SIMPLE EXAMPLES OF AFFINE MANIFOLDS WITH INFINITELY MANY EXOTIC MODELS

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Abstract. We give a simple general method of construction affine varieties with infinitely many exotic models. In particular we show that for every $d > 1$ there exists a Stein manifold of dimension d, which has uncountable many different structures of affine variety.

1. Introduction.

Given any smooth complex affine variety X , one can ask if there exists smooth affine varieties Y non isomorphic to X but which are biholomorphic to X when equipped with their underlying structures of complex analytic manifold. When such exist, these varieties Y could be called exotic models of X.

Examples of affine varieties with exotic models was found in dimension two and three (see [1], [9], [11]). Moreover, in [6] we showed that for every $n \geq 7$ there are $n-$dimensional rational affine manifolds with exotic models. The aim of this note is to give a simple general method of construction of such examples. In particular we show that such examples do exist in any dimension $d > 1$ (for $d = 1$ it is easy to see that such examples do not exist). Here we modify our idea from [6] and we prove:

Theorem 1.1. *Let V be a non-rational smooth affine curve. Then*

(i) the affine surface $Y := V \times \mathbb{C}$ has uncountably many different exotic models.

(ii) for every non $\mathbb{C}-$uniruled smooth affine variety Z the variety $Y \times Z$ has an exotic model. Moreover, if the group $Aut(V \times Z)$ is at most countable, then the Stein $n-$fold $X \times Z$ has uncountably many different structures of affine variety.

and

Theorem 1.2. *Let V be a smooth affine surface, which has a smooth completion $\overline{V}$, such that $H^0(\overline{V}, K_{\overline{V}}) \neq 0$. Then*

(i) the affine fourfold $X := V \times \mathbb{C}^2$ has infinitely many different exotic models.

(ii) for every non $\mathbb{C}-$uniruled smooth affine variety Z the variety $X \times Z$ has an exotic model. Moreover, if the group $Aut(V \times Z)$ is finite, then the Stein $n-$fold $X \times Z$ has infinitely many different structures of affine variety.

Remark 1.3. In particular we can take as V (above) any generic surface $V \subset \mathbb{C}^3$ of degree $d \geq 4$.

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2. Exotic models

Let us recall the definition of a $\mathbb{C}$-uniruled variety which was introduced in our paper [7]. First recall that *a polynomial curve* in X is the image of the affine line $A^1(\mathbb{C})$ under a non-constant morphism $\phi : A^1(\mathbb{C}) \to X$. Now we have:

Definition 2.1. An affine variety X is said to be $\mathbb{C}$-*uniruled* if it is of dimension ≥ 1 and there exists a Zariski open, non-empty subset U of X such that for every point $x \in U$ there is a polynomial curve in X passing through x.

It is well-known, that if X is projectively smooth variety, which is $\mathbb{C}-$uniruled, then $H^0(\overline{X}, K_{\overline{X}}) = 0$, where $K_{\overline{X}}$ denotes the canonical divisor of $\overline{X}$. In the sequel we need the following basic theorem, which was proved in our paper [5]:

Theorem 2.2. *Let X be a non-$\mathbb{C}$-uniruled smooth affine variety. Let $\mathbf{F}$ be an algebraic vector bundle on X of* rank r. *If the total space of $\mathbf{F}$ is isomorphic to $X \times \mathbb{C}^r$, then $\mathbf{F}$ is a trivial vector bundle.*

We have also the following version of this theorem (compare with [5]):

Theorem 2.3. *Let X be a non-$\mathbb{C}$-uniruled smooth affine variety. Let $\mathbf{F}, \mathbf{G}$ be algebraic vector bundles on X of* rank r. *If the total space of $\mathbf{F}$ is isomorphic to the total space of $\mathbf{G}$, then $\mathbf{F}$ is isomorphic to $\sigma^*\mathbf{G}$ for some automorphism $\sigma \in Aut(X)$.*

Proof. Let F denote the total space of $\mathbf{F}$ and G the total space of $\mathbf{G}$. In what follows, we will identify X with the zero section of $\mathbf{F}$ and $\mathbf{G}$. Note that

$$\mathbf{F} \cong TF|_X/TX, \ \mathbf{G} \cong TG|_X/TX.$$

Assume that there exists an isomorphism $\Phi : F \to G$. Let $\pi : \mathbf{G} \to \mathbf{X}$ be the projection and take $f = \pi \circ \Phi$. Since the vector bundle $\mathbf{F}$ is locally trivial in the Zariski topology, Lemma 3.4 in [5] shows that $\Phi(\mathbf{F}_x) = \mathbf{G}_{f(x)}$ for every $x \in X$. Consequently, the mapping $\sigma := f|X : X \to X$ is an bijection. Moreover, it is easy to check that for every $x \in X$ the derivative $d_x\sigma$ is an isomorphism. Consequently the mapping σ is an automorphism. Take $\mathbf{G}' = \sigma^*\mathbf{G}$. Let $\Sigma : G \to G'$ be the induced isomorphism of total spaces (locally given as $U \times \mathbb{C}^n \ni (x, v) \to (\sigma^{-1}(x), v) \in \sigma^{-1}(U) \times \mathbb{C}^n$). Replace Φ by $\Sigma \circ \Phi$ and $\mathbf{G}$ by $\mathbf{G}'$.

Now the mapping $\Phi_{|X} : X \in x \mapsto (x, t(x)) \in G$ is a section. Consider the isomorphism $\Psi : G \ni (x, v) \mapsto (x, v - t(x)) \in \ G$. Again we can replace Φ by $\Psi \circ \Phi$ to obtain $\Phi|X : X \times \{0\} \ni (x, 0) \mapsto (x, 0) \in G$. Hence we can assume that Φ transforms the zero section into the zero section, and moreover it induces the identity on the zero section. Hence $d\Phi(TX) = TX$ and the mapping

$$d\Phi : TF|_X/TX \cong \mathbf{F} \to TG|_X/TX \cong \mathbf{G}$$

is an isomorphism. Consequently, the bundle $\mathbf{F}$ is isomorphic to $\mathbf{G}$. □

Now we review some results about Stein manifolds. It is well-known that a $n-$dimensional Stein manifold X has the homotopy type of a (real) $n-$dimensional CW complex (see [8]). Complex vector bundles on such CW complexes have tha following nice property:

Theorem 2.4. *([2], p. 111) Let Y be a $r-$dimensional CW complex and let $\mathbf{F}$ be a complex vector bundle on Y of rank k. If $r \leq 2k - 1$, then $\mathbf{F}$ has a one dimensional trivial summand.*

Now we are ready to prove our first result:

Theorem 2.5. *Let V be a non-rational smooth affine curve. Then*

(i) the affine surface $Y := V \times \mathbb{C}$ has uncountably many different exotic models.

(ii) for every non $\mathbb{C}-$uniruled smooth affine variety Z the variety $Y \times Z$ has an exotic model. Moreover, if the group $Aut(V \times Z)$ is at most countable, then the Stein $n-$fold $X \times Z$ has uncountably many different structures of affine variety.

Proof. (i) Let $\overline{V}$ be a smooth compactification of V and $\{x_1, ..., x_r\} = \overline{V} \backslash V$. Then $Pic(V) = Pic(\overline{V})/ < x_1, ..., x_r >$. Since the subgroup $< x_1, ..., x_r >$ is countable and $Pic(\overline{V})$ not, we have $Pic(V) \neq 0$ - in fact this group is uncountable. Let $\mathbf{L} \in Pic(V)$ be a non-zero line bundle. Hence it is algebraically non-trivial. However, by Theorem 2.4 $\mathbf{L}$ is holomorphically trivial. Consequently the total space of any line bundle $\mathbf{L} \in Pic(X)$ is biholomorphic to Y.

Note that the total space of every line bundle $\mathbf{L} \in Pic(V)$ determines one affine structure $Y_{\mathbf{L}}$ on Y. Let ρ be the relation on $Pic(V)$ such that $\mathbf{L}$ is in a relation with $\mathbf{L}'$ if and only if there exists an automorphism $\sigma \in Aut(V) : \mathbf{L}' = \sigma^*\mathbf{L}$. Since the group $Aut(V)$ is finite, we see that the set $S := Pic(V)/\rho$ is uncountable. Denote the class of relation of $\mathbf{L}$ with respect to relation ρ by $[\mathbf{L}]$.

Structures $Y_{\mathbf{L}}$ and $Y_{\mathbf{L}'}$ are not isomorphic for $[\mathbf{L}] \neq [\mathbf{L}']$ by Theorem 2.3. This means that there is at least $\#S$ different affine structures on Y.

(ii) Let $\pi : V \times Z \to V$ be a projection. Take $\mathbf{L}' = \pi^*(\mathbf{L})$. Then $\mathbf{L}$ is holomorphically trivial. However, it is algebraically non-trivial. Indeed, take a point $z \in Z$. If we identify V with $V \times \{z\} \subset V \times Z$, then $\mathbf{L}'_{|V} = \mathbf{L}$. Now we can finish as above.

Note that the mapping $\pi^* : Pic(V) \ni \mathbf{L} \to \pi^*\mathbf{L} \in Pic(V \times Z)$ is injective, hence the group $Pic(V \times Z)$ is uncountable. If the group $Aut(V \times Z)$ is at most countable, then the set $S' = A^2(V \times Z)/\rho$ (where ρ is the relation as above) is uncountable. Now we can finish as above. □

Corollary 2.6. *Let $\Gamma_1, ..., \Gamma_r$ be a finite family of smooth non-rational curves ($r \geq 1$) and put $X = \mathbb{C} \times \prod_{i=1}^r \Gamma_i$. Then the Stein manifold X has uncountable many different structures of affine variety. In particular for every $d > 1$ there exists a Stein manifold of dimension d, which has uncountable many different structures of affine variety.*

Proof. Let us note that $\overline{\kappa}(\Gamma_i) = 1$ (where $\overline{\kappa}$ denotes the logarithmic Kodaira dimension). We have $\overline{\kappa}(\prod_{i=1}^r \Gamma_i) = \sum_{i=1}^r \overline{\kappa}(\Gamma_i) = r$ (see [3], Theorem 11.3). Hence the variety $\prod_{i=1}^r \Gamma_i$ is of general type and consequently it has a finite automorphisms group (see Theorem 11.12 in [3]). □

We show that our method can be applied also to affine surfaces. The following Lemma is well known:

Lemma 2.7. *Let X be a smooth affine surface. Let $A^p(X)$ denotes the group of codimension p-cycles modulo rational equivalence. Let $c_1 \in A^1(X)$, $c_2 \in A^2(X)$. Then there exists an algebraic vector bundle $\mathbf{F}$ of rank 2, such that $c_i(\mathbf{F}) = c_i$ for $i = 1, 2$, where $c_i(\mathbf{F})$ is an i^{th} Chern class of $\mathbf{F}$.*

Proof. Let $X = Spec(A)$. Let $\mathbf{L}$ be a line bundle which correspond to c_1. Moreover, let A/I represent c_2, where I is a product of different maximal ideals. Then $Ext^1_A(I, L)$ is cyclic, where L is a module of sections of $\mathbf{L}$. Following [14] we get an exact sequence

$$0 \to L \to F \to I \to 0,$$

where F is a projective module of rank 2. If $\mathbf{F}$ is a vector bundle which correspond to F, we get $c_i(\mathbf{F}) = c_i$ for $i = 1, 2$. □

We have also:

Lemma 2.8. *Let V be a smooth affine surface, which has a smooth completion $\overline{V}$, such that $H^0(\overline{V}, K_{\overline{V}}) \neq 0$. Then there exists an algebraic vector bundle $\mathbf{F}$ of rank two on V, which is algebraically non-trivial, but holomorphically trivial.*

Proof. Note that $H^0(\overline{V}, K_{\overline{V}}) \neq 0$. By Theorem 2.9 we have $A^2(V) \neq 0$. Take nonzero $c_2 \in A^2(V)$. By Lemma 2.7 there is an algebraic vector bundle $\mathbf{F}$ of rank 2 such that $c_1(\mathbf{F}) = 0$ and $c_2(\mathbf{F}) = c_2$. In particular $\mathbf{F}$ is algebraically non-trivial and it has trivial determinant.

We show that $\mathbf{F}$ is holomorphically trivial. We will make use of Grauert's theorem on the Oka principle for vector bundles which says that on Stein spaces the holomorphic and topological classifications coincide. Therefore we can use the topological theory of complex vector bundles. Moreover, since every $n-$dimensional Stein manifold has a homotopy type of a (real) $n-$dimensional CW complex, if we study vector bundles on V, we can assume that V itself is a $2-$dimensional CW complex. In particular by Theorem 2.4 we have $\mathbf{F} = \mathbf{L} \oplus \mathbf{E^1}$, where $\mathbf{E}^1$ denotes the trivial line bundle. Since $\mathbf{E}^1 = \wedge^2 \mathbf{F} = \mathbf{L} \otimes \mathbf{E}^1 = \mathbf{L}$ we have $\mathbf{F} = \mathbf{E}^1 \oplus \mathbf{E}^1$ is holomorphically trivial. □

Moreover, we need the following result of Mumford and Roitman (see [10], [12]):

Theorem 2.9. *Let X be an irreducible, proper, non-singular variety of dimension d over $\mathbb{C}$, such that $H^0(X; K_X) \neq 0$, where K_X is the canonical divisor of X. Then for any affine open subset $V \subset X$, we have $A^d(V) \neq 0$.*

Finally we have:

Theorem 2.10. *Let V be a smooth affine surface, which has a smooth completion $\overline{V}$, such that $H^0(\overline{V}, K_{\overline{V}}) \neq 0$. Then*

(i) the affine fourfold $X := V \times \mathbb{C}^2$ has infinitely many different exotic models.

(ii) for every non $\mathbb{C}-$uniruled smooth affine variety Z the variety $X \times Z$ has an exotic model. Moreover, if the group $Aut(V \times Z)$ is finite, then the Stein $n-$fold $X \times Z$ has infinitely many different structures of affine variety.

Proof. (i) Let $\mathbf{F}$ be a vector bundle as in Lemma 2.8. This vector bundle is algebraically non-trivial but holomorphically trivial.

Let $\mathbf{E}^2$ be a trivial vector bundle of rank two on V. Since V is a non-uniruled variety, by Theorem 2.2 total spaces $\mathcal{F}$ and $\mathcal{E}$ of vector bundles $\mathbf{F}$ and $\mathbf{E}^2$ are not isomorphic as algebraic varieties. However in obvious way $\mathcal{F}$ and $\mathcal{E}$ are biholomorphic as total spaces of the same trivial holomorphic vector bundle.

Note that the total space of every vector bundle $\mathbf{F}$ as above, determines one affine structure $Y_{\mathbf{F}}$ on Y. Let ρ be the relation on $A^2(V)$ such that a is in a relation with b if and only if there exists an automorphism $\sigma \in Aut(V)$ such that $a = \sigma^* b$. Note that the group $A^2(V)$ is infinite, because by [13] we have $A^2(V) \otimes \mathbb{Q} \neq 0$. Since the group $Aut(V)$ is finite (see [4]), we have that the set $S := A^2(V)/\rho$ is infinite. Denote the class of relation of $a \in A^2(V)$ with respect to relation ρ by $[a]$. Structures $Y_{\mathbf{F}}$ and $Y_{\mathbf{F'}}$ are not isomorphic for $[c_2(\mathbf{F})] \neq [c_2(\mathbf{F'})]$ by Theorem 2.3. This means that there is at least $\#S$ different affine structures on Y.

(ii) Since V is not $\mathbb{C}-$uniruled, then also $V \times Z$ is not $\mathbb{C}-$uniruled. Let $\pi : V \times Z \to V$ be a projection. Take $\mathbf{G} = \pi^*(\mathbf{F})$. Then $\mathbf{G}$ is holomorphically trivial. However, it is algebraically non-trivial. Indeed, take a point $z \in Z$. If we identify V with $V \times \{z\} \subset V \times Z$, then $\mathbf{G}_{|V} = \mathbf{F}$. Now we can finish as above.

Note that the mapping $\pi^* : A^2(V) \ni a \to \pi^* a \in A^2(V \times Z)$ is injective, hence the group $A^2(V \times Z)$ is infinite. If the group $Aut(V \times Z)$ is finite, then the set $S' = A^2(V \times Z)/\rho$ (where ρ is the relation as above) is infinite. Now we can finish as above. □

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