On Additive and Multiplicative (Controlled) Poisson Equations

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Abstract

Assuming that the Markov processes satisfy minorization property existence and properties of the solutions to additive and multiplicative Poisson equations are studied using splitting techniques. The problem is then extended to study risk sensitive and risk neutral control problems and corresponding to them Bellman equations.

Key words: risk neutral and risk sensitive control, discrete time Markov processes, splitting, Poisson equations, Bellman equations
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1 Introduction

On a probability space $(\Omega, \mathcal{F}, P)$ consider a Markov process $X = (x_n)$ taking values on a complete separable metric state space E endowed with the Borel σ-algebra $\mathcal{E}$. Assume that (x_n) has a transition operator $P(x_n, \cdot)$ at generic time n. Let $c : E \mapsto R$ be continuous bounded and $\gamma > 0$. We would like to find constants λ and λ_γ such that the functions

$$w(x) := E_x \left\{ \sum_{i=0}^{\infty} (c(x_i) - \lambda) \right\} \tag{1}$$

and

$$e^{w_\gamma(x)} := E_x \left\{ \exp \left\{ \sum_{i=0}^{\infty} \gamma(c(x_i) - \lambda_\gamma) \right\} \right\} \tag{2}$$

are well defined.
The problems above are closely related to the existence of solutions: constant λ and a function w or constant λ_γ and function w_γ to the following equations:
additive Poisson equation (APE)

$$w(x) + \lambda = c(x) + Pw(x) \tag{3}$$

where $Pf(x) := E_x \{f(x(1))\} = \int_E f(y)P(x,dy)$,
and multiplicative Poisson equation (MPE)

$$e^{w_\gamma(x)+\lambda_\gamma} = e^{\gamma c(x)} \int_E e^{w_\gamma(y)} P(x,dy). \tag{4}$$

Sufficient condition for existence of solutions to APE is (see [9] and [5]) uniform ergodicity of (x_n), i.e.

$$\sup_{A\in\mathcal{E}} \sup_{x,z\in E} |P(x,A) - P(z,A)| < 1. \tag{5}$$

In the case of MPE a sufficient condition for the existence of solutions can be formulated as follows (see [2] and [5])

$$\sup_{x,z\in E} h(P(x,\cdot), P(z,\cdot)) < \infty \tag{6}$$

where $h(\mu,\nu) := \sup_{A,B\in\mathcal{E}} \ln \frac{\mu(A)\nu(B)}{\nu(A)\mu(B)}$ is so called Hilbert norm in the space $\mathcal{P}(E)$ of probability measures on E.

In the paper we shall formulate more general conditions for the existence of solutions of APE and MPE and shall explain relationship of these equations to problems (1) and (2).

We will be furthermore interested in control of Markov processes. For this purpose we shall assume that (x_n) has a controlled transition operator $P^{a_n}(x_n,\cdot)$ at generic time n, where a_n is the control at time n taking values on a compact metric space U and adapted to the σ-algebra $\sigma\{x_0, x_1, \dots, x_n\}$.
Let now $c : E\times U \mapsto R$ be continuous bounded. We are looking for control (a_n) minimizing the following cost functionals:
risk neutral (average cost per unit time)

$$J((a_n)) := \limsup_{n\to\infty} \frac{1}{n} E_x^{(a_n)} \left\{ \sum_{i=0}^{n-1} c(x_i, a_i) \right\} \tag{7}$$

or *risk sensitive cost functional*

$$J_\gamma((a_n)) := \frac{1}{\gamma} \limsup_{n\to\infty} \frac{1}{n} \ln E_x^{(a_n)} \left\{ \exp\left\{ \sum_{i=0}^{n-1} \gamma c(x_i, a_i) \right\} \right\}. \tag{8}$$

The study of risk sensitive functional is motivated by the fact that it measures not only average cost but also higher moments of the average cost in particular its variance with weight γ (see [1] for financial motivation of these kind of problems). It can be also considered as a dual problem to minimization of the probability that the average cost is greater that a given benchmark (see [7]).

The following Bellman equations correspond to the cost functionals (7) and (8) respectively

$$w(x) + \lambda = \inf_{a\in U} \left(c(x, a) + P^a w(x) \right) \tag{9}$$

where $P^a f(x) := \int_E f(y) P^a(x, dy)$, and

$$e^{w_\gamma(x)+\gamma\lambda_\gamma} = \inf_{a\in U} \left(e^{\gamma c(x,a)} \int_E e^{w_\gamma(y)} P^a(x, dy) \right). \tag{10}$$

One can expect that λ and λ_γ are optimal values of the cost functionals (7) and (8) respectively.

In what follows we shall assume the following Feller property
(F): $U \times E \ni (a, x) \mapsto P^a f(x)$ is continuous for $f \in C(E)$.
Under (F) and controlled uniform ergodicity of the form

$$\sup_{A\in\mathcal{E}} \sup_{a,a'\in U} \sup_{x,z\in E} |P^a(x, A) - P^{a'}(z, A)| < 1 \tag{11}$$

there is (see [9]) a bounded continuous function w and a unique constant λ which solve the Bellman equation (9). Furthermore

$$\lambda = \inf_{(a_n)} J((a_n)) = J(\hat{u}(x_n)), \tag{12}$$

where $\hat{u}$ is a Borel measurable function for which the infimum on the right hand side of (9) is attained.
If additionally to (F) we have that

$$\sup_{x,z\in E} \sup_{a,a'\in U} h(P^a(x, \cdot), P^{a'}(z, \cdot)) < \infty \tag{13}$$

then there exist (see [2]) a bounded function w_γ and a unique constant λ_γ for which the Bellman equation (10) is satisfied. Moreover

$$\lambda_\gamma = \inf_{(a_n)} J_\gamma((a_n)) = J_\gamma((\hat{u}_\gamma(x_n))), \tag{14}$$

where $\hat{u}_\gamma$ is a function for which the infimum in the right hand side of (10) is attained.

We shall consider the following two classes of controls:
Markov controls $\mathcal{U}_M = \{(a_n) : a_n = u_n(x_n)\}$, where $u_n : E \mapsto U$, and *stationary controls* $\mathcal{U}_s = \{(a_n) : a_n = u(x(n))\}$, where $u : E \mapsto U$. We shall also indentify Markov control $a_n = u_n(x_n)$ with a sequence (u_n) of functions $u_n : E \mapsto U$. Similarly stationary control $a_n = u(x_n)$ with shall identify with function $u : E \mapsto U$. Since we shall use so called splitting of Markov processes technique introduced (see [6]) we shall assume the following *minorization property:*

(A1) $\exists_{\beta>0}$ $\exists_{C compact \in \mathcal{E}}$ $\exists_{\nu \in \mathcal{P}(E)}$ *with* $\nu(C) = 1$ such that $\forall_{A\in\mathcal{E}}$

$$\inf_{x\in C} \inf_{a\in U} P^a(x, A) \geq \beta\nu(A)$$

(A2) C given in (A1) is ergodic, i.e. $\forall_{(a_n)\in\mathcal{U}_M}$ $\forall_{x\in E}$ $E_x^{(a_n)}\{\tau_C\} < \infty$, where $\tau_C = \inf\{i > 0 : x_i \in C\}$ and $\forall_{(a_n)\in\mathcal{U}_M}$

$$\sup_{x\in C} E_x^{(a_n)}\{\tau_C\} < \infty$$

.

Given the set C satisfying (A1) and (A2) and Markov control (u_n) we consider a new state space $\hat{E} = \{C \times \{0\} \cup C \times \{1\} \cup E \setminus C \times \{0\}\}$ and splitting of (x_n) in the form $\hat{x}_n = (x_n^1, x_n^2) \in \hat{E}$ with Markov control of the form $a_n = u_n(x_n^1)$ and dynamics defined below:

(i) when $(x_n^1, x_n^2) \in C\times\{0\}$, x_n^1 moves to y accordingly to $(1-\beta)^{-1}(P^{a_n}(x_n^1, dy) - \beta\nu(dy))$ and whenever $y \in C$, x_n^2 is changed into $x_{n+1}^2 = \beta_{n+1}$, where β_n is i.i.d. $P\{\beta_n = 0\} = 1 - \beta$, $P\{\beta_n = 1\} = \beta$,

(ii) when $(x_n^1, x_n^2) \in C \times \{1\}$, x_n^1 moves to y accordingly to ν and $x_{n+1}^2 = \beta_{n+1}$,

(iii) when $(x_n^1, x_n^2) \in E\setminus C\times\{0\}$, x_n^1 moves to y accordingly to $P^{a_n}(x_n^1, dy)$ and whenever $y \in C$, x_n^2 is changed into $x_{n+1}^2 = \beta_{n+1}$.

Let $C_0 = C \times \{0\}$, $C_1 = C \times \{1\}$. The following properties of the split Markov process are shown in [3]

Lemma 1 *For $n = 1, 2 \dots$ we have P a.e.*

$$P\{\hat{x}_n \in C_0 | \hat{x}_n \in C_0 \cup C_1, \hat{x}_{n-1}, \dots, \hat{x}_0\} = 1 - \beta$$

$$P\{\hat{x}_n \in C_1 | \hat{x}_n \in C_0 \cup C_1, \hat{x}_{n-1}, \dots, \hat{x}_0\} = \beta.$$

Lemma 2 *Under Markov control $(a_n) \in \mathcal{U}_M$ the process $(\hat{x}_n = (x_n^1, x_n^2))$ is Markov with transition operator $\hat{P}^{a_n}(\hat{x}_n, dy)$ defined by (i)-(iii). Furthermore the first coordinate (x_n^1) is also a Markov process with transition operator $P^{a_n}(x_n^1, dy)$.*

Corollary 1 *For any bounded Borel measurable function $f : E^m \mapsto R$, $m = 1, 2, \dots,$ and control $(a_n) \in \mathcal{U}_M$ we have*

$$E_x^{(a_n)}\{f(x_1, x_2, \dots, x_m)\} = \hat{E}_{\delta_x^*}^{(a_n)}\left\{f(x_1^1, x_2^1, \dots, x_m^1)\right\} \tag{15}$$

where $\delta_x^* = \delta_{(x,0)}$ for $x \in E \setminus C$ and $\delta_x^* = (1-\beta)\delta_{(x,0)} + \beta\delta_{(x,1)}$.

2 The study of additive Poisson equation

We start with an obvious lemma which follows directly from the boundedness of c, and conditions (A1) and (A2)

Lemma 3 *Given Borel measurable $u : E \mapsto U$ there is a unique $\lambda(u)$ such that for $x \in C_1$*

$$\hat{E}_x\left\{\left\{\sum_{i=1}^{\tau_{C_1}}\Big(c(x_i^1, a_i) - \lambda(u)\Big)\right\}\right\} = 0. \tag{16}$$

For Borel measurable $u : E \mapsto U$ let

$$\hat{w}^u(x) := \hat{E}_x\left\{\sum_{i=0}^{\tau_{C_1}}\Big(c(x_i^1, u(x_i^1)) - \lambda(u)\Big)\right\}, \tag{17}$$

By an analogy to [4], where more specific case was studied, we can show the following results:

Lemma 4 *Function $\hat{w}^u$ is a unique up to an additive constant solution to the additive Poisson equation (APE) for the split Markov process $(\hat{x}_n)$:*

$$\hat{w}^u(x) = c(x^1, u(x^1)) - \lambda(u) + \int_{\hat{E}} \hat{w}^u(y)\hat{P}_{u(x^1)}(x, dy) \tag{18}$$

Furthermore, if $\hat{w}$ and λ satisfy the equation

$$\hat{w}(x) = c(x^1, u(x^1)) - \lambda + \int_{\hat{E}} \hat{w}(y)\hat{P}^{u(x^1)}(x, dy) \tag{19}$$

then $\lambda = \lambda(u)$ (defined in Lemma 3) and $\hat{w}$ differs from $\hat{w}^u$ by an additive constant.

Corollary 2 *Given solution $\hat{w}^u : \hat{E} \mapsto R$ to APE we have that w^u defined by*

$$w^u(x) := \hat{w}^u(x, 0) + 1_C(x)\beta(\hat{w}^u(x, 1) - \hat{w}^u(x, 0)) \tag{20}$$

is a solution to APE for the original Markov process (x_n)

$$w^u(x) = c(x, u(x)) - \lambda(u) + \int_E w^u(y)P^{u(x)}(x, dy). \tag{21}$$

Furthermore if w^u is a solution to (21) then $\hat{w}^u$ defined by

$$\hat{w}^u(x^1, x^2) = c(x^1, u(x^1)) - \lambda(u) + \hat{E}_{x^1,x^2}\left\{w^u(x^1(1))\right\} \tag{22}$$

is a solution to (18).

Proposition 1 *For Borel measurable $u : E \to U$ the value $\lambda(u)$ defined in Lemma 3 is equal to*

$$\lambda(u) = \lim_{n\to\infty} \frac{1}{n} E_x \left\{\sum_{i=0}^{n-1} c(x_i, u(x_i)).\right\} \tag{23}$$

3 The study of multiplicative Poisson equation

To study MPE we need an assumption stronger than (A2). Fix $\gamma > 0$. We shall impose that

(A3) $\forall_{(a_n)\in\mathcal{U}_s}$ $\exists_d$ s.t. $\forall_{x\in\hat{E}}$

$$\hat{E}_x^{(a_n)}\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}} \gamma\left(c(x_i^1, a_i) - d\right)\right\}\right\} < \infty$$

and for $x \in C_1$

$$\hat{E}_x^{(a_n)}\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}} \gamma\left(c(x_i^1, a_i) - d\right)\right\}\right\} \geq 1.$$

Under (A3) we easily obtain that

Lemma 5 *Under (A3) for Borel measurable $u : E \mapsto U$ and there is a unique $\lambda_\gamma(u)$ such that for*

$$\hat{E}_x^{(a_n)}\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}}\gamma\left(c(x_i^1,a_i)-\lambda_\gamma(u)\right)\right\}\right\}=1 \tag{24}$$

for $x \in C_1$.

For Borel measurable $u : E \mapsto U$ and $\gamma > 0$ for which (A3) holds define

$$e^{\hat{w}_\gamma^u(x)}=\hat{E}_x^u\left\{\exp\left\{\sum_{i=0}^{\tau_{C_1}}\gamma\left(c(x_i^1,u(x_i^1))-\lambda_\gamma(u)\right)\right\}\right\}, \tag{25}$$

We have (see [3] for the proofs)

Lemma 6 *Function $\hat{w}_\gamma^u$ defined in (25) is a unique up to an additive constant solution to the multiplicative Poisson equation (MPE) for the split Markov process $(\hat{x}_n)$:*

$$e^{\hat{w}^u(x)_\gamma}=e^{\gamma(c(x^1,u(x^1))-\lambda_\gamma(u))}\int_{\hat{E}}e^{\hat{w}_\gamma^u(y)}\hat{P}^{u(x^1)}(x,dy) \tag{26}$$

Furthermore, if $\hat{w}$ and λ satisfy the equation

$$e^{\hat{w}(x)}=e^{\gamma(c(x^1,u(x^1))-\lambda)}\int_{\hat{E}}e^{\hat{w}(y)}\hat{P}^{u(x^1)}(x,dy) \tag{27}$$

then $\lambda = \lambda_\gamma(u)$ defined in Lemma 5 and $\hat{w}$ differs from $\hat{w}_\gamma^u$ by an additive constant.

Corollary 3 *If $\hat{w}_\gamma^u : \hat{E} \mapsto R$ is a solution to MPE (26) we have that w_γ^u defined by*

$$e^{w_\gamma^u(x)}:=e^{\hat{w}_\gamma^u(x,0)}+1_C(x)\beta(e^{\hat{w}_\gamma^u(x,1)}-e^{\hat{w}_\gamma^u(x,0)}) \tag{28}$$

is a solution to MPE for the original Markov process $(x(n))$

$$e^{w_\gamma^u(x)}=e^{\gamma(c(x,u(x))-\lambda_\gamma(u))}\int_E e^{w_\gamma^u(y)}P^{u(x)}(x,dy) \tag{29}$$

Furthermore if w_γ^u is a solution to (29) then $\hat{w}_\gamma^u$ defined by

$$e^{\hat{w}_\gamma^u(x^1,x^2)}=e^{\gamma(c(x^1,u(x^1))-\lambda_\gamma(u))}\hat{E}_{x^1,x^2}\left\{e^{w_\gamma^u(x_1^1)}\right\} \tag{30}$$

is a solution to (26).

Recall now Proposition 1 of [3]

Proposition 2 *If for Borel measurable $u : E \mapsto U$*

(B1) $\exists_{d(u)}$ such that for $x \in \hat{E}$, $N = 1, 2, \ldots$

$$\hat{E}_x^u \left\{ \exp \left\{ \sum_{i=1}^{\tau_{C_1} \wedge N} \gamma \left(c(x_i^1, u(x_i^1)) - d(u) \right) \right\} \right\} < \infty \tag{31}$$

and for $z \in C_1$

$$\hat{E}_z^u \left\{ \exp \left\{ \sum_{i=1}^{\tau_{C_1}} \gamma \left(c(x_i^1, u(x_i^1)) - d(u) \right) \right\} \right\} > 1 \tag{32}$$

(B2) for $x \in \hat{E}$

$$\inf_N \hat{E}_x^u \left\{ \exp \left\{ \sum_{i=1}^{\sigma_{C_1} \wedge N-1} \gamma \left(c(x_i^1, u(x_i^1)) - \lambda_\gamma(u) \right) \right\} \right\} > 0 \tag{33}$$

(B3) for $x \in \hat{E}$

$$\sup_N \hat{E}_x^u \left\{ \exp \left\{ \sum_{i=1}^{\sigma_{C_1} \wedge N-1} \gamma \left(c(x_i^1, u(x_i^1)) - \lambda_\gamma(u) \right) \right\} \right\} < \infty \tag{34}$$

with $\sigma_{C_1} = \inf \{ i \geq 0 : \hat{x}(i) \in C_1 \}$

then for $x \in E$

$$\lambda_\gamma(u) = \frac{1}{\gamma} \lim_{n \to \infty} \frac{1}{n} \ln E_x^u \left\{ \exp \left\{ \sum_{i=0}^{n-1} \gamma c(x_i, u(x_i)) \right\} \right\}. \tag{35}$$

Remark 1 *Sufficient condition for (B1)-(B3) is*
(D1): $\hat{E}_x^{(a_n)} \{ \exp \{ \gamma \|c\|_{sp} \tau_{C_1} \} \} < \infty$ for $x \in \hat{E}$, for $a_n = u(x_n^1)$, where $\|c\|_{sp} := \sup_{(x,a) \in E \times U} c(x,a) - \inf_{(x,a) \in E \times U} c(x,a)$. Notice that then we also have satisfied the condition (A3) for fixed stationary control u.

4 Asymptotics of MPEs

Given Borel measurable $u : E \mapsto U$ assume that we have (D1) satisfied for $0 < \gamma < \gamma_0$. Then by the Remark 1 there are solutions $\lambda_\gamma(u)$ and w^u_γ to the MPE (29) and $\lambda_\gamma(u)$ is of the form (35). We are now interested in the limit behavior of $\lambda_\gamma(u)$ and w^u_γ as $\gamma \to 0$.

Proposition 3 *We have that $\lambda_\gamma(u)$ decreases to $\lambda(u)$ and $w^u_\gamma(x)$ converges to $w^u(x)$ for $x \in E$ as $\gamma \downarrow 0$, where $\lambda(u)$ and w^u are solutions to the APE (21).*

Proof. Notice first that by Hölder inequality

$$\frac{1}{\gamma_1} \ln E^u_x \left\{ \exp \left\{ \sum_{i=0}^{n-1} \gamma_1 c(x_i, u(x_i)) \right\} \right\} \leq \frac{1}{\gamma_2} \ln E^u_x \left\{ \exp \left\{ \sum_{i=0}^{n-1} \gamma_2 c(x_i, u(x_i)) \right\} \right\} \tag{36}$$

whenever $0 < \gamma_1 \leq \gamma_2$. Therefore by (35) $\lambda_\gamma(u)$ is decreasing as $\gamma \to 0$. Consequently there is $\lambda_0 = \lim_{\gamma \downarrow 0} \lambda_\gamma(u)$. Consider now the split Markov process $(\hat{x}_n)$ corresponding to stationary control u. Let $\hat{w}^u_\gamma$ be given by (25). Then

$$e^{\hat{w}^u_\gamma(x)} \leq \hat{E}^u_x \left\{ \exp \left\{ \sum_{i=0}^{\tau_{C_1}} \gamma \left(c(x^1_i, u(x^1_i)) - \lambda_0 \right) \right\} \right\} \tag{37}$$

Consequently by de l'Hospital rule we have

$$\begin{aligned} \limsup_{\gamma \downarrow 0} \hat{w}^u_\gamma(x) &\leq \limsup_{\gamma \downarrow 0} \frac{1}{\gamma} \ln \hat{E}^u_x \left\{ \exp \left\{ \sum_{i=0}^{\tau_{C_1}} \gamma \left(c(x^1_i, u(x^1_i)) - \lambda_0 \right) \right\} \right\} \\ &= \hat{E}_x \left\{ \sum_{i=0}^{\tau_{C_1}} \left(c(x^1_i, u(x^1_i)) - \lambda_0 \right) \right\} \end{aligned} \tag{38}$$

Similarly for $\bar{\gamma} < \gamma_0$

$$\begin{aligned} \liminf_{\gamma \downarrow 0} \frac{1}{\gamma} \hat{w}_\gamma(x) &\geq \liminf_{\gamma \downarrow 0} \ln \hat{E}^u_x \left\{ \exp \left\{ \sum_{i=0}^{\tau_{C_1}} \gamma \left(c(x^1_i, u(x^1_i)) - \lambda_{\bar{\gamma}} \right) \right\} \right\} \\ &= \hat{E}_x \left\{ \sum_{i=0}^{\tau_{C_1}} \left(c(x^1_i, u(x^1_i)) - \lambda_{\bar{\gamma}} \right) \right\}. \end{aligned} \tag{39}$$

Therefore

$$\hat{E}_x \left\{ \sum_{i=0}^{\tau_{C_1}} \left(c(x^1_i, u(x^1_i)) - \lambda_{\bar{\gamma}} \right) \right\} \leq \liminf_{\gamma \downarrow 0} \frac{1}{\gamma} \hat{w}_\gamma(x)$$

$$\leq \limsup_{\gamma \downarrow 0} \frac{1}{\gamma} \hat{w}_\gamma(x) \leq \hat{E}_x \left\{ \sum_{i=0}^{\tau_{C_1}} \Big(c(x_i^1, u(x_i^1)) - \lambda_0 \Big) \right\} \tag{40}$$

and letting $\bar{\gamma} \to 0$ we obtain that

$$\lim_{\gamma \downarrow 0} \frac{1}{\gamma} \hat{w}_\gamma^u(x) = \hat{E}_x \left\{ \sum_{i=0}^{\tau_{C_1}} \Big(c(x_i^1, u(x_i^1)) - \lambda_0 \Big) \right\} := w(x). \tag{41}$$

Since $\hat{w}_\gamma^u(x_0) = \gamma(c(x_0, u(x_0)) - \lambda_\gamma(u))$ for $x_0 \in C_1$, we have $w(x_0) = c(x_0, u(x_0)) - \lambda_0$. Therefore by Lemma 4, $\lambda_0 = \lambda(u)$ and $w(x) = \hat{w}^u(x)$. From (20) and (28) we immediately have that $\lim_{\gamma \downarrow 0} w_\gamma^u(x) = w^u(x)$, which completes the proof.

□

5 Approximations of the Markov process

In this section we shall introduce an approximation of Markov transition operator in the form of a transition operator of Markov process satisfying the condition (13). We assume first that
(A4):

$$P^a(x, A) = \int_A p(x, a, y) \nu(dy) \tag{42}$$

where $p > 0$ is a continuous function. Moreover letting $|x| := \rho(x, \theta)$, where ρ is a metric on E and $\theta \in E$ is a fixed point define

$$\tilde{p}_N(x, a, y) = \begin{cases} \frac{p(x,a,y)}{\Delta_N^a(x)} & \text{for} \quad |y| \leq N \\ \frac{p(\theta,\bar{a},y)}{\Delta_N^a(x)} & \text{for} \quad |y| \geq N+1 \\ \frac{p(x,a,y)(N+1-|y|)+p(\theta,\bar{a},y)(|y|-N)}{\Delta_N^a(x)} & \text{elsewhere} \end{cases}$$

with $\Delta_N^a(x) = P^a(x, B_N) + P^{\bar{a}}(\theta, B_{N+1}^c) + \int_{B_{N+1} \setminus B_N} [p(x, a, y)(N+1-|y|) + p(\theta, \bar{a}, y)(|y| - N)] \nu(dy)$, where $B_N = \{x \in E : |x| \leq N\}$ and $\bar{a}$ is a fixed element of U.
Then let

$$p_N(x, a, y) = \tilde{p}_N(x, a, y) \text{ if } |x| \leq N$$

$$p_N(x, a, y) = \tilde{p}_N\left(\frac{x}{|x|} N, a, y\right) \text{ for } |x| > N.$$

and define

$$P_N^a(x, dy) = p_N(x, a, y) \nu(dy) \tag{43}$$

We clearly have that

Lemma 7

$$\sup_{a\in U} \|P_N^a(x,\cdot)-P^a(x,\cdot)\|_{var} \to 0 \tag{44}$$

as $N\to\infty$, uniformly in x from compact sets. Furthermore for each N

$$\sup_{a,a'\in U}\sup_{x,x'\in E}\sup_{y\in E}\frac{p_N(x,a,y)}{p_N(x',a',y)}<\infty \tag{45}$$

which means that (13) is satisfied.

Remark 2 *For controlled Markov process with transition operator $P_N^a(x,dy)$ defined in (43) we clearly have that conditions (F), (11) and (13) are satisfied. Consequently we have solutions $w^{(N)}$, $\lambda^{(N)}$ and $w_\gamma^{(N)}$, $\lambda_\gamma^{(N)}$ to the Bellman equations (9) and (10) respectively with operator P^a replaced by P_N^a. Furthermore, there exist optimal stationary controls $\hat{u}^{(N)}$ and $\hat{u}_\gamma^{(N)}$, which are in fact selectors to the right hand sides of (9) and (10) respectively, for the cost functionals $J^{(N)}$ and $J_\gamma^{(N)}$ which correspond to the functionals J and J_γ with operator P^a replaced by P_N^a.*

6 Solution to Additive Bellman Equation

(A5) $\exists_{\epsilon>0}$ *such that* $\forall_{K \ compact\subset \ \hat{E}}$

$$\sup_{a\in U}\sup_{x\in K}\sup_{N}\hat{E}_x^{a,N}\left\{|\sum_{i=1}^{\tau_{C_1}}(c(x_i^1,\hat{u}^{(N)}(x_i^1))-\lambda^{(N)})|^{1+\epsilon}\right\}=M(K)<\infty, \tag{46}$$

where above we control using in the first moment control $a_0=a$ and $a_n=\hat{u}^{(N)}(x_n^1)$ for $n\geq 1$.

Theorem 1 *Under (A5) there exist λ and a continuous function $w: E\mapsto R$ such that*

$$w(x)=\inf_{a\in U}[c(x,a)-\lambda+\int_E w(y)P^a(x,dy)] \tag{47}$$

Moreover λ is an optimal value of the cost functional (7) within the class $\mathcal{U}_s$. The control $\hat{u}$ for which infimum in (47) is attained, is an optimal control.

If for an admissible control (a_n) we have $\lim_{t\to\infty}\frac{1}{t}E_x^{(a_n)}\{w(x_t)\}=0$ then $\lambda\leq J_x((a_n))$.

Proof. The proof consists of several steps:

Step 1. We prove first that $\sup_N \hat{E}_x^{a,N}\left\{\hat{w}_N^{\hat{u}^{(N)}}(\hat{x}_1)\right\}$, where $\hat{w}_N^{\hat{u}^{(N)}}$ is a solution to APE corresponding to transition operator $P^{\hat{u}^{(N)}}$, is bounded uniformly on compact subsets of $(E_0 \cup C_1) \times U$. In fact,

$$\hat{E}_x^{a,N}\left\{\hat{w}_N^{\hat{u}^{(N)}}(\hat{x}_1)\right\} = \hat{E}_x^{a,N}\left\{\chi_{C_1}(\hat{x}_1)\left(c(x_1^1, \hat{u}^{(N)}(x_1^1)) - \lambda^{(N)}(\hat{u}^{(N)})\right)\right\}$$
$$+\hat{E}_x^{a,N}\left\{\chi_{C_1^c}(\hat{x}_1)\sum_{i=1}^{\tau_{C_1}}\left(c(x_i^1, \hat{u}^{(N)}(x_i^1)) - \lambda^{(N)}(\hat{u}^{(N)})\right)\right\} \tag{48}$$

and by (A5) follows the required boundedness.

Step 2. We show now that for $N = 1, 2\ldots$, the functions $\hat{E}_x^{a,N}\left\{\hat{w}_N^{\hat{u}^{(N)}}(\hat{x}_1)\right\}$ are equicontinuous in x and a from compact subsets of $E_0 \cup C_1$ and U respectively.

Notice first that by (44) for each compact set $K \subset E_0 \cup C_1$, $\varepsilon' > 0$ there is a compact set $K_1 \supset C_0 \cup C_1$ such that

$$\sup_{a\in U}\sup_{x\in K}\sup_N \hat{P}_x^{aN}\left\{\hat{x}_1 \in K_1^c\right\} < \varepsilon' \tag{49}$$

Furthermore by Hölder inequality

$$\sup_{a\in U}\sup_{x\in K}\sup_N |\hat{E}_x^{a,N}\left\{\chi_{C_1^c}(\hat{x}_1)\chi_{K_1^c}(\hat{x}_1)\sum_{i=1}^{\tau_{C_1}}(c(x_i^1, \hat{u}^{(N)}(x_i^1)) - \lambda^N(\hat{u}^{(N)}))\right\}|$$
$$\le \sup_{a\in U}\sup_{x\in K}\sup_N\left(\hat{P}_x^{a,N}\left\{\hat{x}_1 \in K_1^c\right\}\right)^{\frac{\varepsilon}{1+\varepsilon}} \sup_{a\in U}\sup_{x\in K}\sup_N\left(\hat{E}_x^{a,N}\left\{|\sum_{i=1}^{\tau_{C_1}}(c(x_i^1, \hat{u}^N(x_i^1))\right.\right.$$
$$\left.\left.-\lambda^N(\hat{u}_N))|^{(1+\epsilon)}\right\}\right)^{\frac{1}{1+\varepsilon}} \le \varepsilon'^{\frac{\varepsilon}{1+\varepsilon}}\left(M(K)\right)^{\frac{1}{1+\varepsilon}} \tag{50}$$

Consequently by (48)-(50)

$$|\hat{E}_x^{a,N}\left\{\hat{w}_N^{\hat{u}^{(N)}}(\hat{x}_1)\right\} - \hat{E}_{x'}^{a',N}\left\{\hat{w}_N^{\hat{u}^{(N)}}(\hat{x}_1)\right\}|$$
$$\le \|c\|\|\hat{P}^{aN}(x, C_1 \cap \cdot) - \hat{P}^{a'N}(x', C_1 \cap \cdot)\|_{var} + 2\varepsilon'^{\frac{\varepsilon}{1+\varepsilon}}\left(M(K)\right)^{\frac{1}{1+\varepsilon}}$$
$$+|\hat{E}_x^{a,N}\left\{\chi_{K_1}(\hat{x}_1)\hat{w}_N^{\hat{u}^{(N)}}(\hat{x}_1)\right\} - \hat{E}_{x'}^{a',N}\left\{\chi_{K_1}(\hat{x}_1)\hat{w}_N^{\hat{u}^{(N)}}(\hat{x}_1')\right\}| \tag{51}$$

For $\delta > 0$ choose K_1 in (49) such that $\varepsilon'^{\frac{\varepsilon}{1+\varepsilon}}\left(M(K)\right)^{\frac{1}{1+\varepsilon}} < \frac{\delta}{6}$. Since

$$|\hat{E}_x^{a,N}\left\{\chi_{K_1}(\hat{x}_1)\hat{w}_N^{\hat{u}^{(N)}}(\hat{x}_1)\right\} - \hat{E}_{x'}^{a',N}\left\{\chi_{K_1}(\hat{x}_1)\hat{w}_N^{\hat{u}^{(N)}}(\hat{x}_1')\right\}|$$
$$\le \sup_{x\in K_1}|\hat{w}_N^{\hat{u}^{(N)}}(x)|\|\hat{P}^{aN}(x, K_1 \cap \cdot) - \hat{P}^{a'N}(x', K_1 \cap \cdot)\|_{var}$$

for x, $x' \in E_0 \cup C_1$ and a, $a' \in U$ such that

$$\|\hat{P}^{aN}(x, C_1 \cap \cdot) - \hat{P}^{a'N}(x', C_1 \cap \cdot)\|_{var} \leq \frac{\delta}{3\|c\|} \tag{52}$$

and

$$\|\hat{P}^{aN}(x, K_1 \cap \cdot) - \hat{P}^{a'N}(x', K_1 \cap \cdot)\|_{var} \leq \frac{\delta}{3 \sup_{z \in K_1} |\hat{w}_N^{\hat{u}^{(N)}}(z)|} \tag{53}$$

by (51) we obtain that

$$|\hat{E}_x^{a,N} \left\{ \hat{w}_N^{\hat{u}^{(N)}}(\hat{x}_1) \right\} - \hat{E}_{x'}^{a',N} \left\{ \hat{w}_N^{\hat{u}^{(N)}}(\hat{x}_1) \right\}| \leq \delta.$$

Now by (A5) $\sup_{z \in K_1} |\hat{w}_N^{\hat{u}^{(N)}}(z)|$ is bounded in N and therefore by (44) we can choose x, x' and a, a' in (52) and (53) uniformly in N, which completes the proof of equicontinuity.
Step 3. By step 1, 2 and (20) we immediately see that

$$E_x^{a,N} \left\{ w_N^{\hat{u}^{(N)}}(x_1) \right\}$$

is uniformly (in N) bounded and equicontinuous in x and a from compact subsets of $E \times U$. Since $\hat{u}^{(N)}$ is optimal for $P_N^a(x, dy)$ we have that $w_N^{\hat{u}_N} = w^{(N)}$. Therefore by Ascoli theorem (thm. 33 of [8]) there is a subsequence N_k such that

$$E_x^{a,N_k} \left\{ w^{(N_k)}(x_1) \right\}$$

converges uniformly in $a \in U$ and x from compact subsets of E and $\lambda^{(N_k)}(\hat{u}^{(N_k)}) \to \lambda$ (since $\lambda^N(\hat{u}^{(N)}) \in [\inf_{x \in E, a \in U} c(x,a), \sup_{x \in E, a \in U} c(x,a)]$). Consequently there is a continuous function w such that

$$w(x) = \inf_{a \in U} [c(x,a) - \lim_{k \to \infty} \int_E w^{(N_k)}(y) P_{N_k}^a(x, dy)] \tag{54}$$

Step 4. To prove that function w defined in (54) is a solution to the Bellman equation (47) it remains to show that

$$\lim_{k \to \infty} E_x^{a,N_k} \left\{ w^{(N_k)}(x_1) \right\} = E_x^a \left\{ w(x_1) \right\}. \tag{55}$$

In fact, by (A5) and Fatou lemma

$$E_x^a \left\{ w(x_1) \right\} \leq \lim_{k \to \infty} E_x^{a,N_k} \left\{ w^{(N_k)}(x_1) \right\} < \infty \tag{56}$$

By step 1 and 2 one can find a compact set $K_1 \supset C$ such that

$$\sup_N \sup_{a\in U} E_x^{a,N}\left\{\chi_{K_1^c}(x_1)|w^{(N)}(x_1)|\right\} \leq \frac{\varepsilon}{3} \tag{57}$$

and

$$\sup_{a\in U} E_x^a\left\{\chi_{K_1^c}(x_1)|w(x_1))|\right\} \leq \frac{\varepsilon}{3}. \tag{58}$$

Therefore

$$\begin{aligned}
&|E_x^a\{w(x_1)\} - E_x^{a,N_k}\left\{w^{(N_k)}(x_1)\right\}| \leq \\
&|E_x^a\{\chi_{K_1}(x_1)w(x_1)\} - E_x^{a,N_k}\{\chi_{K_1}(x_1)w(x_1)\}| \\
&+|E_x^{a,N_k}\left\{\chi_{K_1}(x_1)\left(w(x_1) - w^{(N_k)}(x_1)\right)\right\}| \\
&+E_x^{a,N_k}\left\{\chi_{K_1^c}(x_1)w^{(N_k)}(x_1)\right\} + E_x^a\left\{\chi_{K_1^c}(x_1)w(x_1)\right\} \\
&\leq \sup_{x\in K_1}|w(x)|\|P^a(x, K_1\cap\cdot) - P^{aN}(x, K_1\cap\cdot)\|_{var} + \sup_{x\in K_1}|w(x) - w^{(N_k)}(x)| + \frac{2\varepsilon}{3}.
\end{aligned}$$

Consequently letting $k\to\infty$ and taking into account that ε may be arbitrarily small we obtain the convergence (55). By continuity on x and a of the right hand side of (47) we have the existence of a Borel measurable function $\hat{u}$ for which the infimum is attained.
Step 5. We shall show now that for Borel measurable $u: E \mapsto U$ we have we have $\lambda(u) \geq \lambda$. In fact, then

$$w(x) \leq c(x, u(x)) - \lambda + \int_E w(y)P^{u(x)}(x, dy). \tag{59}$$

Define following (22)

$$\hat{w}^u(x^1, x^2) = c(x^1, u(x^1)) - \lambda + \hat{E}_{x^1,x^2}^{u(x^1)}\left\{w(x_1^1)\right\} \tag{60}$$

Since by Corollary 1 for $a \in U$

$$\begin{aligned}
&E_x^a\{w(x_1)\} = \hat{E}_{\delta_x^*}^a\left\{w(x_1^1)\right\} = \\
&\chi_C(x)[(1-\beta)\hat{E}_{(x,0)}^a\left\{w(x_1^1)\right\} + \beta\hat{E}_{(x,1)}^a\left\{w(x_1^1)\right\}] + \chi_{E\setminus C}(x)\hat{E}_{(x,0)}^a\left\{w(x_1^1)\right\}
\end{aligned}$$

from (59) we have

$$\begin{aligned}
&w(x) \leq c(x, u(x)) - \lambda + \chi_C(x)[(1-\beta)\hat{E}_{(x,0)}^{u(x)}\left\{w(x_1^1)\right\} \\
&+\beta\hat{E}_{(x,1)}^{u(x)}\left\{w(x_1^1)\right\}] + \chi_{E\setminus C}(x)\hat{E}_{(x,0)}^{u(x)}\left\{w(x_1^1)\right\} = \\
&\chi_C(x)\left((1-\beta)\hat{w}^u(x,0) + \beta\hat{w}^u(x,1)\right) + \chi_{E\setminus C}(x)\hat{w}^u(x,0).
\end{aligned}$$

Therefore

$$\hat{E}^{u(x^1)}_{(x^1,x^2)}\left\{w(x^1_1)\right\} \leq \hat{E}^{u(x^1)}_{(x^1,x^2)}\left\{\chi_C(x^1_1)\Big((1-\beta)\hat{w}^u(x^1_1,0)+\beta\hat{w}^u(x^1_1,1)\Big)+\chi_{E\setminus C}(x^1_1)\hat{w}^u(x^1_1,0)\right\}$$
$$= \hat{E}^{(u(x^1)}_{(x^1,x^2)}\left\{\hat{w}^u(x_1)\right\}. \tag{61}$$

Consequently by (60) we have that

$$\hat{w}(x^1,x^2) \leq \ c(x^1,u(x^1))-\lambda+\hat{E}^{u(x^1)}_{(x^1,x^2)}\left\{\hat{w}^u(x_1)\right\} \tag{62}$$

Iterating the last inequality for $z \in C_1$ we obtain

$$\hat{E}^u_z\left\{\hat{w}(x_1)\right\} \leq \hat{E}^u_z\left\{\sum_{i=1}^{\tau_{C_1}}(c(x^1_i,u(x^1_i))-\lambda)+\hat{E}^u_{x_{\tau_{C_1}}}\left\{w(x_1)\right\}\right\} \tag{63}$$

Since by step 1 we have that $\hat{E}_z\left\{\hat{w}(x_1)\right\} < \infty$ we obtain

$$\hat{E}_z\left\{\sum_{i=1}^{\tau_{C_1}}(c(x^1_i,u(x^1_i))-\lambda)\right\} \geq 0,$$

for $z \in C_1$, which by Lemma 3 implies that $\lambda \leq \lambda(u)$.
Step 6. By Proposition 1 and step 5 we have for any Borel measurable $u: E \mapsto U$

$$\lambda = \lambda(\hat{u}) = J_x(\hat{u}(x_n)) \leq J_x((u(x_n))),$$

which shows optimality of $(\hat{u}(x_n))$ within the class of stationary controls. If for an admissible control (a_n) we have $\limsup_{t\to\infty}\frac{1}{t}E^{(a_n)}_x\left\{w(x_t)\right\}=0$, then iterating (47) we obtain

$$w(x) \leq E^{(a_n)}_x\left\{\sum_{i=0}^{t-1}(c(x_i,a_i)-\lambda)+w(x_t)\right\}$$

and dividing both sides of the last inequality by t and letting t to infinity we obtain that $J_x((a_n)) \geq \lambda$ which completes the proof.

□

7 Solution to Multiplicative Bellman Equation

Assume now that

(A6) $\exists_{\epsilon>0}$ *such that* $\forall_{K\ compact \subset\ \hat{E}}$

$$\sup_{a\in U}\sup_{x\in\hat{K}}\sup_{N}\hat{E}_x^{a,N}\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}}\gamma(c(x_i^1,\hat{u}_\gamma^{(N)}(x_i^1))-\lambda_\gamma^{(N)}(\hat{u}_\gamma^{(N)}))\right\}\right\}^{1+\epsilon}<\infty, \tag{64}$$

where above we control using in the first moment control $a_0 = a$ and $a_n = u_N(x_n^1)$ for $n \geq 1$.

We can now recall Theorem 1 of [3]

Theorem 2 *Under (A1)-(A4) and (A6) there exist λ_γ and a continuous function $w_\gamma : E \mapsto R$ such that*

$$e^{w_\gamma(x)} = \inf_{a\in U}[e^{\gamma(c(x,a)-\lambda_\gamma)}\int_E e^{w_\gamma(y)}P^a(x,dy)] \tag{65}$$

Moreover, under (D1): $\hat{E}_x^{(a_n)}\{\exp\{\gamma\|c\|_{sp}\tau_{C_1}\}\}<\infty$ *for $x\in\hat{E}$, for all $(a_n)\in\mathcal{U}_s$ we have that λ_γ is an optimal value of the cost functional J_x^γ and the control $(\hat{u}_\gamma(x_n))$, where $\hat{u}_\gamma$ is a Borel measurable function for which the infimum in the right hand side of (42) is attained, is an optimal control within the class of controls from $\mathcal{U}_s$.*
Furthermore, if for admissible control (a_n) we have that

$$\limsup_{t\to\infty}E_x^{(a_n)}\left\{\left(E_{x_t}^{a_t}\left\{e^{w_\gamma(x_1)}\right\}\right)^\alpha\right\}<\infty$$

for every $\alpha>1$, then $\lambda_\gamma \leq J_x^\gamma((a_n))$.

8 Asymptotics of Bellman equations

Notice first that by Proposition 3 for any Borel measurable $u : E \mapsto U$ provided that (D1) is satisfied for sufficiently small $\gamma > 0$ we have

$$J_\gamma((u(x(n)))\to J((u(x(n)))) \tag{66}$$

as $\gamma \to 0$, and the limit is decreasing.

Consequently we have

Theorem 3 *Under (A1)-(A6) together with (D1) satisfied for sufficiently small $\gamma > 0$ we have*

$$\lim_{\gamma\to 0}\lambda_\gamma = \lambda. \tag{67}$$

Furthermore, risk neutral ε- optimal control $u \in \mathcal{U}_s$ within the class of stationary controls is nearly optimal for the risk sensitive cost functional with γ close to 0, within the class of stationary controls.

Proof. By Theorem 2

$$\lambda_\gamma = \inf_{u \in \mathcal{U}_s} J_\gamma(u) \tag{68}$$

and

$$\lambda = \inf_{u \in \mathcal{U}_s} J(u). \tag{69}$$

Therefore by (66) we immediately obtain (67). Now, if $u \in \mathcal{U}_s$ is ε- optimal for J within the class of stationary strategies, then by (66) for $0 < \gamma < \gamma_0$ we have

$$J_\gamma(u) \leq J(u) + \varepsilon \leq \lambda + 2\varepsilon \leq \lambda_\gamma + 3\varepsilon. \tag{70}$$

which is required 3ε - optimality.

□

Remark 3 *Although we have convergence (67) it is not clear that w_γ being a solution to the Bellman equation (65) converges, (or at least its subsequence) to w which is a solution to the equation (47), as $\gamma \to 0$, provided that at fixed point $\bar{x} \in E$ we have $w_\gamma(\bar{x}) = w(\bar{x})$.*

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