The Widder spaces, representations of the convolution algebra $L^1(\mathbb{R}^+)$ and one parameter semigroups of operators

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Abstract. A short proof is presented of the isometric isomorphism between the Widder space $W(\mathbb{K}^+;E)$ and the space $L(L^1(\mathbb{R}^+);E)$. The Hille–Yosida generation theorem from the theory of operator semigroups is reproved by an argument involving representations of the convolution algebra $L^1(\mathbb{R}^+)$.

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0. Introduction

Theorem 16a on p. 315 of D. V. Widder's book "The Laplace Transform" says that a C^∞ function f defined on $(0,\infty)$ is the Laplace transform of a function belonging to $L^\infty(0,\infty)$ if and only if

$$\sup\left\{\frac{\lambda^{k+1}}{k!}|f^{(k)}(\lambda)| : \lambda > 0,\ k = 0,1,\dots\right\} < \infty. \tag{0.1}$$

If A is a closed operator from a Banach space X into X such that the resolvent operator $R(\lambda) = (\lambda - A)^{-1}$ exists in $L(X)$ for every $\lambda > 0$, then

$$\frac{\lambda^{k+1}}{k!}R^{(k)}(\lambda) = (-1)^k[\lambda R(\lambda)]^{k+1} \tag{0.2}$$

and a condition analogous to (0.1) takes the form

$$\sup\{\|[\lambda R(\lambda)]^k\| : \lambda > 0,\ k = 1,2,\dots\} < \infty. \tag{0.3}$$

According to the Hille–Yosida generation theorem ([H], p. 238; [Y;1]; [H-P], p. 360; [Y;2], p. 248), a closed operator A from a Banach space X into X is the infinitesimal generator of a bounded one parameter semigroup $(S(t))_{t\geq 0} \subset L(X)$ of class C^0 if and only if the domain of A is dense in X, the resolvent set of A contains $(0,\infty)$, and the resolvent family of A satisfies condition (0.3).

The present paper connects the above-mentioned theorems in the framework of the theory of linear maps from the Banach space $L^1(\mathbb{R}^+)$ to other Banach spaces, and representations of the convolution algebra $L^1(\mathbb{R}^+)$. The paper contains a short proof of Widder's theorem in the operator theoretical version going back to B. Hennig and F. Neubrander [H-N]. Then a result is deduced on representing a pseudoresolvent with values in a Banach algebra A as the homomorphic image of the canonical pseudoresolvent with values in $L^1(\mathbb{R}^+)$. This permits us to establish a connection between representations of $L^1(\mathbb{R}^+)$ and one parameter semigroups of operators, leading to a new proof of the Hille–Yosida theorem, and to an almost trivial proof of the Trotter–Kato theorem on approximation of semigroups.

The role of $L^1(\mathbb{R}^+)$ in the present paper is analogous to the role of L. Schwartz's space of infinitely differentiable rapidly decreasing functions in paper [L] of J. L. Lions concerning the semigroups-distributions.

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Notation. In the first and the second chapter E denotes a Banach space over the field $\mathbb{K}$ equal either to $\mathbb{R}$ or to $\mathbb{C}$. In subsequent chapters $E = A$, an abstract Banach algebra, and $E = L(X)$, the Banach algebra of endomorphisms of a Banach space X. Throughout the paper

$$\mathbb{R}^+ = (0,\infty), \quad \overline{\mathbb{R}^+} = [0,\infty), \quad \mathbb{C}^+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\},$$
$$\mathbb{K}^+ = \mathbb{R}^+ \text{ if } \mathbb{K} = \mathbb{R}, \quad \mathbb{K}^+ = \mathbb{C}^+ \text{ if } \mathbb{K} = \mathbb{C}.$$

1. The Widder spaces $W(\mathbb{R}^+;E)$ and $W(\mathbb{C}^+;E)$

Denote by $W(\mathbb{R}^+;E)$ the Banach space over the field $\mathbb{K}$ whose elements are infinitely differentiable functions $f : \mathbb{R}^+ \to E$ such that $\|f\|_{W(\mathbb{R}^+;E)} < \infty$, where

$$\|f\|_{W(\mathbb{R}^+;E)} = \sup\left\{\frac{\lambda^{k+1}}{k!}\|f^{(k)}(\lambda)\|_E : \lambda \in \mathbb{R}^+,\ k = 0,1,\dots\right\}.$$

If $\mathbb{K} = \mathbb{C}$ then denote by $W(\mathbb{C}^+;E)$ the complex Banach space of holomorphic functions $f : \mathbb{C}^+ \to E$ such that $\|f\|_{W(\mathbb{C}^+;E)} < \infty$, where

$$\|f\|_{W(\mathbb{C}^+;E)} = \sup\left\{\frac{(\operatorname{Re}\lambda)^{k+1}}{k!}\|f^{(k)}(\lambda)\|_E : \lambda \in \mathbb{C}^+,\ k = 0,1,\dots\right\}.$$

We call $W(\mathbb{R}^+;E)$ and $W(\mathbb{C}^+;E)$ the Widder spaces. This is legitimated by Theorem 16a, p. 315, in Chapter VII of Widder's book [W], quoted in our Introduction, and also by other theorems in the same chapter of [W]. Importance of the Widder spaces for the generation theory of cosine operator functions and integrated semigroups manifests itself in the papers of M. Sova [S;1]–[S;4].

1.1. PROPOSITION. *Suppose that* $f \in W(\mathbb{R}^+;E)$.

(A) *If* $\mathbb{K} = \mathbb{R}$ *then* f *is real-analytic on* $\mathbb{R}^+$ *and for every* $\mu \in \mathbb{R}^+$ *the Taylor development of* f *with center at* μ *converges to* f *almost uniformly on the interval* $(0, 2\mu)$.

(B) *If* $\mathbb{K} = \mathbb{C}$ *then* f *extends to an* E*-valued function* $\widetilde{f}$ *holomorphic on* $\mathbb{C}^+$ *such that* $\widetilde{f} \in W(\mathbb{C}^+;E)$ *and* $\|\widetilde{f}\|_{W(\mathbb{C}^+;E)} = \|f\|_{W(\mathbb{R}^+;E)}$.

PROOF. (A) We reproduce our proof presented in [B], pp. 282–283 (see the footnote on p. 281 of [B]). Fix a $\mu \in \mathbb{R}^+$. By Taylor's formula,

$$f(\lambda) = \sum_{k=0}^{l} \frac{(\lambda-\mu)^k}{k!} f^{(k)}(\mu) + R_{l+1}$$

for every $\lambda \in \mathbb{R}^+$ and $l = 0,1,\dots$, where

$$R_{l+1} = \int_{\mu}^{\lambda} \frac{(\lambda-\nu)^l}{l!} f^{(l+1)}(\nu)\, d\nu.$$

Let $M = \|f\|_{W(\mathbb{R}^+;E)}$. Then

$$\|R_{l+1}\|_E \le M\left|\int_\mu^\lambda \frac{(\lambda-\nu)^l}{l!}\cdot\frac{(l+1)!}{\nu^{l+2}}\,d\nu\right| = M\left|\int_\mu^\lambda (l+1)\left(\frac{\lambda}{\nu}-1\right)^l\frac{d\nu}{\nu^2}\right|$$
$$= \frac{M}{\lambda}\int_0^{|\lambda/\mu-1|}(l+1)\sigma^l\,d\sigma = \frac{M}{\lambda}\left|\frac{\lambda-\mu}{\mu}\right|^{l+1},$$

whence $\lim_{l\to\infty}\|R_{l+1}\|_E = 0$ almost uniformly with respect to λ on the interval $(0,2\mu)$. ■

(B) If $\mathbb{K}=\mathbb{C}$ and $f\in W(\mathbb{R}^+;E)$ then

$$\left\|\frac{(\lambda-\mu)^k}{k!}f^{(k)}(\mu)\right\|_E \le \frac{|\lambda-\mu|^k}{\mu^{k+1}}\|f\|_{W(\mathbb{R}^+;E)}$$

for every $\mu\in\mathbb{R}^+$ and $\lambda\in\mathbb{C}$, so that the Taylor series

$$\sum_{k=0}^{\infty}\frac{(\lambda-\mu)^k}{k!}f^{(k)}(\mu) \tag{1.1}$$

converges in the norm of E, almost uniformly with respect to λ in the disc

$$D_\mu = \{\lambda\in\mathbb{C} : |\lambda-\mu|<\mu\}.$$

The sum of this series is an E-valued function holomorphic in D_μ and, as a consequence of (A), it is equal to f on $(0,2\mu) = D_\mu\cap\mathbb{R}$. Since $\bigcup_{\mu>0}D_\mu = \mathbb{C}^+$, it follows that f extends uniquely to an E-valued function $\widetilde{f}$ holomorphic on $\mathbb{C}^+$. Since obviously $\|\widetilde{f}\|_{W(\mathbb{C}^+;E)} \ge \|f\|_{W(\mathbb{R}^+;E)}$, it remains to show that

$$\|\widetilde{f}\|_{W(\mathbb{C}^+;E)} \le \|f\|_{W(\mathbb{R}^+;E)}. \tag{1.2}$$

We shall present two proofs of inequality (1.2), the first employing Widder's Theorems 16a and 16b from pp. 315–316 of [W], and the second based on some direct estimations of the Taylor series (1.1). Notice that in Corollary 2.3, we shall deduce the Widder theorems from the case $\mathbb{K}=\mathbb{R}$ of our Theorem 2.2. Notice also that our proof of this last case is independent of part (B) of Proposition 1.1.

The first proof of inequality (1.2). By the Bohnenblust–Sobczyk complex version of the Hahn–Banach Theorem ([Y;2], Sec. IV.6, pp. 107–108) it is sufficient to show that

$$\|\phi\circ\widetilde{f}\|_{W(\mathbb{C}^+;\mathbb{C})} \le \|\phi\circ f\|_{W(\mathbb{R}^+;\mathbb{C})} \tag{1.3}$$

for every $\mathbb{C}$-linear functional $\phi\in E^*$ such that $\|\phi\|\le 1$. But if $\phi\in E^*$ and $\|\phi\|\le 1$ then $\|\phi\circ f\|_{W(\mathbb{R}^+;\mathbb{C})} \le \|f\|_{W(\mathbb{R}^+;E)} < \infty$ and hence, by Widder's theorems, there is $g\in L^\infty(\mathbb{R}^+;\mathbb{C})$ such that

$$\|g\|_{L^1(\mathbb{R}^+;\mathbb{C})} = \operatorname*{ess\,sup}_{t\in\mathbb{R}^+}|g(t)| = \|\phi\circ f\|_{W(\mathbb{R}^+;\mathbb{C})}$$

and

$$(\phi \circ f)(\lambda) = \int_0^\infty e^{-\lambda t} g(t)\, dt \qquad \text{for every } \lambda \in \mathbb{R}^+.$$

Since $\phi \circ \widetilde{f}$ is holomorphic on $\mathbb{C}^+$ and $\phi \circ \widetilde{f}|\mathbb{R}^+ = \phi \circ f$, and since the Lebesgue integral $\int_0^\infty e^{-\lambda t} g(t)\, dt$ exists for every $\lambda \in \mathbb{C}^+$ and depends holomorphically on λ, it follows that

$$(\phi \circ \widetilde{f})(\lambda) = \int_0^\infty e^{-\lambda t} g(t)\, dt \qquad \text{for every } \lambda \in \mathbb{C}^+.$$

As a consequence,

$$(\phi \circ \widetilde{f})^{(k)}(\lambda) = (-1)^k \int_0^\infty t^k e^{-\lambda t} g(t)\, dt$$

and

$$|(\phi \circ \widetilde{f})^{(k)}(\lambda)| \le \|g\|_{L^\infty(\mathbb{R}^+;\mathbb{C})} \cdot \int_0^\infty t^k e^{-(\mathrm{Re}\,\lambda)t}\, dt = \|\phi \circ f\|_{W(\mathbb{R}^+;\mathbb{C})} \cdot \frac{k!}{(\mathrm{Re}\,\lambda)^{k-1}}$$

for every $\lambda \in \mathbb{C}^+$, proving (1.3). ■

The second proof of inequality (1.2). Fix $\lambda \in \mathbb{C}^+$. If $\mu \in (|\lambda|^2(2\,\mathrm{Re}\,\lambda)^{-1}, \infty)$ then $\lambda \in D_\mu$ and, for $k = 0, 1, \ldots,$

$$\widetilde{f}^{(k)}(\lambda) = \sum_{n=0}^\infty \frac{(\lambda - \mu)^n}{n!} f^{(k+n)}(\mu),$$

so that

$$\|\widetilde{f}^{(k)}(\lambda)\|_E \le \sum_{n=0}^\infty \frac{|\lambda - \mu|^n}{n!} \cdot \frac{(k+n)!}{\mu^{k+n+1}} \|f\|_{W(\mathbb{R}^+;E)}.$$

Hence inequality (1.2) is an immediate consequence of the following

LEMMA. *If $\lambda \in \mathbb{C}^+$ and $k = 0, 1, \ldots$ then the series*

$$\sum_{n=0}^\infty \frac{(k+n)!|\lambda - \mu|^n}{n!\mu^{k+n+1}} \tag{1.4}$$

converges for every real $\mu > |\lambda|^2(2\,\mathrm{Re}\,\lambda)^{-1}$ and

$$\lim_{\mu \to \infty} \sum_{n=0}^\infty \frac{(k+n)!|\lambda - \mu|^n}{n!\mu^{k+n+1}} = \frac{k!}{(\mathrm{Re}\,\lambda)^{k+1}}. \tag{1.5}$$

PROOF. Put

$$x = x(\mu) = \frac{|\lambda - \mu|}{\mu}.$$

Then

$$\lim_{n\to\infty} \sqrt[n]{\frac{(k+n)!|\lambda - \mu|^n}{n!\mu^{k+n+1}}} = x(\mu) \lim_{n\to\infty} \sqrt[n]{\frac{(n+1)(n+2)\ldots(n+k)}{\mu^{k+1}}} = x(\mu)$$

for every $\mu > 0$. If $\mu > |\lambda|^2(2\operatorname{Re}\lambda)^{-1}$ then

$$0 < x(\mu) = \sqrt{1 - \frac{2\operatorname{Re}\lambda}{\mu^2}\left(\mu - \frac{|\lambda|^2}{2\operatorname{Re}\lambda}\right)} < 1, \tag{1.6}$$

and hence the series (1.4) is convergent by the Cauchy convergence test. Furthermore, if $\mu > |\lambda|^2(2\operatorname{Re}\lambda)^{-1}$ then, as a consequence of (1.6),

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(k+n)!|\lambda-\mu|^n}{n!\mu^{k+n+1}} &= \frac{1}{\mu^{k+1}} \sum_{n=0}^{\infty} D^k[x^{n+k}] \\ &= \frac{1}{\mu^{k+1}} D^k\left[\frac{x^k}{1-x}\right] \\ &= \frac{1}{\mu^{k+1}} \sum_{l=0}^{k} \binom{k}{l} \left(D^{k-l}\frac{1}{1-x}\right) D^l[x^k] \\ &= \frac{k!}{[\mu(1-x)]^{k+1}} \sum_{l=0}^{k} \frac{(1-x)^l}{l!} D^l[x^k], \end{aligned} \tag{1.7}$$

where D stands for the derivation operator $\frac{d}{dx}$. Since

$$\lim_{\mu\to\infty} x(\mu) = 1,$$

it follows that

$$\lim_{\mu\to\infty} \sum_{l=0}^{k} \frac{(1-x)^l}{l!} D^l[x^k] = 1. \tag{1.8}$$

Furthermore,

$$\mu(1-x) = \mu - |\lambda-\mu| = \frac{2\mu\operatorname{Re}\lambda - |\lambda|^2}{\mu + |\lambda-\mu|} = \frac{2\operatorname{Re}\lambda - \frac{|\lambda|^2}{\mu}}{1+x},$$

and hence

$$\lim_{\mu\to\infty} \mu(1-x) = \operatorname{Re}\lambda. \tag{1.9}$$

Now, equality (1.5) follows from (1.7), (1.8) and (1.9). ∎

2. Representation theorems for elements of Widder spaces

Consider the Banach space $L^1(\mathbb{R}^+;\mathbb{K})$ of (the equivalence classes of) $\mathbb{K}$-valued functions Lebesgue integrable on $\mathbb{R}^+$. The norm of an element φ of $L^1(\mathbb{R}^+;\mathbb{K})$ is

$$\|\varphi\|_{L^1(\mathbb{R}^+;\mathbb{K})} = \int_0^\infty |\varphi(\xi)|\,d\xi.$$

For every $t \in \mathbb{R}^+$ the characteristic function $1_{(0,t]}$ of the interval $(0,t]$ is an element of $L^1(\mathbb{R}^+;\mathbb{R})$. For every $\lambda \in \mathbb{K}^+$ the exponential function ϕ_λ such that

$$\phi_\lambda(\xi) = e^{-\lambda\xi} \quad \text{for } \xi \in \mathbb{R}^+$$

is an element of $L^1(\mathbb{R}^+;\mathbb{K})$. Furthermore,

the map $\phi_\bullet : \lambda \to \phi_\lambda$ *belongs to* $W(\mathbb{K}^+; L^1(\mathbb{R}^+;\mathbb{K}))$ *and* $\|\phi_\bullet\|_{W(\mathbb{K}^+;L^1(\mathbb{R}^+;\mathbb{K}))} = 1$.

It is sufficient to prove the above claim in the case of $\mathbb{K} = \mathbb{C}$. To this end, observe that, for every $\xi \in \mathbb{R}^+$, $\lambda \in \mathbb{C}^+$ and $h \in \mathbb{C} \setminus \{0\}$ such that $|h| < \operatorname{Re}\lambda$, one has

$$\begin{aligned} |h^{-1}(e^{-(\lambda+h)\xi} - e^{-\lambda\xi}) + \xi e^{-\lambda\xi}| &= |h|\xi^2 \left| \sum_{k=2}^{\infty} \frac{(-h\xi)^{k-2}}{k!} \right| e^{-(\operatorname{Re}\lambda)\xi} \\ &\le \frac{1}{2}|h|\xi^2 e^{(|h|-\operatorname{Re}\lambda)\xi}, \end{aligned}$$

whence

$$\lim_{\mathbb{C}\setminus\{0\}\ni h\to 0} \int_0^\infty |h^{-1}[\phi_{\lambda+h} - \phi_\lambda(\xi)] + \xi\phi_\lambda(\xi)|\, d\xi = 0$$

for every $\lambda \in \mathbb{C}^+$, by the Lebesgue dominated convergence theorem. Hence for every $\lambda \in \mathbb{C}^+$ the complex derivative $\frac{d}{d\lambda}\phi_\lambda$ exists in the sense of the norm topology of $L^1(\mathbb{R}^+;\mathbb{C})$ and $\left[\frac{d}{d\lambda}\phi_\lambda\right](\xi) = -\xi\phi_\lambda(\xi)$ for every $\lambda \in \mathbb{C}^+$ and $\xi \in \mathbb{R}^+$. It follows that $\phi_\bullet$ is an $L^1(\mathbb{R}^+;\mathbb{C})$-valued function holomorphic on $\mathbb{C}^+$ with derivatives satisfying $\left[\left(\frac{d}{d\lambda}\right)^k\phi_\lambda\right](\xi) = (-\xi)^k\phi_\lambda(\xi)$. As a consequence,

$$\left\| \left(\frac{d}{d\lambda}\right)^k \phi_\lambda \right\|_{L^1(\mathbb{R}^+;\mathbb{C})} = \int_0^\infty \xi^k e^{-(\operatorname{Re}\lambda)\xi}\, d\xi = \frac{k!}{(\operatorname{Re}\lambda)^{k+1}},$$

so that $\phi_\bullet$ belongs to the Widder space $W(\mathbb{C}^+; L^1(\mathbb{R}^+;\mathbb{C}))$ and $\|\phi_\bullet\|_{W(\mathbb{C};L^1(\mathbb{R}^+;\mathbb{C}))} = 1$.

2.1. Lemma. *The set* $\{\phi_\lambda : \lambda \in \mathbb{R}^+\}$ *is* $\mathbb{K}$*-linearly dense in* $L^1(\mathbb{R}^+;\mathbb{K})$.

Proof. Because $\operatorname{span}_{\mathbb{K}}\{1_{(0,t]} : t \in \mathbb{R}^+\}$ consists of all the $\mathbb{K}$-valued, left-continuous, piecewise constant functions on $\mathbb{R}^+$ with bounded supports, it follows that $\overline{\operatorname{span}}_{\mathbb{K}}\{1_{(0,t]} : t \in \mathbb{R}^+\} = L^1(\mathbb{R}^+;\mathbb{K})$, the closure being taken in the norm topology of $L^1(\mathbb{R}^+;\mathbb{K})$. Hence Lemma 2.1 will follow once it is shown that $1_{(0,t]} \in \overline{\operatorname{span}}_{\mathbb{R}}\{\phi_\lambda : \lambda \in \mathbb{R}^+\}$ for every $t \in \mathbb{R}^+$. To this end, fix any $t \in \mathbb{R}^+$. Since

$$\left\| \frac{e^{knt}}{k!}\phi_{kn} \right\|_{L^1(\mathbb{R}^+;\mathbb{R})} = \frac{e^{knt}}{k!kn}$$

for every $n = 1, 2, \ldots$, the series

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{e^{knt}}{k!} \phi_{kn}$$

is absolutely convergent in the norm of $L^1(\mathbb{R}^+;\mathbb{R})$, and hence its sum p_n is in $\overline{\operatorname{span}}_{\mathbb{R}}\{\phi_\lambda : \lambda \in \mathbb{R}^+\}$. Therefore in order to prove that $1_{(0,t]} \in \overline{\operatorname{span}}_{\mathbb{R}}\{\phi_\lambda : \lambda \in \mathbb{R}^+\}$ it is sufficient to show that

$$\lim_{n\to\infty} \|p_n - 1_{(0,t]}\|_{L^1(\mathbb{R}^+;\mathbb{R})} = 0. \tag{2.1}$$

Since

$$p_n(\xi) = 1 - \exp(-e^{n(t-\xi)}),$$

it follows that $0 \le p_n(\xi) < 1$ for every $\xi \in \mathbb{R}^+$ and

$$p_n(\xi) = \exp(0) - \exp(-e^{n(t-\xi)}) = \int_{-e^{n(t-\xi)}}^{0} \exp(u)\, du < e^{n(t-\xi)} < e^{t-\xi}$$

for every $\xi \in (t, \infty)$. Hence

$$0 \le p_n(\xi) < \min(1, e^{t-\xi}) \tag{2.2}$$

for every $n = 1, 2, \ldots$ and $\xi \in \mathbb{R}^+$. Furthermore,

$$\lim_{n\to\infty} p_n(\xi) = 1_{(0,t]}(\xi) \quad \text{for every } \xi \in \mathbb{R}^+ \setminus \{t\}. \tag{2.3}$$

Equality (2.1) follows from (2.2) and (2.3) by the Lebesgue dominated convergence theorem. ∎

Remark. The above proof was inspired by the comments on p. 165 of [H-N] concerning the Phragmén real inversion formula for the Laplace–Stieltjes transform. Equality (2.1) is stated there without proof. See also [Y;2], p. 166, Lemma 1.

2.2. Theorem. *Suppose that f is a function defined on $\mathbb{K}^+$ and taking values in E. Let $M \in \mathbb{R}^+$. Then the following three conditions are equivalent*:

(i) *$f \in W(\mathbb{K}^+; E)$ and $\|f\|_{W(\mathbb{K}^+;E)} \le M$;*

(ii) *$\|\sum_{i=1}^{j} c_i f(\lambda_i)\|_E \le M \|\sum_{i=1}^{j} c_i \phi_{\lambda_i}\|_{L^1(\mathbb{R}^+;\mathbb{K})}$ whenever $j = 1, 2, \ldots$, $(c_1, \ldots, c_j) \in \mathbb{K}^j$ and $(\lambda_1, \ldots, \lambda_j) \in (\mathbb{K}^+)^j$;*

(iii) *there exists an operator $T \in L(L^1(\mathbb{R}^+;\mathbb{K}); E)$ such that $\|T\| \le M$ and $T(\phi_\lambda) = f(\lambda)$ for every $\lambda \in \mathbb{K}^+$.*

Remarks. The equivalence (i)⇔(iii) shows that the map $T \to T(\phi_\bullet)$ is an isometric isomorphism of $L(L^1(\mathbb{R}^+;\mathbb{K}); E)$ onto $W(\mathbb{K}^+; E)$. An E-valued function f defined on $\mathbb{K}^+$ belongs to $W(\mathbb{K}^+; E)$ if and only if it may be represented in the form $f = T(\phi_\bullet)$, where $T \in L(L^1(\mathbb{R}^+;\mathbb{K}); E)$ and $\phi_\bullet$ is the "canonical" element of $W(\mathbb{K}^+; L^1(\mathbb{R}^+;\mathbb{K}))$ discussed above. The importance of such a representation of elements of the Widder spaces was emphasised by B. Hennig and F. Neubrander in [H-N], where the equivalence (i)⇔(iii) is proved in the case of $\mathbb{K} = \mathbb{R}$. See [H-N], Section 2, pp. 156–162, in particular Lemma 2.3 and Theorem 2.5. The implication (i)⇒(iii) is established in [H-N] (in the proof of Theorem 2.5, p. 160) by means of an argument similar to one in Widder's original proof of his Theorems 16a and 16b in Chapter VII of [W], pp. 315–316. This argument is based on Widder's "general representation theorem", i.e. Theorem 11a in Chapter VII of [W], p. 303, which is related to the Post–Widder real inversion formula for the Laplace transform. A similar but easier proof of the implication (i)⇒(iii) in the case of $\mathbb{K} = \mathbb{R}$ was presented by A. Bobrowski in [B]. His proof involves another "representation theorem", related to the R. S. Phillips real inversion formula for the Laplace transform (see [Ph]; [H-P], p. 223).

Proof of Theorem 2.2. In the scheme (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i) the proofs of (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i) are routine and obvious. Therefore only the proof of (i)$\Rightarrow$(ii) will be presented. Fix $j = 1, 2, \ldots$, $(c_1, \ldots, c_j) \in \mathbb{K}^j$ and $(\lambda_1, \ldots, \lambda_j) \in (\mathbb{K}^+)^j$, and define

$$g = \sum_{i=1}^{j} c_i \phi_{\lambda_i}.$$

By Proposition 1.1(A), for f in $W(\mathbb{K}^+; E)$ one has

$$f(n - ne^{-\lambda/n}) = \sum_{k=0}^{\infty} e^{-\lambda k/n} \frac{(-n)^k}{k!} f^{(k)}(n)$$

for every $n = 1, 2, \ldots$ and $\lambda \in \mathbb{K}^+$, so that

$$\sum_{i=1}^{j} c_i f(n - ne^{-\lambda_i/n}) = \sum_{k=0}^{\infty} g\left(\frac{k}{n}\right) \frac{(-n)^k}{k!} f^{(k)}(n).$$

Hence condition (i) implies that

$$\Big\| \sum_{i=1}^{j} c_i f(n - ne^{-\lambda_i/n}) \Big\|_E \le M \frac{1}{n} \sum_{k=0}^{\infty} \left| g\left(\frac{k}{n}\right) \right|$$

for every $n = 1, 2, \ldots$ Condition (ii) follows from this inequality by passing to the limit as $n \to \infty$. Indeed, $\lim_{n\to\infty}(n - ne^{-\lambda/n}) = \lambda$ for every $\lambda \in \mathbb{C}$, so that

$$\lim_{n\to\infty} \Big\| \sum_{i=1}^{j} c_i f(n - ne^{-\lambda_i/n}) \Big\|_E = \Big\| \sum_{i=1}^{j} c_i f(\lambda_i) \Big\|_E.$$

Furthermore,

$$\begin{aligned}
\left| \frac{1}{n} \sum_{k=0}^{\infty} \left| g\left(\frac{k}{n}\right) \right| - \Big\| \sum_{i=1}^{j} c_i \phi_{\lambda_i} \Big\|_{L^1(\mathbb{R}^+;\mathbb{K})} \right| &= \left| \int_0^{\infty} \left[\left| g\left(\frac{1}{n}[n\xi]\right) \right| - |g(\xi)| \right] d\xi \right| \\
&\le \sum_{i=1}^{j} |c_i| \int_0^{\infty} |e^{\lambda_i(n\xi - [n\xi])/n} - 1| e^{-(\mathrm{Re}\,\lambda_i)\xi}\, d\xi \\
&\le \sum_{i=1}^{j} \frac{|c_i|}{\mathrm{Re}\,\lambda_i} \sup_{0 \le \theta < 1} |e^{\lambda_i \theta/n} - 1|,
\end{aligned}$$

so that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \left| g\left(\frac{k}{n}\right) \right| = \Big\| \sum_{i=1}^{j} c_i \phi_{\lambda_i} \Big\|_{L^1(\mathbb{R}^+;\mathbb{K})}. \blacksquare$$

2.3. Corollary (D. V. Widder [W], pp. 315–316, Theorems 16a and 16b). *Let f be a function defined on $\mathbb{R}^+$ and taking values in $\mathbb{K}$. Then $f \in W(\mathbb{R}^+; \mathbb{K})$ if and only* *if there is a $\mathbb{K}$-valued function g Lebesgue measurable and essentially bounded on $\mathbb{R}^+$, such that*

$$f(\lambda) = \int_0^\infty e^{-\lambda\xi} g(\xi)\, d\xi \quad \textit{for every } \lambda \in \mathbb{R}^+.$$

Moreover, if $f \in W(\mathbb{R}^+; \mathbb{K})$ and g is as above, then $\operatorname{ess\,sup}_{\xi\in\mathbb{R}^+} |g(\xi)| = \|f\|_{W(\mathbb{R}^+;\mathbb{K})}$.

Proof. The space dual to $L^1(\mathbb{R}^+; \mathbb{K})$, i.e. the space $L(L^1(\mathbb{R}^+; \mathbb{K}); \mathbb{K})$ of continuous linear functionals T on $L^1(\mathbb{R}^+; \mathbb{K})$ is isometrically isomorphic to the space $L^\infty(\mathbb{R}^+; \mathbb{K})$ of (the equivalence classes of) $\mathbb{K}$-valued functions g Lebesgue measurable and essentially bounded on $\mathbb{R}^+$, equipped with the norm $\|g\|_{L^\infty(\mathbb{R}^+;\mathbb{K})} = \operatorname{ess\,sup}_{\xi\in\mathbb{R}^+} |g(\xi)|$. The isomorphism is determined by the equality

$$T(\varphi) = \int_0^\infty \varphi(\xi) g(\xi)\, d\xi \quad \text{for every } \varphi \in L^1(\mathbb{R}^+; \mathbb{K}).$$

See [Y;2], p. 115, Example 3. According to Theorem 2.2, $f \in W(\mathbb{R}^+; \mathbb{K})$ if and only if there is a linear functional $T \in L(L^1(\mathbb{R}^+; \mathbb{K}); \mathbb{K})$ such that $\|T\| = \|f\|_{W(\mathbb{R}^+;\mathbb{K})}$ and $f(\lambda) = T(\phi_\lambda)$ for every $\lambda \in \mathbb{R}^+$. Hence $f \in W(\mathbb{R}^+; \mathbb{K})$ if and only if there is $g \in L^\infty(\mathbb{R}^+; \mathbb{K})$ such that $\operatorname{ess\,sup}_{\xi\in\mathbb{R}^+} |g(\xi)| = \|f\|_{W(\mathbb{R}^+;\mathbb{K})}$ and $f(\lambda) = \int_0^\infty \phi_\lambda(\xi) g(\xi)\, d\xi = \int_0^\infty e^{-\lambda\xi} g(\xi)\, d\xi$ for every $\lambda \in \mathbb{R}^+$. ■

2.4. Corollary (W. Arendt [A], p. 329, Theorem 1.1; B. Hennig and F. Neubrander [H-N], p. 159, Theorem 2.5). *Let f be a function on $\mathbb{R}^+$ taking values in E, and let $M \in \mathbb{R}^+$. Then*

(a) $f \in W(\mathbb{R}^+; \mathbb{K})$ *and* $\|f\|_{W(\mathbb{R}^+;E)} \le M$

if and only if there is a function g defined on $[0, \infty)$ and taking values in E such that

(b) $g(0) = 0$ *and* $\|g(\xi_1) - g(\xi_2)\|_E \le M|\xi_1 - \xi_2|$ *for every ξ_1 and ξ_2 in $\overline{\mathbb{R}^+}$, and*

(c) $f(\lambda) = \lambda \int_0^\infty e^{-\lambda\xi} g(\xi)\, d\xi$ *for every $\lambda \in \mathbb{R}^+$.*

Proof. By Theorem 2.2, condition (a) is equivalent to the existence of a linear operator $T \in L(L^1(\mathbb{R}^+; \mathbb{R}); E)$ such that $\|T\| \le M$ and

(d) $f(\lambda) = T(\phi_\lambda)$ for every $\lambda \in \mathbb{R}^+$.

According to Lemma 2.3 in [H-N], p. 158, there is one-to-one correspondence between functions g satisfying (b) and operators $T \in L(L^1(\mathbb{R}^+; \mathbb{R}); E)$ such that $\|T\| \le M$. This correspondence is determined by the formulas

$$g(\xi) = T(1_{(0,\xi]})$$

for every $\xi \in \mathbb{R}^+$, and

$$T(\varphi) = \int_0^\infty \varphi(\xi)\, dg(\xi) = -\int_0^\infty \varphi'(\xi) g(\xi)\, d\xi$$

for every $\varphi \in L^1(\mathbb{R}^+;\mathbb{R})$ such that $\varphi' \in L^1(\mathbb{R}^+;\mathbb{R})$. The last equality shows that (c) is equivalent to (d). ∎

REMARKS. The above proof of Corollary 2.4 coincides with the proof of Theorem 2.5 in [H-N], p. 160. Arendt's earlier proof consists in applying linear functionals and deducing the result from Widder's theorem. Notice that in Corollary 2.4 formula (c) may be replaced by

$$\text{(c}'\text{)} \qquad f(\lambda) = \int_0^\infty e^{-\lambda\xi} g'(\xi)\, d\xi \quad \text{for every } \lambda \in \mathbb{R}^+$$

only in the case when the Banach space E has the Radon–Nikodym property, and thus, in particular, if the Banach space E is reflexive. See [A], p. 331, Theorem 1.4. The "canonical" element $\phi_\bullet$ of $W(\mathbb{R}^+;L^1(\mathbb{R}^+;\mathbb{R}))$ admits a representation (c) with $g(\xi) = 1_{(0,\xi]}$, i.e.

$$\phi_\lambda = \lambda \int_0^\infty e^{-\lambda\xi} 1_{(0,\xi]}\, d\xi \quad \text{for every } \lambda \in \mathbb{R}^+.$$

The uniformly lipschitzian map $\mathbb{R}^+ \ni \xi \to 1_{(0,\xi]} \in L^1(\mathbb{R}^+;\mathbb{R})$ is nowhere differentiable in the sense of the norm topology of $L^1(\mathbb{R}^+;\mathbb{R})$. Furthermore, it is impossible to represent $f = \phi_\bullet$ in the form (c$'$), with a map $\mathbb{R}^+ \ni \xi \to g'(\xi,\bullet) \in L^1(\mathbb{R}^+;\mathbb{R})$ weakly measurable and weakly essentially bounded on $\mathbb{R}^+$, and with integral in the sense of Pettis (see [D-U], p. 53). Indeed, by Lemma 2.1, such a representation would lead to the equality $\int_0^\infty \varphi(\xi)\psi(\xi)\, d\xi = \int_0^\infty \varphi(\xi)[\int_0^\infty g'(\xi,\eta)\psi(\eta)\, d\eta]\, d\xi$ for every $\varphi \in L^1(\mathbb{R}^+;\mathbb{R})$ and $\psi \in L^\infty(\mathbb{R}^+;\mathbb{R})$, and hence to the conclusion that for every $n = 1,2,\ldots$ the equality $\cos(n\xi) = \int_0^\infty g'(\xi,\eta)\cos(n\eta)\, d\eta$ holds for almost every $\xi \in \mathbb{R}^+$, in the sense of the Lebesgue measure. But the sequence $\cos(n\bullet)$, $n = 1,2,\ldots$, of elements of $L^\infty(\mathbb{R}^+;\mathbb{R})$ converges $*$-weakly to zero, so that the last equality would imply that $\lim_{n\to\infty}\cos(n\xi) = 0$ for almost every $\xi \in \mathbb{R}^+$. However, this contradicts the Egoroff theorem ([Y;2], p. 16), because for every $k = 1,2,\ldots$ and $n = 1,2,\ldots$ the set $\left\{x \in [k\pi,(k+1)\pi] : |\cos(nx)| \geq \frac{1}{2}\right\}$ has Lebesgue measure $\frac{2}{3}\pi$.

3. Pseudoresolvents belonging to Widder spaces as homomorphic images of a canonical pseudoresolvent

Let A be a Banach algebra over the field $\mathbb{K}$. By a *pseudoresolvent* with values in A defined on $\mathbb{K}^+$ we mean any map $r : \mathbb{K}^+ \to A$ satisfying the *resolvent equation*

$$\text{(3.1)} \qquad r(\lambda) - r(\mu) = (\mu - \lambda) r(\lambda) r(\mu)$$

for every λ and μ in $\mathbb{K}^+$. See [D-S;II], Sec. IX.1; [D-M;C], Sec. XII.5; [Y;2], Sec. VII.4; and Appendix I of the present paper.

EXAMPLE (the canonical pseudoresolvent). The Banach space $L^1(\mathbb{R}^+;\mathbb{K})$ becomes a commutative Banach algebra over the field $\mathbb{K}$ when the product of any two elements

φ and ψ of $L^1(\mathbb{R}^+;\mathbb{K})$ is defined as the convolution $\varphi * \psi$, so that

$$(\varphi * \psi)(\xi) = \int_0^\xi \varphi(\xi-\eta)\psi(\eta)\,d\eta = \int_0^\xi \psi(\xi-\eta)\varphi(\eta)\,d\eta$$

for $\xi \in \mathbb{R}^+$. See [P], Sec. 5.1.10; [Y;2], Sec. VI.5. The "canonical" element $\phi_\bullet$ of the Widder space $W(\mathbb{K}^+; L^1(\mathbb{R}^+;\mathbb{K}))$ discussed in Section 2 is a pseudoresolvent defined on $\mathbb{K}^+$ and taking values in the convolution Banach algebra $L^1(\mathbb{R}^+;\mathbb{K})$. Indeed, if $\lambda \in \mathbb{K}^+$, $\mu \in \mathbb{K}^+$ and $\lambda \neq \mu$, then

$$[\phi_\lambda * \phi_\mu](\xi) = e^{-\lambda\xi}\int_0^\xi e^{(\lambda-\mu)\eta}\,d\eta = \frac{1}{\mu-\lambda}(e^{-\lambda\xi} - e^{\mu\xi}) = \frac{1}{\mu-\lambda}[\phi_\lambda - \phi_\mu](\xi)$$

for $\xi \in \mathbb{R}^+$. See [D-M;C], p. 223.

3.1. Lemma. *Let $r : \mathbb{K}^+ \to A$ be a pseudoresolvent with values in a Banach algebra A over the field $\mathbb{K}$. If $\liminf_{\lambda\in\mathbb{K}^+,\,|\lambda|\to\infty} \|\lambda r(\lambda)\|_A < 1$ then r vanishes identically on $\mathbb{K}^+$.*

Proof. Suppose that $\liminf_{\lambda\in\mathbb{K}^+,\,|\lambda|\to\infty} \|\lambda r(\lambda)\|_A = \Theta < 1$. Then, by the resolvent equation (3.1), for every $\mu \in \mathbb{K}^+$ and $\lambda \in \mathbb{K}^+$ one has

$$\|r(\mu)\|_A = \|r(\lambda) + (\lambda-\mu)r(\lambda)r(\mu)\|_A \leq \|\lambda r(\lambda)\|_A\left(\frac{1}{|\lambda|} + \frac{|\lambda-\mu|}{|\lambda|}\|r(\mu)\|_A\right),$$

so that $\|r(\mu)\|_A \leq \Theta\|r(\mu)\|_A$ and hence $r(\mu) = 0$. ∎

3.2. Theorem. *Let A be a Banach algebra over the field $\mathbb{K}$ and let $r : \mathbb{K}^+ \to A$ be a pseudoresolvent. Then*

$$\|r\|_{W(\mathbb{K}^+;A)} = \sup\{(\operatorname{Re}\lambda)^k \|[r(\lambda)]^k\|_A : \lambda \in \mathbb{K}^+,\ k = 1,2,\ldots\}, \tag{3.2}$$

the sides of this equality being either both finite or both equal to ∞. Furthermore, for every $M \in [0,\infty)$ the following two conditions are equivalent:

(I) *$r \in W(\mathbb{K}^+;A)$ and $\|r\|_{W(\mathbb{K}^+;A)} \leq M$;*

(II) *there is a unique homomorphism of Banach algebras $T : L^1(\mathbb{R}^+;\mathbb{K}) \to A$ such that $\|T\| \leq M$ and $T(\phi_\lambda) = r(\lambda)$ for every $\lambda \in \mathbb{K}^+$.*

Proof. As a consequence of the resolvent equation (3.1), for every $\lambda \in \mathbb{K}^+$ and $k = 0,1,\ldots$ one has

$$r^{(k)}(\lambda) = (-1)^k k![r(\lambda)]^{k+1}$$

whence (3.2) follows. The implication (II)⇒(I) follows from the equivalence (i)⇔(iii) of Theorem 2.2. By the same equivalence, if (I) holds then there is a unique linear operator $T \in L(L^1(\mathbb{R}^+;\mathbb{K});A)$ such that $\|T\| \leq M$ and $T(\phi_\lambda) = r(\lambda)$ for every $\lambda \in \mathbb{K}$. This T is a homomorphism of a Banach algebras, that is,

$$T(\varphi * \psi) = T(\varphi)T(\psi) \tag{3.3}$$

for every φ and ψ in $L^1(\mathbb{R}^+;\mathbb{K})$. Indeed, since according to Lemma 2.1 the set $\{\phi_\lambda : \lambda \in \mathbb{K}^+\}$ is $\mathbb{K}$-linearly dense in $L^1(\mathbb{R}^+;\mathbb{K})$, (3.3) will follow if we check that

$$T(\phi_\lambda * \phi_\mu) = T(\phi_\lambda)T(\phi_\mu) \tag{3.4}$$

for every $\lambda \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$. By continuity, one may assume that $\lambda \neq \mu$. But then

$$T(\phi_\lambda * \phi_\mu) = T\left(\frac{1}{\mu-\lambda}[\phi_\lambda - \phi_\mu]\right) = \frac{1}{\mu-\lambda}[r(\lambda) - r(\mu)] = r(\lambda)r(\mu) = T(\phi_\lambda)T(\phi_\mu). \ \blacksquare$$

3.3. Remark. If r is a resolvent such that $\|r\|_{W(\mathbb{K}^+;A)} < 1$, then $r \equiv 0$ on $\mathbb{K}^+$ by Lemma 3.1. If $T : L^1(\mathbb{R}^+;\mathbb{K}) \to A$ is a homomorphism such that $\|T\| < 1$, then $T = 0$. Indeed, for any $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ one has $\lim_{\lambda\to\infty} \|\lambda\phi_\lambda * \varphi - \varphi\|_{L^1(\mathbb{R}^+;\mathbb{K})} = 0$ and hence $\|T(\varphi)\|_A = \lim_{\lambda\to\infty} \|T(\lambda\phi_\lambda)T(\varphi)\|_A \le \|T\|\|T(\varphi)\|_A$ because $\|\lambda\phi_\lambda\|_{L^1(\mathbb{R}^+;\mathbb{K})} = 1$ for every $\lambda \in \mathbb{R}^+$. But the inequality $\|T(\varphi)\|_A \le \|T\|\|T(\varphi)\|_A$ with $\|T\| < 1$ implies that $T(\varphi) = 0$. Thus if $M \in [0,1)$ then the equivalence (I)$\Leftrightarrow$(II) in Theorem 3.2 is trivial.

3.4. Remark (the Yosida approximation of a homomorphism T). The implication (I)$\Rightarrow$(II) in Theorem 3.2 may be proved by the following direct argument which is an adaptation of the proof of Theorem 1, p. 286, from A. Bobrowski's paper [B] to the case of a pseudoresolvent. Let $\widetilde{A}$ be a unital Banach algebra containing A as a subalgebra, such that $\|a\|_{\widetilde{A}} = \|a\|_A$ for every $a \in A$. (For instance $\widetilde{A} = A$ if A is unital, $\widetilde{A} = \{$the unitization of $A\}$ if A is non-unital. See [P], pp. 19–20.) Denote by ε the multiplicative unit of $\widetilde{A}$. Suppose that condition (I) is satisfied. Following K. Yosida's proof of the Hille–Yosida generation theorem ([Y;1]; [Y;2], pp. 246–248), for every $\mu \in \mathbb{R}^+$ define the element $a_\mu = \mu^2 r(\mu) - \mu\varepsilon$ of $\widetilde{A}$, and consider the exponential map

$$\overline{\mathbb{R}^+} \ni t \to \exp(ta_\mu) = \varepsilon + \sum_{n=1}^{\infty} \frac{(ta_\mu)^n}{n!} \in \widetilde{A}.$$

Then

$$\exp(ta_\mu) = e^{-\mu t}\left[\varepsilon + \sum_{n=1}^{\infty} \frac{(\mu t)^n}{n!}(\mu r(\mu))^n\right],$$

and hence condition (I) implies that

$$\|\exp(ta_\mu)\|_{\widetilde{A}} \le M. \tag{3.5}$$

As a consequence, for every $\mu \in \mathbb{R}^+$ there is a linear operator $T_{a_\mu} \in L(L^1(\mathbb{R}^+;\mathbb{K});\widetilde{A})$ such that

$$T_{a_\mu}(\varphi) = \int_0^\infty \varphi(t)\exp(ta_\mu)\,dt$$

for every $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$, with integral in the sense of Bochner ([D-U], pp. 44–52; [H-P], pp. 76–89; [Y;2], pp. 132–136). Inequality (3.5) implies that

$$\|T_{a_\mu}\|_{L(L^1(\mathbb{R}^+;\mathbb{K});\widetilde{A})} \le M. \tag{3.6}$$

If $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $\psi \in L^1(\mathbb{R}^+;\mathbb{K})$, then

$$\begin{aligned} T_{a_\mu}(\varphi * \psi) &= \int_0^\infty \Big(\int_0^t \varphi(t-u)\psi(u)\,du\Big)\exp(ta_\mu)\,dt \\ &= \iint\limits_{\substack{0\le u<\infty \\ 0\le v<\infty}} \varphi(v)\psi(u)\exp((v+u)a_\mu)\,dv\,du \\ &= \iint\limits_{\substack{0\le u<\infty \\ 0\le v<\infty}} \varphi(v)\psi(u)\exp(va_\mu)\exp(ua_\mu)\,dv\,du = T_{a_\mu}(\varphi)T_{a_\mu}(\psi), \end{aligned}$$

so that $T_{a_\mu} : L^1(\mathbb{R}^+;\mathbb{K}) \to \widetilde{A}$ is a homomorphism of Banach algebras. For every $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{R}^+$ one has

$$\begin{aligned} T_{a_\mu}(\phi_\lambda)(\lambda\varepsilon - a_\mu) = (\lambda\varepsilon - a_\mu)T_{a_\mu}(\phi_\lambda) &= \int_0^\infty e^{-\lambda t}(\lambda\varepsilon - a_\mu)\exp(ta_\mu)\,dt \\ &= -\int_0^\infty \frac{d}{dt}[e^{-\lambda t}\exp(ta_\mu)]\,dt = \varepsilon, \end{aligned}$$

so that λ belongs to the resolvent set of a_μ and $(\lambda\varepsilon - a_\mu)^{-1} = T_{a_\mu}(\phi_\lambda)$. Furthermore,

$$\frac{\lambda\mu}{\lambda+\mu} = \frac{\lambda\mu^2 + |\lambda|^2\mu}{|\lambda+\mu|^2} \in \mathbb{K}^+$$

for every $\lambda \in \mathbb{K}^+$ and $\mu \in \mathbb{R}^+$, and hence

$$\begin{aligned} T_{a_\mu}(\phi_\lambda) = (\lambda\varepsilon - a_\mu)^{-1} &= \frac{1}{\lambda+\mu}\left[\varepsilon - \left(\mu - \frac{\lambda\mu}{\lambda+\mu}\right)r(\mu)\right]^{-1} \\ &= \frac{1}{\lambda+\mu}\left[\varepsilon + \left(\mu - \frac{\lambda\mu}{\lambda+\mu}\right)r\left(\frac{\lambda\mu}{\lambda+\mu}\right)\right] \\ &= \frac{1}{\lambda+\mu}\varepsilon + \left(\frac{\mu}{\lambda+\mu}\right)^2 r\left(\frac{\lambda\mu}{\lambda+\mu}\right), \end{aligned}$$

where the third equality follows from the resolvent equation (3.1). (See [D-M; C], p. 312, formula (4.2).) Hence

$$\lim_{\mu\to\infty} T_{a_\mu}(\phi_\lambda) = r(\lambda) \tag{3.7}$$

for every $\lambda \in \mathbb{K}^+$. From (3.6), (3.7) and Lemma 2.1 it follows that as $\mu \to \infty$ the homomorphisms T_{a_μ} converge pointwise on $L^1(\mathbb{R}^+;\mathbb{K})$ to a homomorphism $T : L^1(\mathbb{R}^+;\mathbb{K}) \to \widetilde{A}$ such that $\|T\| \le M$ and $T(\phi_\lambda) = r(\lambda)$ for every $\lambda \in \mathbb{K}^+$. As a consequence, $T(\phi_\lambda) \in A$ for every $\lambda \in \mathbb{K}^+$ and hence, by Lemma 2.1, $T(\varphi) \in A$ for every $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$, so that T is a homomorphism of the convolution Banach algebra $L^1(\mathbb{R}^+;\mathbb{K})$ into the Banach algebra A.

4. Representations of the convolution algebra $L^1(\mathbb{R}^+;\mathbb{K})$ and the associated one parameter semigroups of operators

4.1. Right translations in $L^1(\mathbb{R}^+;\mathbb{K})$ and convolutions. For every $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $t \in \overline{\mathbb{R}^+}$ define the right translate of φ by t as the element $\varphi_t \in L^1(\mathbb{R}^+;\mathbb{K})$ such that

$$\varphi_t(\xi) = \begin{cases} 0 & \text{if } \xi \in (0,t], \\ \varphi(\xi - t) & \text{if } \xi \in (t,\infty). \end{cases}$$

For every $t \in \overline{\mathbb{R}^+}$ the operator of right translation by t, i.e. the operator $U_t : L^1(\mathbb{R}^+;\mathbb{K}) \ni \varphi \to \varphi_t \in L^1(\mathbb{R}^+;\mathbb{K})$, is an isometry of $L^1(\mathbb{R}^+;\mathbb{K})$ into itself, and the operator family $(U_t)_{t\in\overline{\mathbb{R}^+}} \subset L(L^1(\mathbb{R}^+;\mathbb{K}))$ is a one parameter semigroup of class C^0 ([H-P], p. 321; [Y;2], p. 232).

If φ and ψ belong to $L^1(\mathbb{R}^+;\mathbb{K})$ then the $L^1(\mathbb{R}^+;\mathbb{K})$-valued function $t \to \varphi(t)U_t\psi$ is Bochner integrable on $\mathbb{R}^+$ ([D-U], pp. 44–52; [H-P], pp. 76–89; [Y;2], pp. 132–136) and

$$\int_0^\infty \varphi(t)U_t\psi\,dt = \varphi * \psi. \tag{4.1}$$

It follows that if $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $t \in \mathbb{R}^+$ then

$$\int_0^t U_s\varphi\,ds = \int_0^\infty 1_{(0,t]}(s)U_s\varphi\,ds = 1_{(0,t]} * \varphi. \tag{4.2}$$

Since the function $s \to U_s\varphi$ is continuous from $\overline{\mathbb{R}^+}$ to $L^1(\mathbb{R}^+;\mathbb{K})$ with the norm topology, it follows from (4.2) that

$$U_t\varphi = \frac{d}{dt}[1_{(0,1]} * \varphi] \tag{4.3}$$

for every $t \in \mathbb{R}^+$ and $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$, the derivative being computed in the norm of $L^1(\mathbb{R}^+;\mathbb{K})$.

For every $t \in \overline{\mathbb{R}^+}$ one has

$$\|\lambda\phi_\lambda\|_{L^1(\mathbb{R}^+;\mathbb{K})} = 1. \tag{4.4}$$

If $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $\omega(t) = \|U_t\varphi - \varphi\|_{L^1(\mathbb{R}^+;\mathbb{K})}$, then ω is bounded and continuous on $\overline{\mathbb{R}^+}$, and $\omega(0) = 0$. From (4.1) and (4.4) it follows that

$$\begin{aligned}\|\lambda\phi_\lambda * \varphi - \varphi\|_{L^1(\mathbb{R}^+;\mathbb{K})} &= \left\| \int_0^\infty \lambda e^{-\lambda t}(U_t\varphi - \varphi)\,dt \right\| \le \lambda \int_0^\infty e^{-\lambda t}\omega(t)\,dt \\ &\le \max_{0\le t\le\delta} \omega(t) + 2\|\varphi\|_{L^1(\mathbb{R}^+;\mathbb{K})}\, e^{-\lambda\delta}\end{aligned}$$

for every $\lambda \in \mathbb{R}^+$, $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $\delta \in \mathbb{R}^+$, whence

$$\lim_{\lambda\to\infty} \|\lambda\phi_\lambda * \varphi - \varphi\|_{L^1(\mathbb{R}^+;\mathbb{K})} = 0 \tag{4.5}$$

for every $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$.

Equalities (4.4) and (4.5) mean that the net $(\lambda\phi_\lambda)_{\lambda\in\mathbb{R}^+}$ (equipped with the usual order) is a *bounded approximate unit* ([H-R;II], p. 87; [P], p. 520) in the convolution Banach algebra $L^1(\mathbb{R}^+;\mathbb{K})$. Thus, if T is a continuous representation of the convolution algebra $L^1(\mathbb{R}^+;\mathbb{K})$ on a Banach space X over the field $\mathbb{K}$, then

(4.6) the set $Y = \{T(\varphi)x : \varphi \in L^1(\mathbb{R}^+;\mathbb{K}),\ x \in X\}$ is a closed linear subspace of X

by the *factorization theorem for representations of Banach algebras* ([H-R;II], p. 268, Theorem 32.22; [P], p. 535, Theorem 5.2; see also Appendix II of the present paper). Furthermore, from (4.3) it follows that

$$T(U_t\varphi) = \frac{d}{dt}[T(1_{(0,t]})T(\varphi)] \tag{4.7}$$

for every $t \in \overline{\mathbb{R}^+}$ and $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$, with the derivative computed in the norm of $L(X)$.

4.2. Theorem. *Let T be a continuous representation of the convolution Banach algebra $L^1(\mathbb{R}^+;\mathbb{K})$ on a Banach space X over the field $\mathbb{K}$. Let Y be the closed linear subspace of X defined by* (4.6). *Then there is a unique one parameter semigroup $(S_t)_{t\in\overline{\mathbb{R}^+}} \subset L(Y)$ of class C^0 such that*

$$S_tT(\varphi) = T(U_t\varphi) \tag{4.8}$$

for every $t \in \overline{\mathbb{R}^+}$ and $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$. Furthermore,

$$S_ty = \frac{d}{dt}[T(1_{(0,t]}y] \tag{4.9}$$

for every $t \in \overline{\mathbb{R}^+}$ and $y \in Y$, the derivative being computed in the norm of X, and

$$T(\varphi) = \lim_{\lambda\to\infty} \int_0^\infty \varphi(t)S_tT(\lambda\phi_\lambda)\,dt \tag{4.10}$$

for every $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$, the limit being taken in the norm topology of $L(X;Y)$, and the integral of the $L(X;Y)$-valued function $t \to \varphi(t)S_tT(\lambda\phi_\lambda)$ being understood in the sense of Bochner. From (4.9) *it follows that*

$$\sup_{t\in\overline{\mathbb{R}^+}} \|S_t\|_{L(Y)} \le \|T\|_{L(L^1(\mathbb{R}^+;\mathbb{K});L(X))}.$$

Proof. *Existence of a semigroup $(S_t)_{t\in\overline{\mathbb{R}^+}} \subset L(Y)$ of class C^0 satisfying* (4.8) *and* (4.9). According to (4.6) for every $y \in Y$ there are $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $x \in X$ such that

$$y = T(\varphi)x, \tag{4.11}$$

whence

$$\frac{1}{s-t}[T(1_{(0,s]})y - T(1_{(0,t]})y] = \frac{1}{s-t}[T(1_{(0,s]})T(\varphi)x - T(1_{(0,t]})T(\varphi)x]$$

for every $t \in \overline{\mathbb{R}^+}$ and $s \in \overline{\mathbb{R}^+} \setminus \{t\}$. From this equality and from (4.7) it follows that for every $t \in \overline{\mathbb{R}^+}$ and $y \in Y$ the derivative $\frac{d}{dt}[T(1_{(0,t]})y] = T(U_t\varphi)x$ exists in the norm topology of Y inherited from X. Furthermore,

$$\left\| \frac{1}{s-t}[T(1_{(0,s]})y - T(1_{(0,t]})y] \right\|_Y = \frac{1}{|t-s|}\|T(1_{(s\wedge t, s\vee t]})y\|_Y \\ \leq \|T\|_{L(L^1(\mathbb{R}^+;\mathbb{K});L(X))}\|y\|_Y$$

and hence

$$\left\| \frac{d}{dt}[T(1_{(0,t]})y] \right\|_Y \leq \|T\|_{L(L^1(\mathbb{R}^+;\mathbb{K});L(X))}\|y\|_Y.$$

Thus for every $t \in \overline{\mathbb{R}^+}$ there exists an operator $S_t \in L(Y)$ satisfying (4.8) and (4.9). Representing an element $y \in Y$ in the form (4.11) and using (4.8), one concludes that:

1° the map $\overline{\mathbb{R}^+} \ni t \to S_t y = T(U_t\varphi)x \in Y$ is continuous,

2° $S_0 y = S_0 T(\varphi)x = T(U_0\varphi)x = T(\varphi) = y$,

3° $S_{t_1+t_2}y = S_{t_1+t_2}T(\varphi)x = T(U_{t_1+t_2}\varphi)x = T(U_{t_1}[U_{t_2}\varphi])x = S_{t_1}T(U_{t_2}\varphi)x$
$= S_{t_1}S_{t_2}T(\varphi)x = S_{t_1}S_{t_2}y$ for every $t_1 \in \overline{\mathbb{R}^+}$ and $t_2 \in \overline{\mathbb{R}^+}$.

Hence the operator family $(S_t)_{t\in\overline{\mathbb{R}^+}} \subset L(Y)$ is a one parameter semigroup of class C^0.

Uniqueness of $S_t \in L(Y)$ *satisfying* (4.8). Suppose that $t \in \overline{\mathbb{R}^+}$ and S_t is an operator in $L(Y)$ satisfying (4.8). Take any element $y \in Y$ and represent it in the form (4.11). Then $S_t y = S_t T(\varphi)x = T(U_t\varphi)x$ and hence, by (4.7),

$$S_t y = \left(\frac{d}{dt}[T(1_{(0,t]})T(\varphi)]\right)x = \frac{d}{dt}[T(1_{(0,t]})T(\varphi)x] = \frac{d}{dt}[T(1_{(0,t]})y].$$

Thus property (4.8) of an operator $S_t \in L(Y)$ implies (4.9), and (4.9) uniquely determines this operator.

(4.8)⇒(4.10). If $\lambda \in \mathbb{R}^+$ and $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$, then the functions $\mathbb{R}^+ \ni t \to \varphi(t)U_t(\lambda\phi_\lambda) \in L^1(\mathbb{R}^+;\mathbb{K})$ and $\mathbb{R}^+ \ni t \to \varphi(t)T(U_t(\lambda\phi_\lambda)) \in L^1(X;Y)$ are Bochner integrable on $\mathbb{R}^+$, and

$$T\Big(\int_0^\infty \varphi(t)U_t(\lambda\phi_\lambda)\,dt\Big) = \int_0^\infty \varphi(t)T(U_t(\lambda\phi_\lambda))\,dt.$$

From this equality, and from (4.1) and (4.8), it follows that

$$T(\lambda\phi_\lambda * \varphi) = \int_0^\infty \varphi(t)S_t T(\lambda\phi_\lambda)\,dt.$$

This implies (4.10), by virtue of (4.5). ∎

4.3. Corollary. *Suppose that $T : L^1(\mathbb{R}^+;\mathbb{K}) \to A$ is a homomorphism of the convolution Banach algebra $L^1(\mathbb{R}^+;\mathbb{K})$ into a Banach algebra A over the field $\mathbb{K}$. Denote by B the closure of $T(L^1(\mathbb{R}^+;\mathbb{K}))$ in A. Then B is a commutative Banach subalgebra of A and there is a unique one parameter semigroup $(S_t)_{t\in\overline{\mathbb{R}^+}} \subset L(B)$ of class C^0 satisfying* (4.8). *Furthermore, whenever $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$, then*

(4.12) *$T(\varphi)$ is the unique element of B such that $T(\varphi)b = \int_0^\infty \varphi(t)S_t b\,dt$ for every $b \in B$,* *the integral of the B-valued function $t \to \varphi(t)S_t b$ being understood in the sense of Bochner.*

PROOF. Since $T(L^1(\mathbb{R}^+;\mathbb{K}))$ is a commutative subalgebra of A, its closure B is a commutative Banach subalgebra of A. Consider the canonical homomorphism $\varrho : B \to L(B)$. Then $\widetilde{T} = \varrho \circ T$ is a continuous representation of the convolution Banach algebra $L^1(\mathbb{R}^+;\mathbb{K})$ on the Banach space B. Theorem 4.1 implies that there is a unique semigroup $(S_t)_{t\in\overline{\mathbb{R}^+}} \subset L(B)$ of class C^0 such that

$$S_t\widetilde{T}(\varphi)b = \widetilde{T}(U_t\varphi)b$$

for every $t \in \overline{\mathbb{R}^+}$, $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $b \in B$. Applying this to $b = T(\lambda\phi_\lambda)$, and remembering that $T(\varphi) \in B$ and $T(U_t\varphi) \in B$, one obtains

$$\begin{aligned} S_tT(\lambda\phi_\lambda * \varphi) &= S_t[T(\varphi)T(\lambda\phi_\lambda)] = S_t\widetilde{T}(\varphi)T(\lambda\phi_\lambda) \\ &= \widetilde{T}(U_t\varphi)T(\lambda\phi_\lambda) = T(U_t\varphi)T(\lambda\phi_\lambda) = T(\lambda\phi_\lambda * U_t\varphi), \end{aligned}$$

whence (4.8) follows in virtue of (4.5), by passing to the limit as $\lambda \to \infty$. Thus there exists a semigroup $(S_t)_{t\in\overline{\mathbb{R}^+}} \subset L(B)$ of class C^0, satisfying (4.8).

In order to prove that such a semigroup is unique, observe that (4.8) and (4.3) imply that

$$S_tT(\varphi) = T(U_t\varphi) = \frac{d}{dt}[T(1_{(0,t]})T(\varphi)]$$

for every $t \in \overline{\mathbb{R}^+}$ and $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$. This equality uniquely determines S_t on the dense subset $T(L^1(\mathbb{R}^+;\mathbb{K}))$ of B, and hence on B, since $S_t \in L(B)$.

It remains to prove (4.12). Let $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $\psi \in L^1(\mathbb{R}^+;\mathbb{K})$. Then, by (4.10) applied to the representation $\widetilde{T}$ and by (4.5), one has

$$\begin{aligned} T(\varphi)T(\psi) = \widetilde{T}(\varphi)T(\psi) &= \lim_{\lambda\to\infty} \int_0^\infty \varphi(t)S_t\widetilde{T}(\lambda\phi_\lambda)T(\psi)\,dt \\ &= \lim_{\lambda\to\infty} \int_0^\infty \varphi(t)S_t[T(\lambda\phi_\lambda)T(\psi)]\,dt \\ &= \lim_{\lambda\to\infty} \int_0^\infty \varphi(t)S_tT(\lambda\phi_\lambda * \psi)\,dt = \int_0^\infty \varphi(t)S_tT(\psi)\,dt. \end{aligned}$$

This means that the equality

$$T(\varphi)b = \int_0^\infty \varphi(t)S_tb\,dt$$

holds for every $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and every b in the dense subset $T(L^1(\mathbb{R}^+;\mathbb{K}))$ of B. By continuity with respect to b, the equality remains true for every $b \in B$. Suppose now that $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$, $c \in B$, and $cb = \int_0^\infty \varphi(t)S_tb\,dt$ for every $b \in B$. Then $[c - T(\varphi)]b = 0$ for every $b \in B$, and in particular $[c - T(\varphi)]T(\lambda\phi_\lambda) = 0$ for every $\lambda \in \mathbb{R}^+$. Hence $c - T(\varphi) = \lim_{\lambda\to\infty}[c - T(\varphi)]T(\lambda\phi_\lambda) = 0$, because the net $(T(\lambda\phi_\lambda))_{\lambda\in\mathbb{R}^+}$ is an approximate unit in the commutative Banach algebra B. This last fact is an immediate consequence of (4.5) and of the facts that $T(L^1(\mathbb{R}^+;\mathbb{K}))$ is dense in B and $T : L^1(\mathbb{R}^+;\mathbb{K}) \to B$ is a homomorphism of Banach algebras. ■

4.4. Remark concerning the proof of Theorem 3.2 by a method of W. Chojnacki. Assertion (4.12) is the crucial point in the proof of our Theorem 3.2 given by W. Chojnacki in [Ch]. From Lemma 2.1 it follows that the commutative Banach algebra B considered in Corollary 4.3 may be equivalently defined by

$$B = \overline{\operatorname{span}}_{\mathbb{K}}\{r(\lambda) : \lambda \in \mathbb{R}^+\}, \tag{4.13}$$

where $r(\lambda) = T(\phi_\lambda)$. In his proof of the implication (I)$\Rightarrow$(II) of Theorem 3.2, W. Chojnacki assumes (I), defines B by (4.13), and considers the pseudoresolvent

$$\varrho \circ r : \mathbb{R}^+ \to L(B).$$

Then equality (3.2) implies that

$$\|([\varrho \circ r](\lambda))^k\|_{L(B)} = \|\varrho([r(\lambda)]^k)\|_{L(B)} \le \|[r(\lambda)]^k\|_B \le M\lambda^{-k} \tag{4.14}$$

for every $\lambda \in \mathbb{R}^+$ and $k = 1, 2, \ldots$ Furthermore,

$$\lim_{\lambda\to\infty} \lambda\varrho(r(\lambda))r(\mu) = \lim_{\lambda\to\infty} \lambda r(\lambda)r(\mu) = \lim_{\lambda\to\infty} [r(\mu) - r(\lambda) + \mu r(\lambda)r(\mu)] = r(\mu)$$

for every $\mu \in \mathbb{R}^+$, by (I) and (3.2). Thus from (4.13) and (4.14) it follows that

$$\lim_{\lambda\to\infty} \lambda\varrho(r(\lambda))b = b \tag{4.15}$$

for every $b \in B$. Conditions (4.14) and (4.15) imply that $\varrho \circ r$ is the resolvent of a closed densely defined operator A from B into B, satisfying the assumptions of the Hille–Yosida generation theorem. It follows that there is a unique one parameter semigroup $(S_t)_{t\in\overline{\mathbb{R}^+}} \subset L(B)$ of class C^0 such that $\|S_t\|_{L(B)} \le M$ for every $t \in \overline{\mathbb{R}^+}$ and

$$r(\lambda)b = \varrho(r(\lambda))b = \int_0^\infty \phi_\lambda(t)S_t b\,dt \tag{4.16}$$

for every $\lambda \in \mathbb{R}^+$ and $b \in B$. Therefore the formula

$$\widetilde{T}(\varphi)b = \int_0^\infty \varphi(t)S_t b\,dt, \tag{4.17}$$

in which $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $b \in B$, defines a continuous representation $\widetilde{T}$ of the convolution Banach algebra $L^1(\mathbb{R}^+;\mathbb{K})$ on the Banach space B such that

$$\widetilde{T}(\phi_\lambda) = \varrho(r(\lambda)) \tag{4.18}$$

for every $\lambda \in \mathbb{R}^+$. Now the main difficulty of Chojnacki's proof arises: one has to pass from (4.17) to (4.12), i.e. one has to prove that

(4.19) for every $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ there is a unique element $T(\varphi)$ of B such that $T(\varphi)b = \widetilde{T}(\varphi)b$ for every $b \in B$.

The difficulty is overcome in [Ch] by showing that if condition (I) of Theorem 3.2 is satisfied then

(4.20) the homomorphism $\varrho : B \to L(B)$ is an isomorphism of B onto a Banach subalgebra of $L(B)$.

This is proved by renorming the Banach algebra A so that the net $(\lambda r(\lambda))_{\lambda\in\mathbb{R}^+}$ is a metric approximate unit in the Banach algebra B equipped with the new norm ([Ch], p. 4, Theorem 2). As a consequence, $\varrho : B \to L(B)$ is an isometry with respect to the new norm in B and the corresponding new norm in $L(B)$ ([Ch], p. 3, Proposition 1). Thus (4.20) follows, and hence (4.18) and Lemma 2.1 imply (4.19). Since $\widetilde{T}$ is a representation of the convolution Banach algebra $L^1(\mathbb{R}^+;\mathbb{K})$ on the Banach space B, (4.19) implies that $T : L^1(\mathbb{R}^+;\mathbb{K}) \to B$ is a homomorphism of Banach algebras, and the implication (I)⇒(II) of Theorem 3.2 is proved.

5. The Hille–Yosida theorem

Let X be a Banach space, and $L(X)$ the Banach algebra of linear continuous endomorphisms of X. We will consider a *pseudoresolvent* on $\mathbb{R}^+$ with values in $L(X)$, i.e. a map

$$\mathbb{R}^+ \ni \lambda \to R_\lambda \in L(X) \tag{5.1}$$

satisfying the *resolvent equation*

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu \tag{5.2}$$

for every λ and μ in $\mathbb{R}^+$. It follows from (5.2) that $(R_\lambda)_{\lambda\in\mathbb{R}^+}$ is a commutative family of operators, and that the kernel $\mathcal{K}$ and range $\Im$ of R_λ are both independent of λ. See [Y;2], pp. 215–216. The equality

$$G = \{(x, y) \in X \times X : \lambda R_\lambda x - x = R_\lambda y \text{ for every } \lambda \in \mathbb{R}^+\} \tag{5.3}$$

defines a closed linear subspace of $X \times X$. Following [D-M; XII-XVI], p. 243, we will call G the *extended generator* of the pseudoresolvent (5.1). Equation (5.2) implies that

(5.4) if $x \in X$, $y \in X$, and there exists a $\mu \in \mathbb{R}^+$ such that $\mu R_\mu x - x = R_\mu y$, then $(x, y) \in G$.

Indeed, it follows from (5.2) that if $\mu R_\mu x - x = R_\mu y$, then

$$\begin{aligned} R_\lambda y &= [1 + (\mu - \lambda)R_\lambda]R_\mu y = [1 + (\mu - \lambda)R_\lambda][\mu R_\mu x - x] \\ &= \mu R_\mu x - x + \mu(\mu - \lambda)R_\lambda R_\mu x + (\lambda - \mu)R_\lambda x \\ &= \mu R_\mu x - x + \mu(R_\lambda - R_\mu)x + (\lambda - \mu)R_\lambda x = \lambda R_\lambda x - x \end{aligned}$$

for every $\lambda \in \mathbb{R}^+$. The *domain of the extended generator* G is, by definition, the set

$$D(G) = \{x \in X : \text{there exists } y \in X \text{ such that } (x, y) \in G\}. \tag{5.5}$$

It follows that

$$D(G) = \Im. \tag{5.6}$$

Indeed, if $(x, y) \in G$ and $\mu \in \mathbb{R}^+$, then $x = R_\mu(\mu x - y) \in \Im$. Conversely, if $x \in \Im$ and $\mu \in \mathbb{R}^+$, then $x = R_\mu z$ for some $z \in X$, so that $\mu R_\mu x - x = R_\mu y$ for $y = \mu x - z$, whence $(x, y) \in G$ by (5.4).

Appendix I contains a necessary and sufficient condition for a subspace of $X \times X$ to be the extended generator of a pseudoresolvent. If a pseudoresolvent (5.1) is the Laplace transform of a measurable contraction semigroup in a function space, then the extended generator (5.3) coincides with the *full generator* of the semigroup defined in [E-K], pp. 23–24. See also [R-Y], p. 263.

If $N = \{0\}$ then G is the graph of a closed operator from X into X whose *resolvent set* contains $\mathbb{R}^+$, and the pseudoresolvent (5.1) is the *resolvent* of this operator.

According to [D-M;C], p. 314, the *regularity space* of the pseudoresolvent (5.1) is, by definition, the linear set

$$\Re = \{x \in X : \lim_{\lambda \to \infty} \|\lambda R_\lambda x - x\| = 0\}. \tag{5.7}$$

It is obvious that

$$\Re \subset \overline{\Im}, \tag{5.8}$$

where $\overline{\Im}$ denotes the closure of $\Im$ in the norm topology of X. If $x \in \Re \cap \mathcal{K}$, then $x = \lim_{\lambda\to\infty} \lambda R_\lambda x = \lim_{\lambda\to\infty} \lambda \cdot 0 = 0$, so that

$$\Re \cap \mathcal{K} = \{0\}. \tag{5.9}$$

From the commutativity of the family of operators (R_λ), it follows that

$$R_\lambda \Re \subset \Re \tag{5.10}$$

for every $\lambda \in \mathbb{R}^+$.

According to [Hi], p. 98, and [D-M; C], p. 315, the *generator of the pseudoresolvent* (5.1) is defined to be the operator A from X into X with domain $D(A)$ such that

$$x \in G(A) \text{ and } y = Ax \text{ if and only if } \lim_{\lambda\to\infty} \|\lambda(\lambda R_\lambda x - x) - y\| = 0. \tag{5.11}$$

Denote by $G(A)$ the graph of A. Definition (5.11) is equivalent to

$$G(A) = \{(x, y) \in X \times X : \lim_{\lambda\to\infty} \|\lambda(\lambda R_\lambda x - x) - y\| = 0\}. \tag{5.12}$$

It is obvious that

$$D(A) \subset \Re. \tag{5.13}$$

Furthermore,

$$(X \times \Re) \cap G \;\subset\; G(A) \;\subset\; (X \times \overline{\Im}) \cap G. \tag{5.14}$$

Indeed, if $(x, y) \in (X \times \Re) \cap G$, then $\lim_{\lambda\to\infty} \lambda R_\lambda y = y$ and $\lambda R_\lambda x - x = R_\lambda y$ for every $\lambda \in \mathbb{R}^+$, so that $y = \lim_{\lambda\to\infty} \lambda(\lambda R_\lambda x - x)$ and $(x, y) \in G(A)$. Hence $(X \times \Re) \cap G \subset G(A)$. If $(x, y) \in G(A)$, then $x \in \overline{\Im}$, by (5.12) and (5.8), so that $\lambda(\lambda R_\lambda x - x) \in \overline{\Im}$ for every $\lambda \in \mathbb{R}^+$, and hence $y = \lim_{\lambda\to\infty} \lambda(\lambda R_\lambda x - x) \in \overline{\Im}$. Furthermore, if $(x, y) \in G(A)$ and

$\mu \in \mathbb{R}^+$, then

$$R_\mu y = R_\mu \lim_{\lambda\to\infty} \lambda(\lambda R_\lambda x - x) = \lim_{\lambda\to\infty} \lambda(\lambda R_\mu R_\lambda x - R_\mu x) = \lim_{\lambda\to\infty} \lambda(\mu R_\mu R_\lambda x - R_\lambda x)$$
$$= (\mu R_\mu - 1) \lim_{\lambda\to\infty} \lambda R_\lambda x = \mu R_\mu x - x,$$

by (5.2) and (5.13). Hence $G(A) \subset (X \times \overline{\Im}) \cap G$.

EXAMPLE. Consider an operator $B \in L(X)$ such that $B^2 = 0$. The constant map $\mathbb{R}^+ \ni \lambda \to B \in L(X)$ is then a pseudoresolvent for which $\Im \subset \mathcal{K}$, $G = \{(-By, y) : y \in X\}$ and $\Re = \{0\}$.

From now on we will make some additional assumptions on the pseudoresolvent (5.1).

5.1. LEMMA. *If* $\lim_{\lambda\to\infty} \|R_\lambda x\| = 0$ *for every* $x \in X$, *then* $\Im \subset \Re$.

PROOF. Let $x \in \Im$. Fix $\mu \in \mathbb{R}^+$ and choose $z \in X$ such that $x = R_\mu z$. Then, by (5.2), $\lambda R_\lambda x - x = \lambda R_\lambda R_\mu z - R_\mu z = \mu R_\mu R_\lambda z - R_\lambda z$, so that $\|\lambda R_\lambda x - x\| \leq (\mu \|R_\mu\| + 1)\|R_\lambda z\|$, and hence $\lim_{\lambda\to\infty} \|\lambda R_\lambda x - x\| = 0$, which means that $x \in \Re$. ∎

5.2. PROPOSITION. *If*

$$\limsup_{\lambda\to\infty} \lambda \|R_\lambda\|_{L(X)} < \infty, \tag{5.15}$$

then

$$\Re = \overline{\Im} \tag{5.16}$$

and

$$G(A) = (X \times \Re) \cap G. \tag{5.17}$$

PROOF. Equalities (5.16) and (5.17) follow at once from (5.8), (5.14), Lemma 5.1 and the fact that if (5.15) is satisfied, then $\Re$ is closed. To prove this fact, suppose that x belongs to the closure of $\Re$. Then there is a sequence $x_1, x_2, \ldots$ of elements of $\Re$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. Since

$$\|\lambda R_\lambda x - x\| \leq (1 + \lambda \|R_\lambda\|)\|x_n - x\| + \|\lambda R_\lambda x_n - x_n\|,$$

it follows that

$$\limsup_{\lambda\to\infty} \|\lambda R_\lambda x - x\| \leq (1 + \limsup_{\lambda\to\infty} \lambda \|R_\lambda\|)\|x_n - x\|,$$

for every $n = 1, 2, \ldots$, whence $\lim_{\lambda\to\infty} \|\lambda R_\lambda x - x\| = 0$, i.e. $x \in \Re$. ∎

5.3. COROLLARY. *If condition* (5.15) *is satisfied then* $\Re$ *is a closed linear subspace of* X, *and* A *is a closed operator from* X *into* X *with domain and range contained in* $\Re$.

5.4. PROPOSITION. *If condition (5.15) is satisfied and A is treated as an operator from $\Re$ into $\Re$ then the resolvent set of A contains $\mathbb{R}^+$ and*

$$(\lambda - A)^{-1} = R_\lambda|\Re \tag{5.18}$$

for every $\lambda \in \mathbb{R}^+$. Furthermore, $D(A)$ is dense in $\Re$.

PROOF. If $x \in D(A)$ and $\lambda \in \mathbb{R}^+$, then $(x, Ax) \subset G$ by (5.14), whence $\lambda R_\lambda x - x = R_\lambda Ax$. This means that

$$R_\lambda(\lambda - A)x = x \text{ for every } \lambda \in \mathbb{R}^+ \text{ and } x \in D(A). \tag{5.19}$$

If $x \in \Re$ and $\lambda \in \mathbb{R}^+$, then, by Lemma 5.1 and Proposition 5.2, $R_\lambda x \in \Im \subset \Re$ and $(R_\lambda x, \lambda R_\lambda x - x) \in (X \times \Re) \cap G = G(A)$, so that $R_\lambda x \in D(A)$ and $(\lambda - A)R_\lambda x = \lambda R_\lambda x - AR_\lambda x = \lambda R_\lambda x - [\lambda R_\lambda x - x] = x$. Hence

$$R_\lambda x \in D(A) \text{ and } (\lambda - A)R_\lambda x = x \text{ for every } \lambda \in \mathbb{R}^+ \text{ and } x \in \Re. \tag{5.20}$$

From (5.19) and (5.20) it follows that if A is treated as an operator from $\Re$ into $\Re$, then the resolvent set of A contains $\mathbb{R}^+$ and (5.18) holds. As a consequence, if $x \in \Re$ then $\lambda R_\lambda x \in D(A)$ for every $\lambda \in \mathbb{R}^+$ and hence $x = \lim_{\lambda\to\infty} \lambda R_\lambda x$ belongs to the closure of $D(A)$, proving that $D(A)$ is dense in $\Re$. ∎

5.5. THEOREM. *Let X be a Banach space (over the field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$), and $L(X)$ the Banach algebra of linear continuous endomorphisms of X. Suppose that the map $\mathbb{R}^+ \ni \lambda \to R_\lambda \in L(X)$ is a pseudoresolvent such that*

$$\sup\{\lambda^k \|R_\lambda^k\|_{L(X)} : \lambda \in \mathbb{R}^+,\ k = 1, 2, \ldots\} = M < \infty. \tag{5.21}$$

Let A be the generator of this pseudoresolvent, $\Re$ its regularity space, and $\Im$ the range of R_λ (independent of λ). Then:

1° *there is a unique continuous representation T of the convolution Banach algebra $L^1(\mathbb{R}^+; \mathbb{K})$ on the Banach space X such that $\|T\|_{L(L^1(\mathbb{R}^+;\mathbb{K});L(X))} = M$ and $T(\phi_\lambda) = R_\lambda$ for every $\lambda \in \mathbb{R}^+$, where $\phi_\lambda(\xi) = e^{-\lambda\xi}$ for $\xi \in \mathbb{R}^+$;*

2° $\Re = \overline{\Im} = \{T(\varphi)x : \varphi \in L^1(\mathbb{R}^+; \mathbb{K}),\ x \in X\}$;

3° *there is a unique semigroup $(S_t)_{t\in\overline{\mathbb{R}^+}} \subset L(\Re)$ of class C^0 such that $\|S_t\|_{L(\Re)} \leq M$ and*

$$S_tT(\varphi) = T(U_t\varphi)$$

for every $t \in \overline{\mathbb{R}^+}$ and $\varphi \in L^1(\mathbb{R}^+; \mathbb{K})$, where $U_t \in L(L^1(\mathbb{R}^+; \mathbb{K}))$ is the operator of right translation by t;

4° $T(\varphi) = \lim_{\lambda\to\infty} \int_0^\infty \varphi(t)S_t\lambda R_\lambda\, dt$ *for every $\varphi \in L^1(\mathbb{R}^+; \mathbb{K})$, with limit in the norm topology of $L(X; \Re)$, the integral of the $L(X; \Re)$-valued function being understood in the sense of Bochner;*

5° *the domain and the range of A are contained in $\Re$, and A is the infinitesimal generator of the semigroup determined in 3°.*

PROOF. Assertion 1° follows from Theorem 3.2 and Proposition 1.1(B). The equality $\Re = \overline{\Im}$ in 2° follows from Proposition 5.2. From 1° and Lemma 2.1 it follows that, for every $\lambda \in \mathbb{R}^+$, $\Im = T(\phi_\lambda)X \subset T(L^1(\mathbb{R}^+;\mathbb{K}))X \subset \overline{\Im}$, and hence $T(L^1(\mathbb{R}^+;\mathbb{K}))X = \overline{\Im}$, by (4.6). Assertion 2° is thus proved. Assertions 3° and 4° follow from 1°, 2°, and Theorem 4.2.

It remains to prove 5°. From (5.21), Corollary 5.3 and Proposition 5.4 it follows that the domain and range of A are contained in $\Re$, and equality (5.18) holds when A is treated as an operator from $\Re$ into $\Re$. Assertions 1° and 4°, and equality (5.18) imply that

$$(\lambda - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_t x \, dt \tag{5.22}$$

for every $\lambda \in \mathbb{R}^+$ and $x \in \Re$. The integrand $\mathbb{R}^+ \ni t \to e^{-\lambda t} S_t x \in \Re$ in (5.22) is continuous in the norm topology of $\Re$, and is absolutely integrable on $\mathbb{R}^+$, so that the integral may be understood either in the sense of Bochner or as an improper Riemann integral. Following [D-S;I], Sec. VII.1, notice that

$$\begin{aligned} \frac{1}{h}(S_h - 1)\int_0^\infty e^{-\lambda t} S_t x \, dt &= \int_0^\infty e^{-\lambda t} S_t \frac{1}{h}(S_h - 1)x \, dt \\ &= \frac{e^{\lambda h} - 1}{h}\int_0^\infty e^{-\lambda t} S_t x \, dt - e^{\lambda t}\frac{1}{h}\int_0^h e^{-\lambda t} S_t x \, dt \end{aligned} \tag{5.23}$$

for every $h \in \mathbb{R}^+$, $\lambda \in \mathbb{R}^+$ and $x \in \Re$. Let $\widetilde{A}$ be the infinitesimal generator of the semigroup $(S_t)_{t\in\overline{\mathbb{R}^+}}$. By a passage to the limit as $h \downarrow 0$, from (5.23) it follows that

$$\int_0^\infty e^{-\lambda t} S_t(\lambda - \widetilde{A})x \, dt = x \text{ for every } \lambda \in \mathbb{R}^+ \text{ and } x \in D(\widetilde{A}),$$

and

$$\int_0^\infty e^{-\lambda t} S_t x \, dt \in D(\widetilde{A}) \text{ and } (\lambda - \widetilde{A})\int_0^\infty e^{-\lambda t} S_t x \, dt = x \text{ for every } \lambda \in \mathbb{R}^+ \text{ and } x \in \Re.$$

These equalities mean that $\mathbb{R}^+$ is contained in the resolvent set of $\widetilde{A}$, and

$$(\lambda - \widetilde{A})^{-1}x = \int_0^\infty e^{-\lambda t} S_t x \, dt \tag{5.24}$$

for every $\lambda \in \mathbb{R}^+$ and $x \in \Re$. See [D-S;I], Sec. VIII.1, Theorem 11. From (5.22) and (5.24) it follows that $A = \widetilde{A}$. ∎

5.6. COROLLARY (see [D-M;C], Sec. XIII.1.4, p. 311). *Let the map $\mathbb{R}^+ \ni \lambda \to R_\lambda \in L(X)$ be a pseudoresolvent with regularity space $\Re$ and generator A. Let $M \in [1,\infty)$. Then the following two conditions are equivalent*:

(a) *there is a unique semigroup* $(S_t)_{t\in\overline{\mathbb{R}^+}} \subset L(X)$ *of class* C^0 *such that* $\|S_t\| \le M$ *for every* $t \in \overline{\mathbb{R}^+}$ *and*

$$R_\lambda x = \int_0^\infty e^{-\lambda t} S_t x \, dt$$

for every $\lambda \in \mathbb{R}^+$ *and* $x \in X$;

(b) $\sup\{\lambda^k \|R_\lambda^k\| : \lambda \in \mathbb{R}^+,\ k = 1, 2, \ldots\} \le M$ *and* $\Re = X$.

If these conditions are satisfied, then the resolvent set of A *contains* $\mathbb{R}^+$, $R_\lambda = (\lambda - A)^{-1}$ *for every* $\lambda \in \mathbb{R}^+$, *and* A *coincides with the infinitesimal generator of the semigroup* $(S_t)_{t\in\overline{\mathbb{R}^+}}$.

Proof. Condition (a) and equality (0.2) imply that:

1° $\lim_{\lambda\to\infty} \lambda R_\lambda x = \lim_{\lambda\to\infty} \lambda \int_0^\infty e^{-\lambda t} S_t x \, dt = S_0 x = x$ for every $x \in X$, which means that $\Re = X$,

$$2^\circ \ R_\lambda^k x = \frac{(-1)^{k-1}}{(k-1)!} R_\lambda^{(k-1)} x = \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{d}{d\lambda}\right)^{k-1} \int_0^\infty e^{-\lambda t} S_t x \, dt$$
$$= \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-\lambda t} S_t x \, dt$$

for every $\lambda \in \mathbb{R}^+$ and $x \in X$, whence $\|R_\lambda^k\| \le M \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-\lambda t} \, dt = M \frac{1}{\lambda^k}$.

This proves that (a) implies (b). The converse implication and the statements concerning A follow from Theorem 5.5. ∎

5.7. Corollary (the Hille–Yosida theorem). *Let* A *be a linear operator from* X *into* X *with domain* $D(A)$. *Let* $M \in [1, \infty)$. *Then the following two conditions are equivalent*:

(A) A *is the infinitesimal generator of a semigroup* $(S_t)_{t\in\overline{\mathbb{R}^+}} \subset L(X)$ *of class* C^0 *such that* $\|S_t\| \le M$ *for every* $t \in \overline{\mathbb{R}^+}$;

(B) $D(A)$ *is dense in* X, A *is a closed operator from* X *into* X *with resolvent set containing* $\mathbb{R}^+$, *and*

$$\sup\{\lambda^k \|(\lambda - A)^{-k}\| : \lambda \in \mathbb{R}^+,\ k = 1, 2, \ldots\} \le M.$$

Proof. (A)⇒(B). If condition (A) is satisfied then, according to Theorem 11 of Sec. VIII.1 of [D-S;I] (i.e. similarly to our equality (5.24)), the resolvent set of A contains $\mathbb{R}^+$, and

$$(5.25) \qquad (\lambda - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_t x \, dt$$

for every $\lambda \in \mathbb{R}^+$ and $x \in X$. It follows that the operator A is closed, and $\int_0^\infty e^{-\lambda t} S_t x \, dt \in D(A)$ for every $x \in X$ and $\lambda \in \mathbb{R}^+$, whence $x = \lim_{\lambda\to\infty} \lambda \int_0^\infty e^{-\lambda t} S_t x \, dt \in \overline{D(A)}$, showing that $D(A)$ is dense in X. The estimate of the norm of $(\lambda - A)^{-k}$ follows from

(5.25) and from the inequality $\sup_{t\in\overline{\mathbb{R}^+}}\|S_t\| \le M$ by an argument similar to the one used in the proof of Corollary 5.6.

(B)⇒(A). Suppose that (B) holds and for every $\lambda \in \mathbb{R}^+$ define $R_\lambda = (\lambda - A)^{-1}$. Then the map $\mathbb{R}^+ \ni \lambda \to R_\lambda \in L(X)$ is a resolvent such that $\sup\{\lambda^k\|R_\lambda^k\| : \lambda \in \mathbb{R}^+, k = 1, 2, \ldots\} \le M$. Furthermore, by Proposition 5.2, the regularity space of this resolvent is $\Re = \overline{\Im} = \overline{(\lambda - A)^{-1}X} = \overline{D(A)} = X$. Thus condition (b) from Corollary 5.6 is satisfied, and so, according to the implication (b)⇒(a), there is a unique semigroup $(S_t)_{t\in\overline{\mathbb{R}^+}} \subset L(X)$ of class C^0 such that $\|S_t\| \le M$ for every $t \in \overline{\mathbb{R}^+}$, and equality (5.25) holds for this semigroup and for the operator A satisfying (B). Theorem 11 from Sec. VIII.1 of [D-S;I] implies that an analogous equality holds for the same semigroup and for its infinitesimal generator. Therefore this infinitesimal generator is equal to A. ∎

5.8. Corollary (a version of the Trotter–Kato approximation theorem; [E-K], Sec. 1.6; [Y;2], Sec. IX.12). *Let $M \in [1, \infty)$. Suppose that for every $n = 0, 1, \ldots$ the map $\mathbb{R}^+ \ni \lambda \to R_{\lambda,n} \in L(X)$ is a pseudoresolvent with regularity space $\Re_n$ and generator A_n such that*

(i) $\sup\{\lambda^k\|R^k_{\lambda,n}\|_{L(X)} : \lambda \in \mathbb{R}^+,\ k = 1, 2, \ldots,\ n = 0, 1, \ldots\} \le M$,

(ii) *there is $\lambda_0 \in \mathbb{R}^+$ such that $\lim_{n\to\infty}\|R_{\lambda_0,n}x - R_{\lambda_0,0}x\|_X = 0$ for every $x \in X$.*

Then, according to Theorem 5.5, condition (i) *implies that, for every $n = 0, 1, \ldots$, $\Re_n$ is a closed subspace of X and there is a unique semigroup $(S_{t,n})_{t\in\overline{\mathbb{R}^+}} \subset L(\Re_n)$ of class C^0 with infinitesimal generator A_n such that*

$$\sup_{t\in\overline{\mathbb{R}^+}}\|S_{t,n}\|_{L(\Re_n)} \le M.$$

Furthermore, the conjunction (i) & (ii) *implies that*

1° *for every $x_0 \in \Re_0$ there is a sequence $x_1, x_2, \ldots$ such that $x_n \in \Re_n$ for every $n = 1, 2, \ldots$ and $\lim_{n\to\infty}\|x_n - x_0\|_X = 0$,*

2° *if $x_0, x_1, \ldots$ is a sequence such that $x_n \in \Re_n$ for every $n = 0, 1, \ldots$ and $\lim_{n\to\infty}\|x_n - x_0\|_X = 0$, then, for every $a \in \mathbb{R}^+$,*

$$\lim_{n\to\infty}\sup_{0\le t\le a}\|S_{t,n}x_n - S_{t,0}x_0\|_X = 0.$$

Proof. Suppose that conditions (i) and (ii) are satisfied. By Theorem 5.5 for every $n = 1, 2, \ldots$ there is a continuous representation T_n of the convolution Banach algebra $L^1(\mathbb{R}^+; \mathbb{K})$ on the Banach space X such that:

(a) $\{T_n(\varphi)x : \varphi \in L^1(\mathbb{R}^+; \mathbb{K}),\ x \in X\} = \Re_n$,

(b) $\|T_n\|_{L(L^1(\mathbb{R}^+;\mathbb{K}); L(X;\Re_n))} \le M$,

(c) $T_n(\phi_\lambda) = R_{\lambda,n}$ for every $\lambda \in \mathbb{R}^+$,

(d) $S_{t,n}T_n(\varphi) = T_n(U_t\varphi)$ for every $t \in \overline{\mathbb{R}^+}$ and $\varphi \in L^1(\mathbb{R}^+; \mathbb{K})$.

Condition (i) implies that

$$R_{\lambda,n} = R_{\mu,n}(1-(\mu-\lambda)R_{\mu,n})^{-1} = R_{\mu,n} + \sum_{k=1}^{\infty}(\mu-\lambda)^k R_{\mu,n}^{k+1}$$

for every $\mu \in \mathbb{R}^+$ and $\lambda \in (0, 2\mu)$, the series being absolutely convergent in $L(X)$, and its terms having the estimate $\|(\mu-\lambda)^k R_{\mu,n}^{k+1}\|_{L(X)} \leq \frac{1}{\mu}|\frac{\lambda}{\mu} - 1|^k$ independent of n. Therefore (i) & (ii) implies that

(e) $\lim_{n\to\infty} \|R_{\lambda,n}x - R_{\lambda,0}x\|_X = 0$ for every $\lambda \in \mathbb{R}^+$ and $x \in X$.

From (b), (c), (e) and Lemma 2.1 it follows that

(f) $\lim_{n\to\infty} \|T_n(\varphi)x - T_0(\varphi)x\|_X = 0$ for every $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $x \in X$.

1°. If $x_0 \in \Re_0$, then, by (a), there are $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $x \in X$ such that $x_0 = T_0(\varphi)x$. From (a) and (f) it follows that if $x_n = T_n(\varphi)x$, then $x_n \in \Re_n$ and $\lim_{n\to\infty} \|x_n - x\|_X = 0$.

2°. Let $x_0, x_1, \ldots$ be a sequence with $x_n \in \Re_n$ $(n = 0, 1, \ldots)$ and $\lim_{n\to\infty} \|x_n - x_0\|_X = 0$. By (a) there are $\varphi \in L^1(\mathbb{R}^+;\mathbb{K})$ and $x \in X$ such that $T_0(\varphi)x = x_0$. As a consequence of (a) and (d),

$$\begin{aligned} S_{t,n}x_n - S_{t,0}x_0 &= S_{t,n}(x_n - T_n(\varphi)x) + S_{t,n}T_n(\varphi)x - S_{t,0}T_0(\varphi)x \\ &= S_{t,n}(x_n - T_n(\varphi)x) + T_n(U_t\varphi)x - T_0(U_t\varphi)x, \end{aligned}$$

so that

(g) $\|S_{t,n}x_n - S_{t,0}x_0\|_X \leq M\|x_n - x_0\|_X + M\|T_n(\varphi)x - T_0(\varphi)x\|_X$
$+ \|T_n(U_t\varphi)x - T_0(U_t\varphi)x\|_X.$

If $a \in \mathbb{R}^+$, then $\{U_t\varphi : 0 \leq t \leq a\}$ is a compact subset of $L^1(\mathbb{R}^+;\mathbb{K})$, and hence from (b) and (f) it follows that

(h) $\lim_{n\to\infty} \sup_{0\leq t\leq a} \|T_n(U_t\varphi)x - T_0(U_t\varphi)x\|_X = 0.$

Assertion 2° follows from (g) and (h). ■

Appendix I. Pseudoresolvents and their extended generators

Let A be an algebra over a commutative field $\mathbb{K}$. By a *pseudoresolvent* with values in A defined on a non-empty subset Λ of $\mathbb{K}$ we mean a map $r : \Lambda \to A$ such that

(i) $$r(\lambda) - r(\mu) = (\mu - \lambda)r(\lambda)r(\mu)$$

for every $\lambda \in \Lambda$ and $\mu \in \Lambda$. It follows that the range of a pseudoresolvent $r : \Lambda \to A$ consists of mutually commuting elements of A.

I.1. PROPOSITION. *Every pseudoresolvent has a unique maximal extension to a pseudoresolvent.*

PROOF. Consider the binary relations $\to$ and $\sim$ on $\mathbb{K}\times A$ such that if $(\lambda,a)\in\mathbb{K}\times A$ and $(\mu,b)\in\mathbb{K}\times A$, then

$$(\lambda,a)\to(\mu,b)\equiv a-b=(\mu-\lambda)ab$$

and

$$(\lambda,a)\sim(\mu,b)\equiv(\lambda,a)\to(\mu,b)\quad\text{and}\quad(\mu,b)\to(\lambda,a).$$

An equivalent definition of $\sim$ is

$$(\lambda,a)\sim(\mu,b)\equiv(\lambda,a)\to(\mu,b)\quad\text{and}\quad ab=ba.$$

Suppose that (λ,a), (μ,b) and (ν,c) belong to $\mathbb{K}\times A$, $(\lambda,a)\to(\mu,b)$ and $(\mu,b)\to(\nu,c)$. Then $b=a+(\lambda-\mu)ab=c+(\nu-\mu)bc$, so that $a-c=[a-b]+[b-c]=(\mu-\lambda)ab+(\nu-\mu)bc=(\mu-\lambda)a[c+(\nu-\mu)bc]+(\nu-\mu)[a+(\lambda-\mu)ab]c=(\nu-\lambda)ac$, which means that $(\lambda,a)\to(\nu,c)$. Thus $\to$ is transitive, and hence $\sim$ is an equivalence. It follows that

(ii) if $r:\Lambda\to A$ is a pseudoresolvent, $\lambda_0\in\Lambda$, $\lambda\in\Lambda$ and $a\in A$ then $a=r(\lambda)$ if and only if $(\lambda,a)\sim(\lambda_0,r(\lambda_0))$.

As a consequence of (ii), if $r:\Lambda\to A$ is a pseudoresolvent and λ_0 is any element of Λ, then the graph of r is equal to the set

$$\{(\lambda,a)\in\Lambda\times A:(\lambda,a)\sim(\lambda_0,r(\lambda_0))\},$$

while the graph of the maximal extension of r to a pseudoresolvent is the whole equivalence class

$$\{(\lambda,a)\in\mathbb{K}\times A:(\lambda,a)\sim(\lambda_0,r(\lambda_0))\}.\ \blacksquare$$

By a *maximal pseudoresolvent* we mean a pseudoresolvent which is equal to its maximal extension to a pseudoresolvent.

I.2. PROPOSITION. *If $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ and A is a Banach algebra over the field $\mathbb{K}$, then every maximal pseudoresolvent with values in A is an analytic function defined on an open subset of $\mathbb{K}$.*

PROOF. Let $r:\Lambda\to A$ be a maximal pseudoresolvent. If $r(\lambda)=0$ for some $\lambda\in\Lambda$, then $r\equiv 0$ on Λ and hence $\Lambda=\mathbb{K}$, because $r:\Lambda\to A$ is maximal. Thus we are reduced to proving the proposition under the additional assumption that $r(\lambda)\neq 0$ for every $\lambda\in\Lambda$. Suppose that $r:\Lambda\to A$ is a maximal pseudoresolvent such that $r(\lambda)\neq 0$ for every $\lambda\in\Lambda$. Take any $\lambda_0\in\Lambda$ and let $B=\{\lambda\in\mathbb{K}:|\lambda-\lambda_0|<\|r(\lambda_0)\|^{-1}\}$. For every $\lambda\in B$ one has $\|(\lambda_0-\lambda)^k[r(\lambda_0)]^{k+1}\|\leq\|r(\lambda_0)\|\theta_\lambda^k$ for $k=1,2,\ldots$, where $\theta_\lambda=|\lambda-\lambda_0|\|r(\lambda_0)\|\in(0,1)$. Hence for every $\lambda\in B$ the series $r(\lambda_0)+(\lambda_0-\lambda)[r(\lambda_0)]^2+(\lambda_0-\lambda)^2[r(\lambda_0)]^3+\ldots$ is absolutely convergent and its sum s is an element of A such that $s-r(\lambda_0)=(\lambda_0-\lambda)sr(\lambda_0)=(\lambda_0-\lambda)r(\lambda_0)s$, i.e. $(\lambda,s)\sim(\lambda_0,r(\lambda_0))$. Since the pseudoresolvent $r:\Lambda\to A$ is maximal, it follows that $B\subset\Lambda$ and

$$r(\lambda)=r(\lambda_0)+(\lambda_0-\lambda)[r(\lambda_0)]^2+(\lambda_0-\lambda)^2[r(\lambda_0)]^3+\ldots\qquad\text{for every }\lambda\in B.\ \blacksquare$$

Suppose now that X is a Banach space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Denote by $L(X)$ the Banach algebra of continuous linear endomorphisms of X. Let $\emptyset \neq \Lambda \subset \mathbb{K}$ and suppose that the map

$$\text{(iii)} \qquad \Lambda \ni \lambda \to R_\lambda \in L(X)$$

is a pseudoresolvent, i.e.

$$\text{(iv)} \qquad R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$$

for every $\lambda \in \Lambda$ and $\mu \in \Lambda$. Then $(R_\lambda)_{\lambda \in \Lambda}$ is a commutative family of operators. Furthermore, the kernel N and the range $\Im$ of R_λ are both independent of $\lambda \in \Lambda$. See [Y;2], pp. 215–216. Define

$$\text{(v)} \qquad G(R_\bullet) = \{(x, y) \in X \times X : \lambda R_\lambda x - x = R_\lambda y \text{ for every } \lambda \in \Lambda\}.$$

Then $G(R_\bullet)$ is a closed subspace of $X \times X$. Following [D-M;XII-XVI], p. 243, we call $G(R_\bullet)$ the *extended generator* of the pseudoresolvent (iii). It is easy to prove that, similarly to (5.4),

$$\text{(vi)} \qquad G(R_\bullet) = \{(x, y) \in X \times X : \text{ there exists } \lambda \in \Lambda \text{ such that } \lambda R_\lambda x - x = R_\lambda y\}.$$

I.3. THEOREM. *Let $\emptyset \neq \Lambda \subset \mathbb{K}$ and let G be a closed linear subspace of $X \times X$. Then the following two conditions are equivalent*:

(vii) *G is the extended generator of a pseudoresolvent with values in $L(X)$ defined on Λ,*

(viii) *for every $\lambda \in \Lambda$ and every $x \in X$ there exists exactly one $y \in X$ such that $(y, \lambda y - x) \in G$.*

Furthermore, if condition (viii) *is satisfied then the pseudoresolvent $\Lambda \ni \lambda \to R_\lambda \in L(X)$ with extended generator G is unique, and, for every $\lambda \in \Lambda$, $x \in X$ and $y \in X$,*

$$\text{(ix)} \qquad y = R_\lambda x \text{ if and only if } (y, \lambda y - x) \in G.$$

PROOF. STEP 1: $G = G(R_\bullet) \Rightarrow$ (ix). Suppose that $G = G(R_\bullet)$. Take any $\lambda \in \Lambda$ and $x \in X$. If $y = R_\lambda x$, then $\lambda R_\lambda y - y = R_\lambda(\lambda y - x)$, whence $(y, \lambda y - x) \in G$, according to (v). Conversely, if $(y, \lambda y - x) \in G$, then, again by (v), $\lambda R_\lambda y - y = R_\lambda(\lambda y - x)$, i.e. $y = R_\lambda x$.

STEP 2: $G = G(R_\bullet) \Rightarrow$ (viii). Indeed, according to Step 1, the equality $G = G(R_\bullet)$ implies (ix), and (ix) implies (viii).

STEP 3: *If condition* (viii) *is satisfied then for every $\lambda \in \Lambda$ there is exactly one operator $R_\lambda \in L(X)$ such that*

$$\text{(x)} \qquad G = \{(x, y) \in X \times X : \lambda R_\lambda x - x = R_\lambda y\}.$$

Indeed, if (viii) is satisfied, then for every $\lambda \in \Lambda$ there is a unique map $R_\lambda : X \to X$ such that

$$\text{(xi)} \qquad (R_\lambda x, \lambda R_\lambda x - x) \in G$$

for every $x \in X$. Since G is a closed linear subspace of $X \times X$, it follows that $R_\lambda : X \to X$ is linear and closed. Hence the closed graph theorem ([D-S;I], Sec. II.2, Theorem 4; [Y;2], Sec. II.6, Theorem 1) shows that $R_\lambda \in L(X)$. If $(x, y) \in G$ and $\lambda \in \Lambda$, then $(x, \lambda x - (\lambda x - y)) \in G$, whence, according to (xi), $R_\lambda(\lambda x - y) = x$, i.e. $\lambda R_\lambda x - x = R_\lambda y$. Conversely, if $\lambda R_\lambda x - x = R_\lambda y$, then $x = R_\lambda(\lambda x - y)$, whence, according to (xi), $(x, y) = (x, \lambda x - (\lambda x - y)) = (R_\lambda(\lambda x - y), \lambda R_\lambda(\lambda x - y) - (\lambda x - y)) \in G$. Thus the map $R_\lambda : X \to X$ defined by (xi) belongs to $L(X)$ and satisfies (x). Furthermore, for every $\lambda \in \Lambda$ the operator $R_\lambda \in L(X)$ satisfying (x) is unique. Indeed, if R_λ satisfies (x) then $(R_\lambda(\lambda x - y), \lambda R_\lambda(\lambda x - y) - (\lambda x - y)) = (x, y)$ for every $(x, y) \in G$, whence, by (viii), R_λ is uniquely determined on the set $\{\lambda x - y : (x, y) \in G\}$. Thus the uniqueness of R_λ satisfying (x) follows from the fact that if condition (viii) holds, then

$$\{\lambda x - y : (x, y) \in G\} = X \tag{xii}$$

for every $\lambda \in \Lambda$. For the proof of (xii) take any $\lambda \in \Lambda$ and $z \in X$. By (viii) there is $x \in X$ such that $(x, \lambda x - z) \in G$, so that, if $y = \lambda x - z$, then $(x, y) \in G$ and $\lambda x - y = z$.

Step 4: *The map $\Lambda \ni \lambda \to R_\lambda \in L(X)$ determined in Step 3 is a pseudoresolvent such that $G(R_\bullet) = G$.* Indeed, let $\lambda \in \Lambda$, $\mu \in \Lambda$ and $x \in X$. Define $y = R_\mu x$, $z = \mu R_\mu x - x$. Then $\mu R_\mu y - y = R_\mu z$, and hence, by Step 3, $(y, z) \in G$. Furthermore, since $(y, z) \in G$, again by Step 3, it follows that $\lambda R_\lambda y - y = R_\lambda z$. Hence $\lambda R_\lambda R_\mu x - R_\mu x = \lambda R_\lambda y - y = R_\lambda z = \mu R_\lambda R_\mu x - R_\lambda x$, so that $R_\lambda x - R_\mu x = (\mu - \lambda) R_\lambda R_\mu x$, proving that the map $\Lambda \ni \lambda \to R_\lambda \in L(X)$ is a pseudoresolvent. The equality $G(R_\bullet) = G$ follows now from (x) and (vi). ■

Remark. If G is a linear subspace of $X \times X$ and

$$D(G) = \{x \in X : \text{there is } y \in X \text{ such that } (x, y) \in G\},$$

then G may be treated as a multivalued operator with domain $D(G)$ which to every $x \in D(G)$ assigns the set

$$G(x) = \{y \in X : (x, y) \in G\}.$$

If $G = G(R_\bullet)$ is the extended generator of a pseudoresolvent (iii), then $D(G) = \Im$ and, as a consequence of (ix) and (v), $R_\lambda = (\lambda - G)^{-1}$ for every $\lambda \in \Lambda$, in the sense that

1° $\lambda R_\lambda x - G(R_\lambda x) = x + \mathcal{K}$ for every $x \in X$,
2° $R_\lambda(\lambda x - G(x)) = \{x\}$ for every $x \in \Im$.

Appendix II. Factorization theorem for representations of Banach algebras

Consider a Banach algebra A with left approximate unit bounded by a number $M \in [1, \infty)$, and a continuous representation T of A on a Banach space X. A *left approximate unit* for A is, by definition, a net $(e_\iota)_{\iota \in I} \subset A$ such that $\lim_\iota \|e_\iota a - a\|_A = 0$ for every $a \in A$. Boundedness by M means that $\|e_\iota\|_A \le M$ for every $\iota \in \Im$. Notice that an approximate unit cannot be bounded by a number strictly less than 1.

Let

$$T(A)X = \{T(a)x : a \in A,\ x \in X\},$$

and denote by $\operatorname{span} T(A)X$ the set of finite linear combinations of elements of $T(A)X$, and by $\overline{\operatorname{span}}\, T(A)X$ its closure in X. Since $(e_\iota)_{\iota \in I}$ is a left approximate unit for A, and the representation T is continuous, it follows that

$$(*) \qquad \lim_\iota \sup_{a \in B} \|e_\iota a - a\|_A = 0$$

for every finite subset B of A, and

$$\binom{*}{*} \qquad \lim_\iota \sup_{y \in C} \|T(e_\iota)y - y\|_X = 0$$

for every finite subset C of $\operatorname{span} T(A)X$. Since the left approximate unit $(e_\iota)_{\iota \in I}$ is bounded, the equalities $(*)$ and $\binom{*}{*}$ remain true for every compact subset B of A, and every compact subset C of $\overline{\operatorname{span}}\, T(A)X$.

LEMMA. *Let A be a Banach algebra with left approximate unit bounded by $M \in [1, \infty)$, and let T be a continuous representation of A on a Banach space X. For every $y \in \overline{\operatorname{span}}\, T(A)X$, every $\varepsilon > 0$, and every sequence $\delta_1, \delta_2, \ldots$ of strictly positive numbers, there is a sequence $e_1, e_2, \ldots$ of elements of A such that*:

(i) $\|e_n\|_A \le M$ *for every* $n = 1, 2, \ldots,$

(ii) $\|T(e_n)y - y\|_X < \delta_n$ *for every* $n = 1, 2, \ldots,$

(iii) $\|e_n e_{i_k} \ldots e_{i_1} - e_{i_k} \ldots e_{i_1}\|_A < \varepsilon / 2^{n-1}$ *whenever* $n = 2, 3, \ldots,$ $k = 1, \ldots, n-1$ *and* $1 \le i_1 < \ldots < i_k < n$,

(iv) $\|e_{i_k} \ldots e_{i_1}\|_A < M + \varepsilon$ *whenever* $k = 1, 2, \ldots$ *and* $1 \le i_1 < \ldots < i_k$.

PROOF. Suppose that $y \in \overline{\operatorname{span}}\, T(A)X$, $\varepsilon > 0$, and $\delta_n > 0$, $n = 1, 2, \ldots,$ are given. A sequence e_n, $n = 1, 2, \ldots,$ satisfying (i)–(iii) will be defined inductively. By $\binom{*}{*}$, there is $e_1 \in A$ such that $\|e_1\| \le M$ and $\|T(e_1)y - y\| < \delta_1$. If $n > 1$ and $e_1, \ldots, e_{n-1}$ are already defined, then $\{e_{i_k} \ldots e_{i_1} : k = 1, \ldots, n-1,\ 1 \le i_1 < \ldots < i_k < n\}$ is a finite subset of A, and hence, by $(*)$ and $\binom{*}{*}$, there exists $e_n \in A$ satisfying (i), (ii) and (iii). Property (iv) follows from (i) and (iii), because if $k \ge 2$ and $1 \le i_1 < \ldots < i_k$, then

$$\begin{aligned}\|e_{i_k} e_{i_{k-1}} \ldots e_{i_1}\| &\le \|e_{i_1}\| + \sum_{m=2}^{k} \|e_{i_m} e_{i_{m-1}} \ldots e_{i_1} - e_{i_{m-1}} \ldots e_{i_1}\| \\ &\le M + \sum_{m=2}^{k} \frac{\varepsilon}{2^{m-1}} < M + \varepsilon. \ \blacksquare\end{aligned}$$

THE FACTORIZATION THEOREM ([H-R;II], p. 268, Theorem 32.22; [P], p. 535, Theorem 5.2). *Let A be a Banach algebra with left approximate unit bounded by $M \in [1, \infty)$, and let T be a continuous representation of the algebra A on a Banach space X. Then*:

(I) *$T(A)X$ is a closed linear subspace of X, i.e.* $T(A)X = \overline{\operatorname{span}}\, T(A)X$,

(II) *for every* $y \in T(A)X$ *and every* $\delta > 0$ *there are* $a \in A$ *and* $x \in \overline{T(A)y}$ *such that* $\|a\|_A \le M$, $\|x - y\|_X \le \delta$ *and* $T(a)x = y$.

REMARK. Condition (II) is equivalent to

(III) *for every* $y \in T(A)X$ *and every* $\varepsilon > 0$ *there are* $a \in A$ *and* $x \in \overline{T(A)y}$ *such that* $\|a\|_A \le M + \varepsilon$, $\|x - y\|_X \le \varepsilon$, *and* $T(a)x = y$.

Indeed, obviously (II) implies (III). Conversely, if (III) holds, then, given $y \in T(A)X$ and $\delta > 0$ choose $\varepsilon > 0$ so small that $\varepsilon M^{-1}(\|y\| + M + \varepsilon) \le \delta$. By (III), there are $a \in A$ and $x \in \overline{T(A)y}$ such that $\|a\| \le M + \varepsilon$, $\|x - y\| \le \varepsilon$ and $T(a)x = y$. Define $\widetilde{a} = \frac{M}{M+\varepsilon}a$, $\widetilde{x} = \frac{M+\varepsilon}{M}x$. Then $T(\widetilde{a})\widetilde{x} = T(a)x = y$, $\|\widetilde{a}\| \le M$ and

$$\begin{aligned}\|\widetilde{x} - y\| \le \|\widetilde{x} - x\| + \|x - y\| &= \frac{\varepsilon}{M}\|x\| + \|x - y\| \\ &\le \frac{\varepsilon}{M}\|y\| + \frac{M+\varepsilon}{M}\|x - y\| \le \varepsilon M^{-1}(\|y\| + M + \varepsilon) \le \delta.\end{aligned}$$

Finally, notice that if A is a unital Banach algebra with unit e, then $[T(e)]^2 = T(e)$, so that $T(e)$ is a continuous projection of X onto its closed subspace $Y = T(e)X$, whence (I) $T(A)X = Y$ and (II) $y = T(e)y$ for every $y \in T(A)X$. Thus the factorization theorem is trivial for unital Banach algebras.

Proof of the factorization theorem. Suppose that the assumptions of the theorem are satisfied, the Banach algebra A being non-unital. Let A_{u} be the unitization of A ([H-R;I], p. 470, Theorem C.3; [P], pp. 18–20). This means that A_{u} is the unital Banach algebra such that:

1° as a linear space, A_{u} is equal to the direct sum $\mathbb{K} + A$, where $\mathbb{K}$ is the field of scalars of A,

2° $A_{\mathrm{u}} = \mathbb{K} + A$ is equipped with the norm $\| \ \|_{A_u}$ such that $\|\lambda + a\|_{A_u} = |\lambda| + \|a\|_A$ for every $\lambda \in \mathbb{K}$ and $a \in A$,

3° multiplication in A_{u} is defined by $(\lambda + a)(\mu + b) = \lambda\mu + (\mu a + \lambda b + ab)$, where $\lambda, \mu \in \mathbb{K}$ and $a, b \in A$.

The unit in A_{u} is $1 = 1 + 0 \in \mathbb{K} + A$. Let $\widetilde{T}$ be the continuous representation of the Banach algebra A_{u} on the Banach space X, such that

$$\widetilde{T}(\lambda + a) = \lambda 1 + T(a)$$

for every $\lambda + a \in A_{\mathrm{u}}$, where $1 \in L(X)$ is the identity operator.

Statement (I) of the factorization theorem will follow once we show that for every $y \in \overline{\mathrm{span}}\, T(A)X$ there exist $\theta \in (0,1)$ and a sequence $a_1, a_2, \ldots$ of elements of A such that the elements $b_n = (1-\theta)^n + a_n$ of A_{u} are invertible and both the limits below exist:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = a \in A \tag{1}$$

and

$$\lim_{n\to\infty} \widetilde{T}(b_n^{-1})y = x. \tag{2}$$

Indeed, since $b_n \in A_{\mathrm{u}}$ are invertible,

$$T(a_n)\widetilde{T}(b_n^{-1})y = \widetilde{T}(b_n)\widetilde{T}(b_n^{-1})y - (1-\theta)^n\widetilde{T}(b_n^{-1})y = y - (1-\theta)^n\widetilde{T}(b_n^{-1})y \tag{3}$$

for every $n = 1, 2, \ldots$, whence

$$T(a)x = y, \tag{4}$$

by (1) and (2), proving that $\overline{\operatorname{span}}\, T(A)X \subset T(A)X$. Moreover, statement (II), or (III), may be deduced from some additional properties of the elements a_n and $\widetilde{T}(b_n^{-1})y$.

The above idea of the proof goes back to P. Cohen [C], who used the formulas

$$b_n = (1-\theta)^n + a_n, \qquad a_n = \theta\sum_{k=1}^{n}(1-\theta)^{k-1}e_k, \tag{5}$$

with some $\theta \in (0,1)$ and $e_k \in \{e_\iota : \iota \in I\}$. See [C], the last line of p. 200, where $\theta = \gamma$. Formulas (5) are also used in the proofs of the factorization theorem presented in [H-R;II] and [P]. See [H-R;II], p. 266, Lemma 32.21, where $\theta = \frac{1}{2d+1}$; [P], p. 536, where $\theta = \frac{P}{2M} = \frac{1}{2M+1}$.

If the elements a_n are defined by (5), then the existence of the limit (1) with $\|a\|_A \le M$ is evident, but the proof of existence of the limit (2) is troublesome. We will use another construction of $a_n \in A$ and $b_n = (1-\theta)^n + a_n \in A_{\mathrm{u}}$, going back to M. Altman [A;1]–[A;3].

In order to prove statements (I) and (III), suppose that $y \in \overline{\operatorname{span}}\, T(A)X$ and $\varepsilon > 0$ are given. Fix any $\theta \in \left(0, \frac{1}{M+1}\right)$ and for every $n = 1, 2, \ldots$ define

$$\delta_n = \frac{\varepsilon}{\theta\|\widetilde{T}\|}\left(\frac{1-\theta(M+1)}{2}\right)^n. \tag{6}$$

Then take a sequence $e_1, e_2, \ldots$ of elements of A satisfying conditions (i)–(iv) of the Lemma, and for every $n = 1, 2, \ldots$ define

$$b_n = (1-\theta+\theta e_n)(1-\theta+\theta e_{n-1})\ldots(1-\theta+\theta e_1). \tag{7}$$

Then $b_n \in A_{\mathrm{u}}$ and

$$b_n = (1-\theta)^n + a_n, \tag{8}$$

where

$$a_n = \sum_{k=1}^{n}\theta^k(1-\theta)^{n-k}\Big(\sum_{1\le i_1<\ldots<i_k\le n} e_{i_k}e_{i_{k-1}}\ldots e_{i_1}\Big) \in A. \tag{9}$$

Since $\theta \in \left(0, \frac{1}{M+1}\right)$, from (i) it follows that $\|\theta e_n\|_A < \frac{M}{M+1} < 1-\theta$, so that

$$\left\|\frac{\theta}{1-\theta}e_n\right\|_A < 1$$

for every $n = 1, 2, \ldots$ As a consequence, for every $n = 1, 2, \ldots$ the element $1-\theta+\theta e_n = (1-\theta)\big(1 - \frac{\theta}{\theta-1} e_n\big)$ of A_{u} is invertible, and its inverse is the sum of the absolutely convergent series

$$(1-\theta+\theta e_n)^{-1} = \frac{1}{1-\theta}\left(1 + \frac{\theta}{\theta-1} e_n + \left[\frac{\theta}{\theta-1} e_n\right]^2 + \ldots\right),$$

so that

$$\begin{aligned} \|(1-\theta+\theta e_n)^{-1}\|_{A_{\mathrm{u}}} &\le \frac{1}{1-\theta}\left(1 + \frac{\theta M}{1-\theta} + \left[\frac{\theta M}{1-\theta}\right]^2 + \ldots\right) \\ &= \frac{1}{1-\theta} \cdot \frac{1}{1 - \frac{\theta M}{1-\theta}} = \frac{1}{1-\theta(M+1)}. \end{aligned}$$

As a consequence, every element $b_n \in A_{\mathrm{u}}$ has inverse $b_n^{-1} \in A_{\mathrm{u}}$ such that

$$\|b_n^{-1}\|_{A_{\mathrm{u}}} \le (1-\theta(M+1))^{-n}. \tag{10}$$

Existence of the limit (1) *and the inequality* $\|a\|_A \le M + \varepsilon$. According to (7), (8) and (9),

$$\begin{aligned} &b_{n+1} - b_n \\ &\quad = (1-\theta+\theta e_{n+1})b_n - b_n = \theta(e_{n+1}b_n - b_n) \\ &\quad = \theta(1-\theta)^n(e_{n+1}-1) + \theta\sum_{k=1}^{n} \theta^k(1-\theta)^{n-k}\Big(\sum_{1\le i_1<\ldots<i_k\le n} (e_{n+1}-1)e_{i_k}\ldots e_{i_1}\Big), \end{aligned}$$

whence, by (i) and (iii),

$$\begin{aligned} \|b_{n+1} - b_n\|_{A_{\mathrm{u}}} &\le \theta(1-\theta)^n(M+1) + \theta\sum_{k=1}^{n}\theta^k(1-\theta)^{n-k}\binom{n}{k}\frac{\varepsilon}{2^n} \\ &< \theta(M+1)(1-\theta)^n + \theta\frac{\varepsilon}{2^n}. \end{aligned}$$

Since $\theta \in \big(0, \frac{1}{M+1}\big)$ it follows that the series $\sum_{n=1}^{\infty}\|b_{n+1}-b_n\|_{A_{\mathrm{u}}}$ is convergent, and hence both the sequences $b_1, b_2, \ldots$ and $a_1, a_2, \ldots$ converge to the common limit a. Furthermore, from (9) and (iv) it follows that

$$\|a_n\|_A \le \sum_{k=1}^{n}\theta^k(1-\theta)^{n-k}\binom{n}{k}(M+\varepsilon) < M + \varepsilon$$

for every $n = 1, 2, \ldots$, whence $\|a\|_A \le M + \varepsilon$.

Existence of the limit (2) *with* $x \in \overline{T(A)y}$ *and* $\|x-y\|_X \le \varepsilon$. Define $x_0 = y$, $x_n = \widetilde{T}(b_n^{-1})y$ for $n = 1, 2, \ldots$ Then, according to (7),

$$x_n - x_{n-1} = \widetilde{T}(b_n^{-1})[y - \widetilde{T}(1-\theta+\theta e_n)y] = \theta\widetilde{T}(b_n^{-1})[y - T(e_n)y],$$

so that, by (10), (ii) and (6),

$$\|x_n - x_{n-1}\|_X \le \theta\|\widetilde{T}\|(1-\theta(M+1))^{-n}\delta_n = \frac{\varepsilon}{2^n}$$

for every $n = 1, 2, \ldots$ It follows that the limit (2) exists, and

$$\|x - y\|_X \le \sum_{n=1}^{\infty} \|x_n - x_{n-1}\|_X \le \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Furthermore, for every $n = 1, 2, \ldots$, $b_n^{-1} = \lambda_n + c_n$, where $\lambda_n \in \mathbb{K}$ and $c_n \in A$, so that $x_n = \lambda_n y + T(c_n)y \in \overline{T(A)y}$, because $T(c_n)y \in T(A)y$ and $y \in \overline{T(A)y}$ by (ii), or $\binom{*}{*}$. It follows that $x = \lim_{n\to\infty} x_n \in \overline{T(A)y}$.

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