-2} \pmod{n^3}.$$

The first congruence is the same as that in [9] and the second one is an immediate consequence of that in [9]. The third congruence follows immediately from Theorem 11(ii) of [9] for $k = 3$; then $n^2 T_{4,3}(n) \equiv 27n^2 \widehat{B}_{n^2\phi(n)-2} \pmod{n^3}$ and it suffices to use (5).

The part (i) of the Theorem for $s = 1$ follows immediately from Proposition 1 and Theorems 14(ii), 15(ii) of [9]; then

$$T_{4,1}(n) \equiv \frac{3}{2}Q_2''(n) - n\widetilde{E}_{n\phi(n)-2} - \frac{7}{8}n^2\widetilde{B}_{n\phi(n)-2} \pmod{n^2},$$

$$nT_{4,2}(n) \equiv 4n\widetilde{E}_{n\phi(n)-2} + 7n^2\widetilde{B}_{n\phi(n)-2} \pmod{n^2},$$

and so

$$T_{4,1}(n) \equiv \frac{3}{2}Q_2''(n) - n\widetilde{E}_{n\phi(n)-2} \pmod{n^2}, \ nT_{4,2}(n) \equiv 4n\widetilde{E}_{n\phi(n)-2} \pmod{n^2}$$

if $3 \nmid n$. The part (i) of the Theorem for $s = 0$ is an obvious consequence of Theorem 14(iii) of [9]; then $T_{4,1}(n) \equiv \frac{3}{2}Q_2'(n) \pmod{n}$.

4. If $r = 6$, then part (i) of the Theorem for $s = 2$ is an immediate consequence of Proposition 1, Theorems 19(i), 20(i) and Theorem 16(ii) of [9]; then for $n > 5$, $3 \nmid n$ we have

$$T_{6,1}(n) \equiv Q_2(n) + Q_3(n) - \frac{5}{4}n\widetilde{D}_{n^2\phi(n)-2} - \frac{91}{72}n^2\widetilde{B}_{n^2\phi(n)-2} \pmod{n^3},$$

$$nT_{6,2}(n) \equiv \frac{15}{2}n\widetilde{D}_{n^2\phi(n)-2} + \frac{91}{6}n^2\widetilde{B}_{n^2\phi(n)-2} \pmod{n^3}$$

and

$$n^2T_{6,3}(n) \equiv -45n^2\widetilde{B}_{n^2\phi(n)-2} \pmod{n^3}.$$

The first congruence is the same as that in [9] and the second one is an immediate consequence of that in [9]. The third congruence follows from the congruence $n^2T_{6,3}(n) \equiv 90n^2\widehat{B}_{n^2\phi(n)-2} \pmod{n^3}$ and (5).

The part (i) of the Theorem for $s = 1$ follows immediately from Proposition 1 and Theorems 19(ii), 20(ii) of [9]; then

$$T_{6,1}(n) \equiv Q_2''(n) + Q_3''(n) - \frac{5}{4}n\widetilde{D}_{n\phi(n)-2} \pmod{n^2},$$

$$nT_{6,2}(n) \equiv \frac{15}{2}n\widetilde{D}_{n\phi(n)-2} \pmod{n^2}.$$

The part (i) of the Theorem for $s = 0$ follows at once from Proposition 1 and Theorem 19(iii) of [9]; then $T_{6,1}(n) \equiv Q_2'(n) + Q_3'(n) \pmod{n}$.

5. If $r = 8$, then part (ii) of the Theorem for $s = 2$ follows from Proposition 1, Theorems 24(i), 25(i) and Theorem 21(ii) of [9]; then for $n > 7$ odd we have

$$T_{8,1}(n) \equiv 2Q_2(n) + 2\widetilde{A}_{n^2\phi(n)-1} - n\widetilde{E}_{n^2\phi(n)-2} - 2n\widetilde{C}_{n^2\phi(n)-2}$$

$$-\frac{7}{8}n^2\widetilde{B}_{n^2\phi(n)-2}+2n^2\widetilde{A}_{n^2\phi(n)-3}\pmod{n^3},$$

$$nT_{8,2}(n)\equiv 8n\widetilde{E}_{n^2\phi(n)-2}+16n\widetilde{C}_{n^2\phi(n)-2}+14n^2\widetilde{B}_{n^2\phi(n)-2}-32n^2\widetilde{A}_{n^2\phi(n)-3}\pmod{n^3}$$

and

$$n^2T_{8,3}(n)\equiv -\frac{111}{2}n^2\widetilde{B}_{n^2\phi(n)-2}+128n^2\widetilde{A}_{n^2\phi(n)-3}\pmod{n^3}.$$

The first congruence is the same as that in [9], the second one follows from that in [9] and the third one is an immediate consequence of Theorem 21(ii) of [9] for $k=3$; then $n^2T_{8,3}(n)\equiv 111n^2\widehat{B}_{n^2\phi(n)-2}+128n^2\widetilde{A}_{n\phi(n)-3}\pmod{n^3}$ and the congruence follows from (5).

The part (ii) of the Theorem for $s=1$ follows immediately from Proposition 1 and Theorems 24(ii), 25(ii) of [9]; then

$$T_{8,1}(n)\equiv 2Q_2''(n)+2\widetilde{A}_{n\phi(n)-1}-n\widetilde{E}_{n\phi(n)-2}-2n\widetilde{C}_{n\phi(n)-2}-\frac{7}{8}n^2\widetilde{B}_{n\phi(n)-2}\pmod{n^2},$$

$$nT_{8,2}(n)\equiv 8n\widetilde{E}_{n\phi(n)-2}+16n\widetilde{C}_{n\phi(n)-2}+14n^2\widetilde{B}_{n\phi(n)-2}\pmod{n^2},$$

and so

$$T_{8,1}(n)\equiv 2Q_2''(n)+2\widetilde{A}_{n\phi(n)-1}-n\widetilde{E}_{n\phi(n)-2}-2n\widetilde{C}_{n\phi(n)-2}\pmod{n^2},$$

$$nT_{8,2}(n)\equiv 8n\widetilde{E}_{n\phi(n)-2}+16n\widetilde{C}_{n\phi(n)-2}\pmod{n^2}$$

if $3\nmid n$. The part (ii) of the Theorem for $s=0$ is an easy consequence of Theorem 24(iii) of [9]; then $T_{8,1}(n)\equiv 2Q_2'(n)+2\widetilde{A}_{\phi(n)-1}\pmod{n}$.

6. If $r=12$, then part (iii) of the Theorem for $s=2$ follows at once from Proposition 1, Theorems 29(i), 30(i) and Theorem 26(ii) of [9]; then for $n>11$ odd we have

$$T_{12,1}(n)\equiv \frac{3}{2}Q_2(n)+Q_3(n)+3\widetilde{F}_{n^2\phi(n)-1}-\frac{5}{4}n\widetilde{D}_{n^2\phi(n)-2}-\frac{5}{3}n\widetilde{E}_{n^2\phi(n)-2}$$
$$-\frac{91}{72}n^2\widetilde{B}_{n^2\phi(n)-2}+3n^2\widetilde{F}_{n^2\phi(n)-3}\pmod{n^3},$$

$$nT_{12,2}(n)\equiv 15n\widetilde{D}_{n^2\phi(n)-2}+20n\widetilde{E}_{n^2\phi(n)-2}+\frac{91}{3}n^2\widetilde{B}_{n^2\phi(n)-2}$$
$$-72n^2\widetilde{F}_{n^2\phi(n)-3}\pmod{n^3}$$

and

$$n^2T_{12,3}(n)\equiv -\frac{363}{2}n^2\widetilde{B}_{n^2\phi(n)-2}+432n^2\widetilde{F}_{n\phi(n)-3}\pmod{n^3}.$$

The first congruence is the same as that in [9], the second one is implied by that in [9] and the third one follows from Theorem 26(ii) of [9] for $k = 3$; then $n^2T_{12,3}(n) \equiv 363n^2\widehat{B}_{n^2\phi(n)-2} + 432n^2\widetilde{F}_{n\phi(n)-3} \pmod{n^3}$ and it suffices to use (5). The part (iii) of the Theorem for $s = 1$ follows at once from Proposition 1 and Theorems 29(ii), 30(ii) of [9]; then

$$T_{12,1}(n) \equiv \frac{3}{2}Q_2''(n) + Q_3''(n) + 3\widetilde{F}_{n\phi(n)-1} - \frac{5}{4}n\widetilde{D}_{n\phi(n)-2} - \frac{5}{3}n\widetilde{E}_{n\phi(n)-2} \pmod{n^2},$$

$$nT_{12,2}(n) \equiv 15n\widetilde{D}_{n\phi(n)-2} + 20n\widetilde{E}_{n\phi(n)-2} \pmod{n^2}.$$

Part (iii) of the Theorem for $s = 0$ follows easily from Proposition 1 and Theorem 29(iii) of [9]; then $T_{12,1}(n) \equiv \frac{3}{2}Q_2'(n) + Q_3'(n) + 3\widetilde{F}_{\phi(n)-1} \pmod{n}$.

7. If $r = 24$, then part (iv) of the Theorem for $s = 2$ follows from Proposition 1, Theorems 34(i), 35(i) and Theorem 31(ii) of [9]; then for $n > 23$ odd we have

$$\begin{aligned} T_{24,1}(n) \equiv\ & 2Q_2(n) + Q_3(n) + 3\widetilde{F}_{n^2\phi(n)-1} + 3\widetilde{G}_{n^2\phi(n)-1} + 4\widetilde{A}_{n^2\phi(n)-1} \\ & - \frac{5}{4}n\widetilde{D}_{n^2\phi(n)-2} - \frac{5}{3}n\widetilde{E}_{n^2\phi(n)-2} - 3n\widetilde{H}_{n^2\phi(n)-2} - \frac{8}{3}n\widetilde{C}_{n^2\phi(n)-2} \\ & - \frac{91}{72}n^2\widetilde{B}_{n^2\phi(n)-2} + 3n^2\widetilde{F}_{n^2\phi(n)-3} + 3n^2\widetilde{G}_{n^2\phi(n)-3} + \frac{28}{9}n^2\widetilde{A}_{n^2\phi(n)-3} \pmod{n^3}, \end{aligned}$$

$$\begin{aligned} nT_{24,2}(n) \equiv\ & 30n\widetilde{D}_{n^2\phi(n)-2} + 40n\widetilde{E}_{n^2\phi(n)-2} + 72n\widetilde{H}_{n^2\phi(n)-2} + 64n\widetilde{C}_{n^2\phi(n)-2} \\ & + \frac{182}{3}n^2\widetilde{B}_{n^2\phi(n)-2} - 144n^2\widetilde{F}_{n^2\phi(n)-3} - 144n^2\widetilde{G}_{n^2\phi(n)-3} \\ & - \frac{448}{3}n^2\widetilde{A}_{n^2\phi(n)-3} \pmod{n^3} \end{aligned}$$

and

$$\begin{aligned} n^2T_{24,3}(n) \equiv\ & -\frac{1455}{2}n^2\widetilde{B}_{n^2\phi(n)-2} + 1728n^2\widetilde{F}_{n^2\phi(n)-3} \\ & + 1728n^2\widetilde{G}_{n^2\phi(n)-3} + 1792n^2\widetilde{A}_{n^2\phi(n)-3} \pmod{n^3}. \end{aligned}$$

Again the first congruence is the same as that in [9], the second one follows immediately from that in [9] and the third one follows from Theorem 26(ii) of [9] for $k = 3$; then

$$\begin{aligned} n^2T_{24,3}(n) \equiv\ & 1455n^2\widehat{B}_{n^2\phi(n)-2} + 1728n^2\widetilde{F}_{n^2\phi(n)-3} \\ & + 1728n^2\widetilde{G}_{n^2\phi(n)-3} + 1792n^2\widetilde{A}_{n^2\phi(n)-3} \pmod{n^3} \end{aligned}$$

and it suffices to use (5).

Part (iv) of the Theorem for $s = 1$ follows immediately from Proposition 1 and Theorems 34(ii), 35(ii) of [9]; then we have

$$T_{24,1}(n) \equiv 2Q_2''(n) + Q_3''(n) + 3\widetilde{F}_{n\phi(n)-1} + 3\widetilde{G}_{n\phi(n)-1} + 4\widetilde{A}_{n\phi(n)-1} - \frac{5}{4}n\widetilde{D}_{n\phi(n)-2} - \frac{5}{3}n\widetilde{E}_{n\phi(n)-2} - 3n\widetilde{H}_{n\phi(n)-2} - \frac{8}{3}n\widetilde{C}_{n\phi(n)-2} \pmod{n^2},$$

$$nT_{24,2}(n) \equiv 30n\widetilde{D}_{n\phi(n)-2} + 40n\widetilde{E}_{n\phi(n)-2} + 72n\widetilde{H}_{n\phi(n)-2} + 64n\widetilde{C}_{n\phi(n)-2} \pmod{n^2}.$$

Part (iii) of the Theorem for $s = 0$ is implied by Proposition 1 and Theorem 34(iii) of [9]; then

$$T_{24,1}(n) \equiv 2Q_2'(n) + Q_3'(n) + 3\widetilde{F}_{\phi(n)-1} + 3\widetilde{G}_{\phi(n)-1} + 4\widetilde{A}_{\phi(n)-1} \pmod{n}.$$

This completes the proof of the Theorem. □

5 Concluding remarks

Let $p \geq 3$ be a prime number and let r be a natural number such that $1 < r < p$. In the next part of the paper we are going to derive some new congruences for the sums $U_r(p) = \sum_{i=1}^{[p/r]} \frac{1}{p-ri}$ modulo p^{s+1} for $s \in \{0, 1, 2\}$ and for all divisors r of 24. We shall use the congruences obtained in the present paper in the case when $n = p$ is an odd prime as well as Kummer's congruences for the generalized Bernoulli numbers.

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