Discrete Time Portfolio Selection with Proportional Transaction Costs

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Abstract

In the paper discrete time portfolio selection with maximization of a discounted satisfaction functional is studied. In section 2 the case without transaction costs is considered and explicit solutions for special satisfaction functions are given. In section 3 the problem with proportional transaction costs is investigated and optimal strategies are characterized.

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1 Introduction

Suppose on a given probability space (Ω, F, P) there is a sequence (ξ_n) of i.i.d. random variables taking values from the interval $[-1, \infty)$. Consider a market with two investment possibilities: in bank with a constant nonnegative deterministic rate of return r or in stock with random rate of return ξ_n at time n, assuming furthermore that ξ_n are not concentrated in a point and $\infty > E\xi_n \geq r$. Denote by x_n the total capital (wealth) we have at time n and by c_n the consumption at time n. Clearly $0 \leq c_n \leq x_n$. The remaining capital $x_n - c_n$ we invest in bank or in stock. Let b_n be a part of the remaining capital invested in bank. Then starting with an initial capital $x_0 = x$ we have the following recursive formula

$$x_{n+1} = [b_n(1+r) + (1-b_n)(1+\xi_n)](x_n - c_n) \tag{1}$$

or equivalently

$$x_{n+1} = [1 + b_n r + (1-b_n)\xi_n](x_n - c_n). \tag{2}$$

With consumption c_n is associated a certain satisfaction $g(c_n)$, where g is a given satisfaction function.

The portfolio selection strategy is characterized by a sequence $u = (b_n, c_n)$ consisting of parts b_n of capital invested in bank and consumption c_n at time n. The purpose is to maximize so called discounted (with discount factor γ, $0 < \gamma < 1$) satisfaction

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$$J_x(u) = E_x^u \left\{ \sum_{i=0}^{\infty} \gamma^i g(x_i) \right\}. \tag{3}$$

The problem formulated above was considered first for a finite horizon with $g(c) = c^\alpha$, $0 < \alpha < 1$ in [?]. Then continuous time diffusion model was studied in [?]. The case with transaction costs was investigated for continuous time diffusion model with $g(c) = c^\alpha$ or $g(c) = lnc$ in [?] and [?]. In this paper we present a discrete time counterpart of [?] and [?]. The advantage of discrete time approach is that we allow a general rate of return and still obtain the characterization of optimal strategies in a form of no transaction cone as in continuous time.

2 The case without transaction costs

We consider in this section the case with no transaction costs.
Define the value function

$$w(x) := \sup_u J_x(u) \tag{4}$$

with supremum over all admissible control strategies $u = (b_n, c_n)$.
Let

$$\mathcal{W} = \left\{ f : [0, \infty) \mapsto R \text{ continuous such, that } ||f||_{\mathcal{W}} := \sup_{x \in [0,\infty)} \frac{|f(x)|}{x+1} < \infty \right\}. \tag{5}$$

We have

Theorem 1 *For $g \in \mathcal{W}$ and γ such that $\gamma(1 + E\xi_1) < 1$ there is a unique function $w \in \mathcal{W}$ satisfying the following Bellman equation*

$$w(x) = \sup_{c \in [0,x]} \left\{ g(c) + \gamma \sup_{b \in [0,1]} E \left\{ w(((1+r)b + (1-b)(1+\xi_1))(x-c)) \right\} \right\}. \tag{6}$$

Furthermore w coincides with the value function defined in (4) and the optimal strategies $(\hat{b}_n, \hat{c}_n)$ are of the form $\hat{b}(x_n)$ and $\hat{c}(x_n)$ with $\hat{b}$, $\hat{c}$ being selectors for which the suprema in (6) are achieved.

Proof. For $f \in \mathcal{W}$ define the operator

$$T_\gamma f(x) := \sup_{c \in [0,x]} \left\{ g(c) + \gamma \sup_{b \in [0,1]} E \left\{ f(((1+r)b + (1-b)(1+\xi_1))(x-c)) \right\} \right\}. \tag{7}$$

Since for each $x \geq 0$, $f(x) \leq ||f||_{\mathcal{W}}(x+1)$, and by assumption $E\xi < \infty$, we clearly have that $T_\gamma f \in \mathcal{W}$. Furthermore using the fact that $E\xi_1 \geq r$ we easily obtain that T_γ is a contraction operator with constant $\gamma(1 + E\xi_1)$ in the space $\mathcal{W}$. Consequently by Banach contraction principle there is a unique in $\mathcal{W}$ solution w to the equation (6). Iterating (6) we obtain that

$$w(x) = \sup_u E_x^u \left\{ \sum_{i=0}^{n-1} \gamma^i g(c_i) + \gamma^n w(x_n) \right\}.$$

Since $\gamma^n|E_x^u w(x_n)| \leq \gamma^n E_x^u(1+x_n) \leq \gamma^n(1+E\xi)^n x \to 0$, as $n \to 0$, we obtain (4). The remaining part of the proof follows in a standard way from the proof of Theorem 4.2.3 of [?].

□

For a particular form of function g we obtain the solution to the equation (6) in an explicit form.

Proposition 1 *If $g(c) = c^\alpha$ with $0 < \alpha < 1$ and $\gamma < \frac{1}{\hat{\lambda}}$, where*

$$\hat{\lambda} := \sup_{b \in [0,1]} E\left\{((1+r)b + (1-b)(1+\xi_1))^\alpha\right\} \tag{8}$$

the function w defined in (4) is a solution to (6) and is of the form

$$w(x) = x^\alpha \left(\frac{1}{1-(\gamma\hat{\lambda})^{\frac{1}{1-\alpha}}}\right)^{1-\alpha} \tag{9}$$

Moreover optimal strategies are $(\hat{b}_n, \hat{c}_n)$

$$\hat{b}_n = \hat{b}, \quad \hat{c}_n = (1-(\gamma\hat{\lambda})^{\frac{1}{1-\alpha}})x_n, \tag{10}$$

where $\hat{b}$ is a value of $0 \leq b \leq 1$ for which the supremum in the definition of $\hat{\lambda}$ is achieved.

If $g(c) = \ln c$, the function w defined in (4) is for $x > 0$ a solution to (6) for any $0 < \gamma < 1$ and is of the form

$$w(x) = \frac{1}{1-\gamma}\ln x + \bar{\lambda}\frac{\gamma}{(1-\gamma)^2} + \frac{\gamma}{(1-\gamma)^2}\ln\gamma + \frac{1}{1-\gamma}\ln(1-\gamma), \tag{11}$$

where

$$\bar{\lambda} = \sup_{b \in [0,1]} E\left\{ln((1+r)b + (1-b)(1+\xi_1))\right\}, \tag{12}$$

Furthermore optimal strategies are

$$\hat{b}_n = \hat{b}, \quad \hat{c}_n = \frac{1}{1+\gamma}x_n \tag{13}$$

with $\hat{b}$ being this time the value of $b \in [0,1]$ for which supremum in the definition of $\bar{\lambda}$ is achieved.

Proof. Consider succesive iterates $T_\gamma^n h$ with T_γ defined in (7) corresponding to function $g(c) = c^\alpha$ and function $h \equiv 0$. Clearly $T_\gamma^n h(x)$ is a nondecreasing sequence. Moreover one can show that $T_\gamma h(x) = x^\alpha$ and for $n = 1, \ldots,$

$$T_\gamma^{n+1} h(x) = x^\alpha (1 + \sum_{i=0}^n (\gamma\hat{\lambda})^{\frac{i}{1-\alpha}})^{1-\alpha}. \tag{14}$$

Therefore when $\gamma < \frac{1}{\hat{\lambda}}$ the sequence $T_\gamma^n h(x)$ converges to $w(x)$ of the form (9) and w is a solution to (6). Since for the strategy (10) we have $0 \leq \gamma^n E_x w(x_n) = \gamma^n \hat{\lambda}^n (\gamma\hat{\lambda})^{\frac{n\alpha}{1-\alpha}} w(x) \to$

0, as $n \to \infty$, while for any other strategy u, $w(x) \geq J_x(u)$, using the arguments of the proof of Theorem 4.2.3 of [?] we obtain that the strategies $(\hat{b}_n, \hat{c}_n)$ given in (10) are optimal and w defined in (6) coincides with that in (9).

In the case when $g(c) = \ln c$, the corresponding iterations of the operator T_γ with initial function $h \equiv 0$ form a sequence and

$$T_\gamma^n h(x) = a_n \ln x + d_n \bar{\lambda} + e_n, \tag{15}$$

where $a_1 = 1$, $a_{n+1} = 1 + \gamma a_n$, $d_1 = 0$, $d_{n+1} = \gamma(d_n + a_n)$, $e_1 = 0$, $e_{n+1} = \gamma e_n + \ln \frac{(\gamma a_n)^{\gamma a_n}}{(a_n+1)^{a_n+1}}$, for $n = 1, \ldots$.
Consequently $a_n = \frac{1-\gamma^n}{1-\gamma}$, $d_n = \sum_{i=1}^{n-1} i\gamma^i$ and $e_n = \sum_{i=1}^{n-1} \gamma^{n-i-1} \ln \frac{(\gamma a_i)^{\gamma a_i}}{(a_i+1)^{a_i+1}}$. Therefore letting $n \to \infty$ we obtain that $a_n \to \frac{1}{1-\gamma}$, $d_n \to \frac{\gamma}{(1-\gamma)^2}$ and $e_n \to \frac{\gamma}{(1-\gamma)^2} \ln \gamma + \frac{1}{1-\gamma} \ln(1-\gamma)$, from which the form (11) of the limit function w follows. By direct substitution one can verify that w defined in (11) is a solution to the equation (6). Moreover for the strategies defined in (13)

$$\gamma^n E_x w(x_n) = \gamma^n n \bar{\lambda} + \gamma^n n \frac{\gamma}{1+\gamma} \frac{1}{1-\gamma} + \gamma^n w(x) \to 0,$$

as $n \to \infty$, and for any n, $E_x^u \left\{ \sum_{i=0}^n \gamma^i g(c_i) \right\} \leq T_\gamma^n h(x)$, so that since $T_\gamma^n h(x) \to w(x)$ we have $w(x) \geq J_x(u)$. Using the arguments of the proof of Theorem 4.2.3 of [?] again, we verify that w is also the value function defined in (4).

□

Remark 1 *It should be pointed out that in the case $g(c) = \ln c$ the solution w to (6) exists for any $\gamma \in (0,1)$ although neither the function $\ln c$ nor w is in the class $\mathcal{W}$. When $g(c) = c^\alpha$ to have a solution to the Bellman equation (6) in the class $\mathcal{W}$, γ should be less than $\frac{1}{\bar{\lambda}}$, which is a weaker assumption than that from Theorem 1. In both cases the optimal strategies (10) and (13) are such that we have to consume a fixed (independently of time) part of the capital (wealth) and the part of the remaining capital invested in bank also should be fixed as an argument of supremum either in (8) or in (12). The fact that optimal b_n is constant follows from separability of $(x - c)$ and $((1 + r)b + (1 - b)\xi_1)$ under expectation sign on the right hand side of (6), which happens when either $w(xy) = w(x)w(y)$ or $w(xy) = w(x) + w(y)$ for $x, y \geq 0$. Studying the so called Cauchy equation (see e.g. [?] Theorem 14.4) one can show that kx^α and $k_1 \ln(k_2 xy)$ are the only functions with above property. Consequently one can expect that the satisfaction functions considered in Proposition 1 are the only ones for which optimal strategies b_n are constant.*

3 The case with proportional transaction costs

Assume now that investing in stock an amount x we have to pay λx ($\lambda \in (0,1)$) transaction costs so that in fact we have to spend $(1 + \lambda)x$. Similarly selling actions for an amount x we obtain only $(1 - \mu)x$ with $\mu \in (0,1)$. Let for $x, y \geq 0$

$$R(x,y) := \begin{cases} x + (1-\mu)y & \text{if } y \geq 0 \\ x + (1+\lambda)y & \text{if } y < 0 \end{cases} \tag{16}$$

and

$$\mathcal{C} := \left\{(x,y) \in R^2 : R(x,y) \geq 0\right\}. \tag{17}$$

Denoting by the pair $(x,y) \in R^2$ the position of an investor with amounts x in bank and y in stock respectively, we see that if $(x,y) \in \mathcal{C}$ he is able to repay possible debts in bank or stock, and therefore we shall call $\mathcal{C}$ a nonbankruptcy cone. Let l_n, m_n be the amounts for which we increase or decrease the stock position at time n. In other words at time n we sell actions for m_n and receive in banking account $(1-\mu)m_n$ and buy actions for the amount l_n paying from our banking account $(1+\lambda)l_n$. Assume we additionally consume the amount c_n at time n. Starting with an initial capital $(x_0, y_0) = (x,y)$ consisting from the amount x on banking account and y in stock we obtain the following recursive formulae for our investment position (x_n, y_n) at time n

$$x_{n+1} = (1+r)(x_n + (1-\mu)m_n - (1+\lambda)l_n - c_n) \tag{18}$$

$$y_{n+1} = (1+\xi_n)(y_n - m_n + l_n) \tag{19}$$

Consequently our investment strategy u now consists of a sequence of three terms (l_n, m_n, c_n). Given a satisfaction function $g \in \mathcal{W}$, discount factor γ and initial position (x,y) we have the following analog of discounted satisfaction functional (3)

$$J_{xy}(u) = E^u_{xy}\left\{\sum_{i=0}^{\infty}\gamma^i g(c_i)\right\}. \tag{20}$$

In what follows we shall consider only initial positions (x,y) in the nonbankruptcy cone $\mathcal{C}$, and our portfolio selection strategy $u = (l_n, m_n, c_n)$ will be admissible if and only if

$$c_n \in [0, R(x_n, y_n)] \quad \text{and} \quad (l_n, m_n) \in \mathcal{A}(x_n, y_n, c_n) \tag{21}$$

with

$$\begin{aligned}&\mathcal{A}(x,y,c) := \{(l,m) \in [0,\infty) \times [0,\infty) : \forall_{\xi \in supp\xi_1} \\ &R\left(\, (1+r)(x + (1-\mu)m - (1+\lambda)l - c), (1+\xi)(y+l-m)\,\right) \geq 0\,\},\end{aligned} \tag{22}$$

where by $supp\xi_1$ we denote the support of the measure generated by the random variable ξ_1.

In other words we consume at time n not more than we owe on the banking account after liquidation of the stock account (paying possible debts there) and the remaining capital $(x_n - c_n, y_n)$ we invest in such way that for each change of the stock prices at the moment $n+1$ we shall stay in the nonbankruptcy cone $\mathcal{C}$.

By analogy to (4) let

$$w(x,y) := \sup_u J_{xy}(u) \tag{23}$$

with supremum considered over all admissible $u = (l_n, m_n, c_n)$.

Bellman equation corresponding to the cost functional (20) is now of the form

$$w(x,y) = \sup_{c\in[0,R(x,y)]} \Big\{ g(c) + \gamma \sup_{(l,m)\in\mathcal{A}(x,y,c)}$$
$$E\,\{\; w\,(\;(1+r)(x+(1-\mu)m-(1+\lambda)l-c),(1+\xi_1)(y+l-m)\;)\;\}\Big\}\,. \tag{24}$$

We shall look for a solution to the equation (24) studying the operator $\bar{T}_\gamma$ in a certain space $\bar{\mathcal{W}}$ defined respectively as follows

$$\bar{T}_\gamma f(x,y) := \sup_{c\in[0,R(x,y)]} \Big\{ g(c) + \gamma \sup_{(l,m)\in\mathcal{A}(x,y,c)}$$
$$E\,\{\; f\,(\;(1+r)(x+(1-\mu)m-(1+\lambda)l-c),(1+\xi_1)(y+l-m)\;)\;\}\;\Big\}\,. \tag{25}$$

and

$$\bar{\mathcal{W}} := \left\{ f \in C(\mathcal{C}) : \|f\|_{\bar{R}} := \sup_{(x,y)\in\mathcal{C}} \frac{|f(x,y)|}{1+\bar{R}(x,y)} < \infty \right\}, \tag{26}$$

where $C(\mathcal{C})$ is the space of continuous bounded functions on $\mathcal{C}$ and

$$\bar{R}(x,y) := \begin{cases} R(x,y) & \text{if} \quad x \geq 0 \\ R(x,y) + x^- & \text{if} \quad x < 0 \end{cases} \tag{27}$$

with $x^- = \max\{0,-x\}$.

Denote by z_1, z_2 the lower and upper bounds of the support of the measure generated by ξ_1 respectively. We shall need the following auxiliary result that characterizes orthogonal pairs (l,m) i.e. such pairs for which $lm = 0$, in the set $\mathcal{A}(x,y,0)$

Lemma 1 *For $(x,y) \in \mathcal{C}$ the set $\mathcal{A}(x,y,0)$ consists of the following orthogonal pairs (l,m):*
i) if $l = 0$, $y < m$ and $y \geq 0$ we have $m \in (y, y + \frac{(1+r)(x+(1-\mu)y)}{(1+\lambda)(1+z_2)-(1+r)(1-\mu)}]$,
ii) if $l = 0$, $y < m$ and $y < 0$ we have $m \in [0, y + \frac{(1+r)(x+(1-\mu)y)}{(1+\lambda)(1+z_2)-(1+r)(1-\mu)}]$ only when additionaly $x \geq \frac{-(1+\lambda)(1+z_2)y}{1+r}$, in the case when $x < \frac{-(1+\lambda)(1+z_2)y}{1+r}$ there are no admissible m,
iii) if $l = 0$, $y \geq m$ we have $m \in [-\frac{(1+r)x+(1-\mu)(1+z_1)y}{(1-\mu)(r-z_1)}, y]$ when $x \leq -\frac{(1-\mu)(1+z_1)y}{1+r}$ and $m \in [0,y]$ whenever $x \geq -\frac{(1-\mu)(1+z_1)y}{1+r}$,
iv) if $m = 0$ and $y + l < 0$ we have $l \in [-\frac{(1+r)x+(1+\lambda)(1+z_2)y}{(1+\lambda)(z_2-r)}, -y)$ when $x \leq -\frac{(1+\lambda)(1+z_2)y}{1+r}$ and $l \in [0,-y)$ whenever $x \geq -\frac{(1+\lambda)(1+z_2)y}{1+r}$,
v) if $m = 0$ and $y + l \geq 0$ we have $l \in [max\{-y,0\}, \frac{(1+r)x+(1-\mu)(1+z_1)y}{(1+r)(1+\lambda)-(1+z_1)(1-\mu)}]$.

Proof. Notice that the following system of inequalities should be satisfied

$$R\,((1+r)(x+(1-\mu)m-(1+\lambda)l),(1+z_1)(y+l-m)) \geq 0$$
$$R\,((1+r)(x+(1-\mu)m-(1+\lambda)l),(1+z_2)(y+l-m)) \geq 0.$$

Letting $l = 0$ or $m = 0$ in the above inequalities by direct calculations we obtain the result. □

Remark 2 *Notice that after possible transactions at each moment of time we should be in the cone*

$$\bar{\mathcal{C}} = \left\{ (x,y) \in \mathcal{C} : y \geq -\frac{1+z_1}{1+r}\frac{1}{1+\lambda}x, \ \ and \ \ y \geq -\frac{1+z_2}{1+r}\frac{1}{1-\mu}x \right\}$$

since otherwise we leave $\mathcal{C}$ with positive probability.

Given functions $f_1, f_2 \in \bar{\mathcal{W}}$ we have that

$$\begin{aligned} & \frac{|\bar{T}_\gamma f_1(x,y) - \bar{T}_\gamma f_2(x,y)|}{1+\bar{R}(x,y)} \leq \sup_{c\in[0,R(x,y)]} \gamma \sup_{(l,m)\in\mathcal{A}(x,y,c)} \\ & \frac{|E\left\{ (f_1 - f_2)\left((1+r)(x+(1-\mu)m-(1+\lambda)l-c),(1+\xi_1)(y+l-m) \right) \right\}|}{1+\bar{R}(x,y)} \\ \leq \;\; & \gamma \|f_1 - f_2\|_{\bar{R}} \frac{1+E\left\{ \bar{R}\left((1+r)(x+(1-\mu)m-(1+\lambda)l-c),(1+\xi_1)(y+l-m) \right) \right\}}{1+\bar{R}(x,y)} \\ := \;\; & \gamma \|f_1 - f_2\|_{\bar{R}} \sup_{c\in[0,R(x,y)]} \sup_{(l,m)\in\mathcal{A}(x,y,c)} M_{xy}(l,m,c), \end{aligned} \tag{28}$$

where we implicitely have defined $M_{xy}(l,m,c)$. Using Lemma 1 we obtain

Lemma 2 *We have*

$$\sup_{c\in[0,R(x,y)]} \sup_{(l,m)\in\mathcal{A}(x,y,c)} M_{xy}(l,m,c) \leq L \tag{29}$$

with $L = \max\left\{ 1+E\xi_1, \frac{1+r}{r-z_1}\max\left\{ 1+E\xi_1, E\xi_1 - z_1 \right\} \right\}$.

Proof. Since $M_{xy}(l,m,c)$ is nonincreasing in c, the maximum over c is attained for $c=0$. Furthermore, one can easily see that

$$\sup_{(l,m)\in\mathcal{A}(x,y,0)} M_{xy}(l,m,0)$$

is attained for orthogonal pairs (l,m). Therefore we use Lemma 1 to obtain the following bounds B for $\sup_{(l,m)\in\mathcal{A}(x,y,0)} M_{xy}(l,m,0)$:
i) when $l=0$, $y<m$ and $y\geq 0$, $B=1+r$,
ii) when $l=0$, $y<m$ and $y<0$, also $B=1+r$,
iii) when $l=0$, $y\geq m$ we consider four subcases:
  a)if $x+(1-\mu)m<0$ and $x\leq -\frac{(1-\mu)(1+z_1)y}{1+r}$, $B=\max\left\{1,\frac{(1+r)(1+E\xi_1)}{r-z_1}\right\}$,
  b) if $x+(1-\mu)m<0$ and $x\geq -\frac{(1-\mu)(1+z_1)y}{1+r}$, $B=1+E\xi_1$,
  c) if $x+(1-\mu)m\geq 0$ and $x\leq -\frac{(1-\mu)(1+z_1)y}{1+r}$, $B=\max\left\{1,\frac{(1+r)(E\xi_1-z_1)}{r-z_1}\right\}$
  d) if $x+(1-\mu)m\geq 0$ and $x\geq -\frac{(1-\mu)(1+z_1)y}{1+r}$, $B=1+E\xi_1$,
iv) when $m=0$ and $y+l<0$, $B=1+r$,
v) when $m=0$ and $y+l\geq 0$ we have the following subcases:
  a) if $x-(1+\lambda)l<0$ and $y\leq 0$, $B=\max\left\{1,\frac{(1+E\xi_1)(1+r)(1-\mu)}{(1+r)(1+\lambda)-(1+z_1)(1-\mu)}\right\}$,
  b) if $x-(1+\lambda)l<0$ and $y>0$, $B=\max\left\{1,\frac{(1+E\xi_1)(1+r)(1+\lambda)}{(1+r)(1+\lambda)-(1+z_1)(1-\mu)}\right\}$,

c) if $x-(1+\lambda)l \geq 0$ and $(1-\mu)(1+E-\xi_1) \geq (1+r)(1+\lambda)$,

$$B = \max\left\{1, \frac{(E\xi_1 - z_1)(1+r)(1-\mu)}{(1+r)(1+\lambda)-(1+z_1)(1-\mu)}\right\},$$

d) if $x-(1+\lambda)l \geq 0$, $(1-\mu)(1+E-\xi_1) < (1+r)(1+\lambda)$ and $y<0$, $B = 1+r$,
e) if $x-(1+\lambda)l \geq 0$, $(1-\mu)(1+E-\xi_1) < (1+r)(1+\lambda)$ and $y \geq 0$, $B = 1+E\xi_1$.
Noticing that

$$\frac{(1+E\xi_1)(1+r)(1+\lambda)}{(1+r)(1+\lambda)-(1+z_1)(1-\mu)} \leq \frac{(1+E\xi_1)(1+r)}{r-z_1},$$

we obtain (29).

□

Remark 3 *Notice that we can not consider $1+R(x,y)$ as the weight in the norm $\|\cdot\|_{\bar{R}}$, since then we have a problem to evaluate in Lemma 2 the value $(1+r)x+(1-\mu)(1+E\xi_1)y+1$, when $x<0$, in terms of a constant multiplied by $1+R(x,y)$.*

We are now in position to formulate one of the main results of this section.

Theorem 2 *For $g \in \mathcal{W}$ and*

$$\gamma < \min\left\{\frac{1}{1+E\xi_1}, \frac{r-z_1}{1+r}\min\left\{\frac{1}{1+E\xi_1}, \frac{1}{E\xi_1 - z_1}\right\}\right\} \tag{30}$$

there is a unique solution $w \in \bar{\mathcal{W}}$ to the Bellman equation (24). Moreover w is a value function (23) corresponding to the cost functional (20). Furthermore given selectors $c : \mathcal{C} \mapsto [0, R(x,y)]$, $l : \mathcal{C} \times [0, R(x,y)] \mapsto [0,\infty)$ and $m : \mathcal{C} \times [0, R(x,y)] \mapsto [0,\infty)$ for which the suprema on the right hand side of (24) are attained, the strategy $\hat{c}_n = c(x_n, y_n)$, $\hat{l}_n = l(x_n, y_n, \hat{c}_n)$, $\hat{m}_n = m(x_n, y_n, \hat{c}_n)$ is optimal.

Proof Notice first that since $\mathcal{A}(x,y,c) = \mathcal{A}(x-c,y,0)$ the multivalued function $(x,y) \to \{(c,l,m) : c \in [0,R(x,y)], (l,m) \in \mathcal{A}(x,y,c)\}$ is continuous in the Haussdorf topology so that by D6(b) of [**?**] for continuous bounded w the function $\bar{T}w$ is also continuous. By (28) and (29) we see that for γ satisfying (30) the operator $\bar{T}_\gamma$ is a contraction in $\bar{\mathcal{W}}$. Therefore the is a unique solution to (23) in the space $\bar{\mathcal{W}}$. Moreover by the definition of $\bar{\mathcal{W}}$ and Lemma 2

$$|E_{xy}\left[\gamma^n w(x_n, y_n)\right]| \leq \gamma^n \|w\|_{\bar{R}} E_{xy}\left[1+\bar{R}(x_n,y_n)\right]$$

$$\leq \gamma^n \|w\|_{\bar{R}} E_{xy}\left[1+\bar{R}(x_{n-1},y_{n-1})\right] L \leq \gamma^n L^n \|w\|_{\bar{R}} \bar{R}(x,y) \to 0$$

as $n \to \infty$. The remaining part of the proof follows from the proof of Theorem 4.2.3 of [**?**].

□

From Lemma 2 for a particular form of the noise ξ_n we obtain

Corollary 1 *If the lower bound z_1 of the support of the measure generated by random variable ξ_1 is equal to -1 then the assertions of Theorem 2 are true for $\gamma < \frac{1}{1+E\xi_1}$. Furthermore for each admissible strategy we have that $x_n \geq 0$.*

Proof. The first part of Corollary immediately follows from Lemma 2. For the second part notice only that the admissible strategy to have (x_n, y_n) in the nonbankruptcy cone $\mathcal{C}$ should not allow x_n be negative.

$\square$

Next two Corollaries characterize the properties of the value function w defined in (23) using Bellman equation (24)

Corollary 2 *Under the assumptions of Theorem 2, if $g \in \mathcal{W}$ is nondecreasing, the function w is also nondecreasing with respect both x and y variable i.e. for $h > 0$ we have*

$$w(x+h, y) \geq w(x, y) \quad \text{and} \quad w(x, y+h) \geq w(x, y).$$

Proof. The function w can be obtained as a limit of the nondecreasing sequence $\bar{T}_\gamma^n h$ with $h \equiv 0$. One can show inductively that $\bar{T}_\gamma^n h$ in nondecreasing with respect to both variables for each $n = 1, 2, \ldots$. Consequently the above property inherits also w.

$\square$

Corollary 3 *Under the assumptions of Theorem 2, if $g \in \mathcal{W}$ is nondecreasing and convex then w is convex i.e. given $\beta \in [0,1]$ and $(x_1, y_1), (x_2, y_2) \in \mathcal{C}$ we have that*

$$w(\beta x_1 + (1-\beta)x_2, \beta y_1 + (1-\beta)y_2) \geq \beta w(x_1, y_1) + (1-\beta)w(x_2, y_2).$$

Proof. By convexity of R and monotonicity and convexity of g, we have that $\bar{T}_\gamma h(x,y) = g(R(x,y))$ with $h \equiv 0$ is convex. Assuming inductively that $\bar{T}_\gamma^n h$ is convex, let l_1, m_1, c_1 and l_2, m_2, c_2 be the strategies for which suprema in $\bar{T}_\gamma^{n+1} h(x_1, y_1)$, $\bar{T}_\gamma^{n+1} h(x_2, y_2)$ are attained respectively. Then

$$\begin{aligned}
&g(\beta c_1 + (1-\beta)c_2) + \gamma E\left\{\bar{T}_\gamma^n h\left((1+r)(\beta x_1 + (1-\beta)x_2 + (1-\mu)(\beta m_1 + (1-\beta)m_2)\right.\right. \\
&-(1+\lambda)(\beta l_1 + (1-\beta)l_2) - (\beta c_1 + (1-\beta)c_2)), (1+\xi_1)((\beta y_1 + (1-\beta)y_2) \\
&+(\beta l_1 + (1-\beta)y_2) - (\beta m_1 + (1-\beta)m_2))\} \\
&\geq \beta\left\{g(c_1) + \gamma E\left\{\bar{T}_\gamma^n h\left((1+r)(x_1 + (1-\mu)m_1 - (1+\lambda)l_1 - c_1\right),\right.\right. \\
&\quad (1+\xi_1)(y_1 + l_1 - m_1))\}\} + (1-\beta)\{g(c_2) \\
&+\gamma E\left\{\bar{T}_\gamma^n h\left((1+r)(x_2 + (1-\mu)m_2 - (1+\lambda)l_2 - c_2), (1+\xi_1)(y_2 + l_2 - m_2)\right)\right\}\Big\}
\end{aligned}$$

Since $\beta c_1 + (1-\beta)c_2 \in [0, R(\beta x_1 + (1-\beta)x_2, \beta y_1 + (1-\beta)y_2)]$ and

$$(\beta l_1 + (1-\beta)l_2, \beta m_1 + (1-\beta)m_2) \in \mathcal{A}(\beta x_1 + (1-\beta)x_2, \beta y_1 + (1-\beta)y_2, \beta c_1 + (1-\beta)c_2)$$

we finally have convexity of $\bar{T}_\gamma^{n+1}$, which by induction completes the proof.

$\square$

Next Corollary explains intuitively an obvious fact that orthogonal strategies are within the class of optimal strategies.

Corollary 4 *Under the assumptions of Theorem 2, if $g \in \mathcal{W}$ is nondecreasing there is an optimal strategy $(\bar{l}_n, \bar{m}_n, \bar{c}_n)$ for the cost functional (20) which is orthogonal i.e. $\bar{l}_n \bar{m}_n = 0$.*

Proof. Given the optimal selectors c, l and m defined in Theorem 2 let $\bar{c}_n = c(x_n, y_n)$ and put $\bar{l}_n = l(x_n, y_n, c(x_n, y_n)) - m(x_n, y_n, c(x_n, y_n))$ $\bar{m}_n = 0$ if $l(x_n, y_n, c(x_n, y_n)) \geq m(x_n, y_n, c(x_n, y_n))$ and $\bar{l}_n = 0$, $\bar{m}_n = m(x_n, y_n, c(x_n, y_n)) - l(x_n, y_n, c(x_n, y_n))$ when $m(x_n, y_n, c(x_n, y_n)) \geq l(x_n, y_n, c(x_n, y_n))$. Clearly the strategy $(\bar{l}_n, \bar{m}_n) \in \mathcal{A}(x_n, y_n, c_n)$. By Corollary 2 and the Bellman equation (24) for $n = 1, 2, \ldots$ we then obtain

$$\begin{aligned} & w(x_n, y_n) \leq \{g(\bar{c}_n) + \\ & \gamma E\left\{w\left((1+r)(x_n + (1-\mu)\bar{m}_n - (1+\lambda)\bar{l}_n - \bar{c}_n), (1+\xi_1)(y_n + \bar{l}_n - \bar{m}_n)\right)\right\}\}. \end{aligned} \tag{31}$$

Consequently $w(x, y) \leq J_{xy}(\bar{u})$ with $\bar{u} = (\bar{l}_n, \bar{m}_n, \bar{c}_n)$ and finally $w(x, y) = J_{xy}(\bar{u})$ i.e. the strategy $\bar{u}$ is optimal.

□

Properties of the function w for a particular form of the satisfaction function g are shown in the following

Proposition 2 *If $g(c) = c^\alpha$ with $0 < \alpha < 1$ and $\gamma < \frac{1}{\hat{\lambda}}$ with $\hat{\lambda}$ defined in (6) the function w defined in (23) is lower semicontinuous and is a solution to the Bellman equation (24). Moreover w is convex and for $\rho > 0$ and $(x, y) \in \mathcal{C}$ we have:*

$$w(\rho x, \rho y) = \rho^\alpha w(x, y). \tag{32}$$

Proof. If $g(c) = c^\alpha$ the sequence $\bar{T}_\gamma^n h$ with $h \equiv 0$ is nondecreasing and bounded from above by the optimal value of the cost functional J_{x+y} corresponding to zero transaction costs, which by Proposition 1 is finite. Consequently the limit function $w(x, y)$ is a solution to (24). Moreover w is also of the form (23). Since c^α is convex and each function $\bar{T}_\gamma^n h(x, y)$ is convex, by the arguments of the proof of Corollary 3, w is also convex. Lower semicontinuity of w follows from the continuity of $\bar{T}_\gamma^n h(x, y)$. To complete the proof it remains to notice that for $\rho > 0$, $n = 1, 2, \ldots$ and $(x, y) \in \mathcal{C}$ we have $\bar{T}_\gamma^n h(\rho x, \rho y) = \rho^\alpha \bar{T}_\gamma^n h(x, y)$.

□

Remark 4 *When γ satisfies (30), since $c^\alpha \in \mathcal{W}$, w is a unique solution to (24) and $w \in \bar{\mathcal{W}}$.*

Since the function $g(c) = \ln c$ is not in the class $\mathcal{W}$ this case has to be studied separately.

Proposition 3 *If $g(c) = \ln c$, for any $\gamma \in (0, 1)$ the function w defined in (23) is lower semicontinuous and is a solution to the equation (24). Moreover w is convex and for $\rho > 0$ and $(x, y) \in \mathcal{C}$ we have*

$$w(\rho x, \rho y) = \frac{\ln \rho}{1 - \gamma} + w(x, y). \tag{33}$$

Proof. Notice first that the optimal value of the cost functional J_{xy} does not exceed the optimal value of the cost functional J_{x+y} (without transaction costs) and therefore by Proposition 1 we have that for any $\gamma \in (0, 1)$, $sup_u J_{xy}(u) < \infty$. For $h(x, y) = \frac{1}{1-\gamma} \ln R(x, y) + \frac{1}{(1-\gamma)} \ln \frac{r}{1+r}$, the sequence $\bar{T}_\gamma^n h(x, y)$ is nondecreasing, continuous in $(x, y) \in \mathcal{C}$ and its limit function w is lower semicontinuous and is a solution to (24). Moreover

$h(x, y) = J_{xy}(\tilde{u})$, where the strategy $\tilde{u}$ is to keep everytime the stock account equal to 0 and consume $c_n = \frac{r}{1+r} R(x_n, y_n)$. Consequently $\bar{T}_\gamma^n h(x, y)$ is the value function corresponding to the optimal strategy over horizon $[0, n]$ and then the strategy $\tilde{u}$, and therefore $w(x, y) \leq sup_u J_{xy}(u)$. For an ϵ - optimal strategy $\bar{u} = (\bar{l}_n, \bar{m}_n, \bar{c}_n)$ we have that $\gamma^n E_{xy}^{\bar{u}} \{g(\bar{c}_n)\} \to \infty$.
Therefore $\liminf_{n\to\infty} \gamma^n E_{xy}^{\bar{u}} \{\ln R(x_n, y_n)\} \geq 0$, since $\bar{c}_n \leq R(x_n, y_n)$. Consequently

$$\liminf_{n\to\infty} \gamma^n E^{\bar{u}} \{h(x_n, y_n)\} \geq 0$$

and

$$\lim_{n\to\infty} \bar{T}_\gamma^n h(x, y) \geq \liminf_{n\to\infty} E_{xy}^{\bar{u}} \left\{ \sum_{i=0}^{n-1} g(\bar{c}_i) + \gamma^n h(x_n, y_n) \right\} \geq E_{xy}^{\bar{u}} \left\{ \sum_{i=0}^{\infty} \gamma^i g(\bar{c}_i) \right\}$$

Therefore $w(x, y) \geq E_{xy}^{\bar{u}} \{\sum_{i=0}^{\infty} \gamma^i g(\bar{c}_i)\} \geq \sup_u J_{xy}(u) - \epsilon$ and finally $w(x, y) = \sup_u J_{xy}(u)$. The convexity of w follows from the convexity of h and $\bar{T}_\gamma^n h(x, y)$ similarly as in the proof of Corollary 3.
Since

$$\bar{T}_\gamma^n h(\rho x, \rho y) = \frac{1 - \gamma^n}{1 - \gamma} \ln \rho + \bar{T}_\gamma^n h(x, y) + \frac{\gamma^n}{(1 - \gamma)} \ln \frac{r}{1 + r},$$

letting $n \to \infty$ we see that the property (33) holds.

□

From the Bellman equation (24), Propositions 2 and 3 we can get the following form of optimal strategies

Theorem 3 *If the value function (23) is a solution to the Bellman equation (24), then the cone $\mathcal{C}$ can be split into the following three zones:*

$$S := \left\{ (x, y) : w(x, y) = \sup_{c \in [0, R(x,y)]} \right.$$
$$\left. \left\{ g(c) + \gamma \sup_{(0,m) \in \mathcal{A}(x,y,c)} E\left\{ w\left((1 + r)(x + (1 - \mu)m - c), (1 + \xi_1)(y - m)\right)\right\} \right\} \right\} \quad (34)$$
$$NT := \left\{ (x, y) : w(x, y) = \sup_{c \in [0, R(x,y)]} \right.$$
$$\left. \{g(c) + \gamma E\{w((1 + r)(x - c), (1 + \xi_1)(y))\}\} \right\} \quad (35)$$
$$B := \left\{ (x, y) : w(x, y) = \sup_{c \in [0, R(x,y)]} \right.$$
$$\left. \left\{ g(c) + \gamma \sup_{(l,0) \in \mathcal{A}(x,y,c)} E\left\{ w\left((1 + r)(x - (1 + \lambda)l - c), (1 + \xi_1)(y + l)\right)\right\} \right\} \right\} \quad (36)$$

called respectively: S - selling, NT - nontransaction, B - buying zones. In particular cases when $g(c) = c^\alpha$, with $\alpha \in (0, 1)$ or $g(c) = \ln c$ the zones S, NT and B are cones. If the zones are closed sets the optimal strategy is: when $(x_n, y_n) \in S$, sell assets until we leave the zone S, when $(x_n, y_n) \in NT$ do nothing, and when $(x_n, y_n) \in B$, buy assets until we leave B.

Proof. Let for $(x, y) \in \mathcal{C}$

$$t(x,y) := \sup_{c \in [0, R(x,y)]} \{g(c) + \gamma E \{w((1+r)(x-c), (1+\xi_1)(y))\}\} \tag{37}$$

If $(x, y) \in S$, then

$$\hat{m} := \inf \{m \geq 0 : t(x + (1-\mu)m, y - m) = w(x,y)\} \tag{38}$$

is the optimal sale, while if $(x, y) \in B$, then

$$\hat{l} := \inf \{l \geq 0 : t(x - (1+\lambda)l, y + l) = w(x,y)\} \tag{39}$$

is the optimal amount of assets we have to buy.
By the properties (32) and (33) for $g(c) = c^\alpha$ or $g(c) = \ln c$ the boundaries of the zones are lines and therefore the zones are cones.

□

References

[1] P. Billingsley, *Probability and Measure*, Wiley 1979,

[2] M.H.A. Davis, A.R. Norman, Portfolio Selection with Transaction Costs, *Mathematics of Oper. Res.* **15**, (1990), 676-713.

[3] M.H.A. Davis, V.G. Panas, T. Zariphopoulou, European Option Pricing with Transaction Costs, *SIAM J. Control Optimiz.* **31**, (1993), 470-493.

[4] O. Hernández - Lerma, J.B. Lasserre, *Discrete - Time Markov Control Processes*, Springer 1995.

[5] R.C. Merton, Optimum Consumption and Portfolio Rules in a Continuous - Time Model, *J. Economic Theory* **3**, (1971), 373-413.

[6] P. Samuelson, Lifetime Portfolio Selection by the Dynamic Stochastic Programming, *Rev. Econom. Statistics* **51**, (1969), 239-246.