From Stäckel systems to Painlevé hierarchies

Maciej Błaszak
Faculty of Physics, Department of Mathematical Physics and Computer Modelling, Adam Mickiewicz University, Poznań, Poland
`blaszakm@amu.edu.pl`

Among all second order nonlinear integrable ordinary differential equations (ODE's) there are two distinguished classes, playing important roles in modern mathematics and physics. The first class is represented by nonlinear equations of *Stäckel-type*, with an autonomous Hamiltonian representation on a symplectic manifold

$$\frac{d\xi}{dt_r} = X_r(\xi) \equiv \pi dh_r(\xi), \quad r = 1, \ldots, n, \tag{1}$$

which are Frobenius integrable (also known as Liouville integrable in this case)

$$[X_r, X_s] = 0, \quad r, s = 1, \ldots, n.$$

Moreover, the autonomous equations (1) are represented by the isospectral Lax equations

$$\frac{d}{dt_k} L(\lambda; \xi) = [U_k(\lambda; \xi), L(x; \xi)], \quad k = 1, \ldots n. \tag{2}$$

The second class is represented by nonlinear ordinary differential equations of *Painlevé-type*, with a non-autonomous Hamiltonian representation

$$\frac{d\xi}{dt_r} = Y_r(\xi, t) = \pi dH_r(\xi, t), \quad r = 1, \ldots, n, \tag{3}$$

where $t = (t_1, \ldots, t_n)$. The set of n equations (3) constitutes a non-autonomous Painlevé-type system if is Frobenius integrable

$$\frac{\partial X_s}{\partial t_r} - \frac{\partial X_r}{\partial t_s} + \{X_r, X_s\} = 0, \quad r, s = 1, \ldots, n \tag{4}$$

and the system is represented by the isomonodromic Lax representation

$$\frac{d}{dt_k} L(x; \xi, t) = [U_k(x; \xi, t), L(x; \xi, t)] + \frac{\partial U_k(x; \xi, t)}{\partial x}, \quad k = 1, \ldots, n. \tag{5}$$

In this lecture we present a systematic deformation of autonomous Stäckel-type systems to non-autonomous Painlevé-type hierarchies. In particular we construct the infinite hierarchies of Painlevé I (P_I), Painlevé II (P_{II}), Painlevé III (P_{III}) and Painlevé IV (P_{IV}).

[1] Błaszak M., Marciniak K., Sergyeyev A., *Deforming Lie algebras to Frobenius integrable non-autonomous Hamiltonian systems*, Rep Math Phys **87** (2021) 249-263

[2] Błaszak M., Marciniak K., Domański Z., *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. I. Frobenius integrability*, Stud. Appl. Math. **148** (2022) 1208-1250

[3] Błaszak M., Marciniak K., Domański Z., *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. II. Isomonodromic Lax representation*, Stud. Appl. Math. **149** (2022) 364-415

[4] Błaszak M., *Multi-component Painlevé ODEs and related non-autonomous KdV stationary hierarchies*, Stud. Appl. Math. **151** (2023) 5-34