1 Preliminary results

1.1 The control problems

Given an underlying probability space (Ω, F, P), let E be a locally compact separable metric space. On E consider a state/signal process (x_n) assumed to be Markov with initial distribution μ and transition kernel $P^a(x, dz)$, where a is the control parameter with $a \in U$ a compact metric space.

The process (x_n) is observed through an observation process (y_n), $y_n \in R^d$, which is statistically dependent on x_n via

$$P\{y_{n+1} \in A | x_0, x_1, \ldots, x_{n+1}, Y^n\} = \int_A r(x_{n+1}, y)\, dy \tag{1.1}$$

for $n = 0, 1, 2, \ldots$, where $A \in \mathcal{B}(R^d)$ with $\mathcal{B}(R^d)$ denoting the family of Borel subsets of R^d, $Y^n = \sigma\{y_1, \ldots, y_n\}$, $Y^0 = \{\emptyset, \Omega\}$ and $r \colon E \times R^d \to [0, \infty)$ is a fixed Borel measurable function.

Notice that (1.1) includes observation models with additive noise as e.g.

$$y_n = h(x_n) + \sigma w_n$$

with (w_n) i.i.d. standard Gaussian for which

$$r(x_n, y) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\Big\{ -\frac{1}{2\sigma^2}(y - h(x_n))^2 \Big\} \tag{1.2}$$

Although we are mainly interested in the original model having a nonfinite number of states and observations, all results can be easily transformed to the case when the state is $E = \{1, 2, \ldots m\}$, and the observations take values in the finite set $D = \{d_1, \ldots d_s\}$. In this case we assume that there exists a function $r \colon E \times D \to [0, 1]$ such that the D-valued observation process (y_n) of (x_n) satisfies the following relation

$$P\{y_{n+1} = d_i | x_0, x_1, \ldots, x_{n+1} = j, Y^n\} = r(j, d_i) \tag{1.3}$$

for $j \in E, 1 \leq i \leq s$.

We shall consider as admissible controls, sequences $u = (a_0, a_1, a_2, \ldots)$ of U-valued random variables a_n adapted to the observation σ-field (Y^n). The values taken by these random variables will generically be denoted by the letter a.

Our study concerns the following three types of stochastic control problems.

I) Finite horizon problem with cost functional

$$J_\mu^T(u) = E_\mu^u\Big\{\sum_{n=0}^{T-1} c_n(x_n, a_n) + b(x_T)\Big\} \tag{1.4}$$

where $T > 0$ is given, E_μ^u denotes expectation given the initial measure μ and the admissible control u

II) Infinite horizon problem with discounting with cost functional

$$J_\mu^\beta(u) = \sum_{n=0}^{\infty} \beta^n E_\mu^u\{c(x_n, a_n)\} \tag{1.5}$$

where $\beta \in (0, 1)$ is the discount factor.

III) Infinite horizon ergodic cost problem with cost functional

$$J_\mu(u) = \limsup_{T\to\infty} T^{-1} \sum_{n=0}^{T-1} E_\mu^u\{c(x_n, a_n)\} \tag{1.6}$$

The purpose is that of determining a nearly-optimal (ε-optimal) control for each of the three problems, namely a control for which the cost function comes within ε of its infimum. This will be achieved by means of an approximation approach.

1.2 The filter process

To study the stochastic control problems with partial observation described in the previous subsection, it will be convenient to associate with each of them a corresponding complete observation problem where the new state is given by the filtering process (π_n^u) which, for a given admissible control u, $A \in \mathcal{B}(E)$ and a given initial law μ for (x_n), is defined as

$$\begin{cases} \pi_0^u(A) = \mu(A) \\ \pi_n^u(A) = P_\mu^u\{x_n \in A | Y^n\} \end{cases} \tag{1.7}$$

where, analogously to E_μ^u, P_μ^u denotes the probability induced by (x_n), given the control u and the initial law μ.

Lemma 1.1 *Given* (1.1), *for each admissible control* u *we have the following representation of the filtering process*

$$\pi^u_{n+1}(A) = \frac{\int\limits_A r(z, y_{n+1}) P^{a_n}(\pi^u_n, dz)}{\int\limits_E r(z, y_{n+1}) P^{a_n}(\pi^u_n, dz)} := M^{a_n}(y_{n+1}, \pi^u_n)(A) \tag{1.8}$$

P_μ *a.e., with the mapping* $M^a(y, \pi)$ *defined implicitly.*

P r o o f. Let $F \in b\mathcal{B}((R^d)^n)$ and $A,\ C \in \mathcal{B}(R^d)$. By (1.1), the properties of conditional expectations, and Fubini's Lemma, we have

$$\int\limits_\Omega M^{a_n}(y_{n+1}, \pi^u_n)(A)\chi_C(y_{n+1}) F(y_1, \dots, y_n)\, dP =$$

$$\int\limits_\Omega E[M^{a_n}(y_{n+1}, \pi^u_n)(A)\chi_C(y_{n+1}) | x_{n+1}, Y^n] F(y_1, \dots, y_n)\, dP =$$

$$\int\limits_\Omega \int\limits_C M^{a_n}(y, \pi^u_n)(A) r(x_{n+1}, y)\, dy F(y_1, \dots, y_n)\, dP =$$

$$\int\limits_\Omega \int\limits_C M^{a_n}(y, \pi^u_n)(A) E[E[r(x_{n+1}, y)|Y^n, x_n]|Y^n]\, dy F(y_1, \dots, y_n)\, dP =$$

$$\int\limits_\Omega \int\limits_C M^{a_n}(y, \pi^u_n)(A) E[\int\limits_E r(z, y) P^{a_n}(x_n, dz)|Y^n]\, dy F(y_1, \dots, y_n)\, dP =$$

$$\int\limits_\Omega \int\limits_C M^{a_n}(y, \pi^u_n)(A) \int\limits_E r(z, y) P^{a_n}(\pi^u_n, dz)\, dy F(y_1, \dots, y_n)\, dP =$$

$$\int\limits_\Omega \int\limits_C \int\limits_A r(z, y) P^{a_n}(\pi^u_n, dz)\, dy F(y_1, \dots, y_n)\, dP =$$

$$\int\limits_\Omega E[\int\limits_A \int\limits_C r(z, y)\, dy P^{a_n}(x_n, dz)|Y^n] F(y_1, \dots, y_n)\, dP =$$

$$\int\limits_\Omega E[E[\int\limits_C r(x_{n+1}, y)\, dy \chi_A(x_{n+1})|Y^n, x_n]|Y^n] F(y_1, \dots, y_n)\, dP =$$

$$\int\limits_\Omega \int\limits_C r(x_{n+1}, y)\, dy \chi_A(x_{n+1}) F(y_1, \dots, y_n)\, dP =$$

$$\int\limits_\Omega E[\chi_C(y_{n+1})|Y^n, x_{n+1}]\chi_A(x_{n+1}) F(y_1, \dots, y_n)\, dP =$$

$$\int_\Omega \chi_C(y_{n+1})\chi_A(x_{n+1})F(y_1,\ldots,y_n)\,dP =$$
$$\int_\Omega \pi_{n+1}(A)\chi_C(y_{n+1})F(y_1,\ldots,y_n)\,dP$$

Therefore by the definition of conditional expectation we obtain (1.8).

■

In the case when the state and observation processes take their values in finite sets E and D respectively and (1.3) holds, by analogy to Lemma 1.1 we obtain

Corollary 1.2 *For the case of finite state and observations spaces under (1.3), for each admissible control u we have the following representation of the filtering process*

$$\pi^u_{n+1}(j) = \frac{r(j,y_{n+1})P^{a_n}(\pi^u_n,j)}{\sum_{k=1}^m r(k,y_{n+1})P^{a_n}(\pi^u_n,k)} := M^{a_n}(y_{n+1},\pi^u_n)(j) \tag{1.9}$$

P_μ a.e., for $1 \le j \le m$, where, similarly as in (1.8), we define implicitly the mapping $M^a(y,\pi)$.

We point out that all the results obtained in the sequel for general E and $D = R^d$ hold also for the case of $E = \{1, 2 \ldots m\}$, $D = \{d_1, \ldots d_s\}$, provided the assumptions and the statements are appropriately reformulated replacing e.g. integrals over E and R^d by suitable summations. Given the filter process (π^u_n) corresponding to an admissible control u, we can rewrite the three cost functionals of the preceding subsection as

I)

$$\begin{aligned} J^T_\mu(u) = \ & E^u_\mu\Big\{\sum_{n=0}^{T-1}\int_E c_n(x,a_n)\pi^u_n(dx) \\ & + \int_E b(x)\pi^u_T(dx)\Big\} \end{aligned} \tag{1.10}$$

II)

$$J^\beta_\mu(u) = \sum_{n=0}^{\infty} \beta^n E^u_\mu \Big\{ \int\limits_E c(x, a_n) \pi^u_n(dx) \Big\} \tag{1.11}$$

III)

$$J_\mu(u) = \limsup_{T\to\infty} T^{-1} \sum_{n=0}^{T-1} E^u_\mu \Big\{ \int\limits_E c(x, a_n) \pi^u_n(dx) \Big\} \tag{1.12}$$

provided $c_n, c \in b\mathcal{B}(E \times U)$ and $b \in b\mathcal{B}(E)$,

The recursive relation (1.8) together with the cost functionals (1.10)–(1.12) defines the complete observation problems corresponding to the partially observed problems with the respective cost functionals (1.4)–(1.6).

By applying the Bellman equations (see section 3.2 below) to the problems corresponding to case II), we find that among the optimal controls (as far as they exist) there is a stationary feedback control given by a function of the current filter process, i.e. such that

$$a_n = u(\pi_n) \tag{1.13}$$

with $u \in \mathcal{B}(P(E), U)$ and which will be referred to as "control function". (For simplicity, in what follows, we shall drop the upper index u in π^u_n, when writing $u(\pi_n)$).

Assuming controls of the type (1.13) we obtain the Markov property of the filtering process.

Lemma 1.3 *Given u satisfying (1.13), the filter process (π^u_n) is Markov with respect to the σ-field Y^n and has transition operator*

$$\prod\nolimits^{u(\nu)}(\nu, F) = \int\limits_E \int\limits_{R^d} F(M^{u(\nu)}(y, \nu)) r(z, y)\, dy P^{u(\nu)}(\nu, dz) \tag{1.14}$$

where $\nu \in P(E)$ and $F \in b\mathcal{B}(P(E))$.

P r o o f. By (1.1) we easily obtain

$$\begin{aligned} E[F(\pi^u_{n+1})|Y^n] &= E[F(M^{u(\pi_n)}(y_{n+1}, \pi^u_n))|Y^n] \\ &= E[E[F(M^{u(\pi_n)}(y_{n+1}, \pi_n))|Y^n, x_{n+1}]|Y^n] \end{aligned}$$

$$= E\Big[\int_{R^d} F(M^{u(\pi_n)}(y,\pi_n))r(x_{n+1},y)\,dy|Y^n\Big]$$
$$= E\Big[\int_{R^d} E[F(M^{u(\pi_n)}(y,\pi_n))r(x_{n+1},y)|Y^n,x_n]\,dy|Y^n\Big]$$
$$= E\Big[\int_{R^d} F(M^{u(\pi_n)}(y,\pi_n))\int_E r(z,y)P^{u(\pi_n)}(x_n,dz)\,dy|Y^n\Big]$$
$$= \int_E\int_{R^d} F(M^{u(\pi_n)}(y,\pi_n))r(z,y)\,dyP^{u(\pi_n)}(\pi_n,dz)$$

Therefore π_n^u is in fact Markov with the transition operator of the form (1.14).

■

We obtain further properties of the transition operator Π of the filtering process making first the assumptions:

(A1) $P^a(x,\cdot)$ is Feller for fixed $a\in U$, that is for any $\varphi\in C(E)$, $x_n\to x$ we have
$$P^a(x_n,\varphi)\to P^a(x,\varphi)\quad \text{as } n\to\infty \tag{1.15}$$

(A2) if $U\ni a_n\to a$, then for $\varphi\in C(E)$, and compact set $K\subset E$
$$\sup_{x\in K}|P^{a_n}(x,\varphi)-P^a(x,\varphi)|\to 0\quad \text{as } n\to\infty \tag{1.16}$$

(A3) $r\in C(E\times R^d)$

(A4) for $E\ni x_n\to x$, $\varphi\in C(R^d)$ we have
$$R(x_n,\varphi):=\int_{R^d} r(x_n,y)\varphi(y)\,dy\to R(x,\varphi), \text{as } n\to\infty \tag{1.17}$$

Proposition 1.4 *Under (A1)–(A4), for $\varphi\in C(E)$, $F\in C(P(E))$, the mappings*
$$U\times R^d\times P(E)\ni(a,y,\nu)\mapsto M^a(y,\nu) \tag{1.18}$$
and
$$U\times P(E)\ni(a,\nu)\mapsto\int_E\int_{R^d} F(M^a(y,\nu))r(z,y)\,dyP^a(\nu,dz) \tag{1.19}$$
are continuous.

P r o o f. Let us first notice that the mapping

$$U \times P(E) \ni (a,\nu) \mapsto P^a(\nu,\varphi) \tag{1.20}$$

for $\varphi \in C(E)$ is continuous.

In fact, let $U \ni a_m \to a$ and $P(E) \ni \nu_m \Rightarrow \nu$, denoting by $\Rightarrow$ the weak convergence on $P(E)$. Clearly the set $\Gamma = \{\nu, \nu_1, \nu_2, \ldots\}$ is compact in $P(E)$. Therefore by Prokhorov's theorem (see Theorem 1.6.2 of [6]) Γ is tight that is for any $\varepsilon > 0$ one can find a compact set $K \subset E$ such that $\mu(K^c) < \varepsilon$ for each $\mu \in \Gamma$.

For given $\varepsilon > 0$ choose the compact set $K \subset E$ as above. We then have

$$\begin{aligned} &|P^{a_m}(\nu_m,\varphi) - P^a(\nu,\varphi)| \le |P^{a_m}(\nu_m,\varphi) - P^a(\nu_m,\varphi)| \\ &\quad + |P^a(\nu_m,\varphi) - P^a(\nu,\varphi)| \le 2\varepsilon\|\varphi\| + \sup_{x\in K} |P^{a_m}(x,\varphi) \\ &\quad - P^a(x,\varphi)| + |P^a(\nu_m,\varphi) - P^a(\nu,\varphi)| \end{aligned}$$

By (A2) the second term on the right hand side converges to 0. By (A1) the mapping $x \mapsto P^a(x,\varphi)$ belongs to $C(E)$ and therefore the third term also converges to 0. Since ε can be chosen arbitrarily small we obtain (1.20).

We show now the continuity of the mapping (1.18). One can easily see that it is sufficient to show the continuity of the numerator in (1.8) i.e. to prove that for $U \ni a_m \to a$, $R^d \ni y_m \to y$, $P(E) \ni \nu_m \Rightarrow \nu$ we have

$$\int_E \varphi(z) r(z,y_m) P^{a_m}(\nu_m, dz) \to \int_E \varphi(z) r(z,y) P^a(\nu, dz) \tag{1.21}$$

for $\varphi \in C(E)$.

To show (1.21) notice that by (1.20) the set $\{P^a(\nu,\cdot), P^{a_n}(\nu_n,\cdot)\ n = 1,2,\ldots,\}$ is compact in $P(E)$ and consequently, by Prokhorov's theorem, is tight.

Therefore for given $\varepsilon > 0$ we can find a compact set $K \subset E$ such that $P^{a_m}(\nu_m, K^c) < \varepsilon$, $P^a(\nu, K^c) < \varepsilon$ for $m = 1,2,\ldots$.

We have

$$\begin{aligned} &\Big| \int_E \varphi(z) r(z,y_m) P^{a_m}(\nu_m,dz) - \int_E \varphi(z) r(z,y) P^a(\nu,dz) \Big| \\ &\quad \le \int_E |\varphi(z)|\ |r(z,y_m) - r(z,y)| P^{a_m}(\nu_m,dz) \end{aligned}$$

$$+\Big|\int_E \varphi(z)r(z,y)(P^{a_m}(\nu_m,dz)-P^a(\nu,dz))\Big|$$
$$\leq 2\|\varphi\|\ \|r\|\varepsilon+\|\varphi\|\sup_{z\in K}|r(z,y_m)-r(z,y)|$$
$$+\Big|\int_E \varphi(z)r(z,y)(P^{a_m}(\nu_m,dz)-P^a(\nu,dz))\Big|$$

By (A3) and (1.20) the second and the third terms on the right hand side converge to 0, as $m\to\infty$. Therefore (1.21) holds and the continuity of (1.18) is established.

It remains to show the continuity of the mapping (1.19). Let $U\ni a_m\to a$ and $P(E)\ni\nu_m\Rightarrow\nu$.

We have

$$\Big|\int_E\int_{R^d} F(M^{a_m}(y,\nu_m))r(z,y)\,dyP^{a_m}(\nu_m,dz)$$
$$-\int_E\int_{R^d} F(M^{a_m}(y,\nu))r(z,y)\,dyP^a(\nu,dz)\Big|$$
$$\leq\int_E\int_{R^d}|F(M^{a_m}(y,\nu_n))-F(M^a(y,\nu))|r(z,y)\,dyP^{a_m}(\nu_m,dz)$$
$$+\Big|\int_E\int_{R^d} F(M^a(y,\nu))r(z,y)\,dy(P^{a_m}(\nu_m,dz)-P^a(\nu,dz)\Big|$$
$$=I_m+II_m$$

Since the mapping $y\mapsto F(M^a(y,\nu))$ belongs to $C(R^d)$, by (A4) and (1.20) we obtain $II_m\to 0$ as $m\to\infty$.

Given $\varepsilon>0$, by the tightness of $\{P^{a_m}(\nu_m,\cdot),\ m=1,2,\ldots\}$ there is a compact set $K\subset E$ such that $P^{a_m}(\nu_m,K^c)<\varepsilon$ for $m=1,2,\ldots$. By (A4) the family of measures $\{R(x,\cdot)\ x\in K\}$ is compact in $P(R^d)$, and therefore tight. So one can find a compact set $L\subset R^d$ such that $R(x,L^c)<\varepsilon$ for $x\in K$.

We can now evaluate I_m. We have

$$\begin{aligned}I_m &\leq 2\|F\|\varepsilon+\int_K\int_{R^d}|F(M^{a_m}(y,\nu_m))-F(M^a(y,v))|r(z,y)\,dyP^{a_m}(\nu_m,dz)\\ &\leq 4\|F\|\varepsilon+\sup_{y\in L}|F(M^{a_m}(y,\nu_m))-F(M^a(y,\nu))|\end{aligned}$$

By the continuity of the mapping (1.18), the second term on the right hand side converges to 0, as $m \to \infty$. Since ε could be chosen arbitrarily small, $I_m \to 0$, as $m \to \infty$. This completes the proof of the continuity of the mapping (1.19).

$\blacksquare$

From Proposition 1.4 we immediately have

Corollary 1.5 *Under (A1)–(A4), for* u *of type (1.13) and* $u \in C(P(E), U)$ *the transition operator*

$$\prod^{u(\nu)}(\nu, \cdot) \quad \textit{is Feller}$$

i.e. for $F \in C(P(E))$ *the mapping* $P(E) \ni \nu \mapsto \prod^{u(\nu)}(\nu, F)$ *is continuous.*

For a given $a \in U$ consider finally the transition operator $\prod^a(\nu, \cdot)$ that corresponds to $\prod^{u(\nu)}(\nu, \cdot)$ as defined in (1.14) for $u(\nu) \equiv a$. We shall now formulate a property of $\prod^a(\nu, \cdot)$ that is associated with concave continuous functions $v : P(E) \to R$

We say that $v : P(E) \to R$ is concave if for $\nu = \lambda\nu_1 + (1-\lambda)\nu_2$, $\nu_1, \nu_2 \in P(E), \lambda \in [0,1]$ we have

$$v(\nu) \geq \lambda v(\nu_1) + (1-\lambda)v(\nu_2)$$

We need the following

Lemma 1.6 *Let* $\pi(\omega)$ *be a* $P(E)$ *valued random variable on* (Ω, F, P) *and* F' *a sub-*σ*-field of* F*. Then for any concave continuous function* $v : P(E) \to R$ *we have*

$$v(E(\pi|F')) \geq E(v(\pi)|F')$$

P r o o f. It is well known (see Theorem 11.7(a) of [23]) that a concave continuous function v is a lower envelope of affine functions $v_i(\nu) = \alpha_i + \beta_i \nu(f_i)$, $f_i \in C(P(E)), i \in I$ a certain family of indices, i.e.

$$v(\nu) = \inf_{i \in I} v_i(\nu)$$

Therefore

$$v_i(E(\pi|F') = E\{v_i(\pi)|F'\} \geq E\{v(\pi)|F'\}$$

and

$$v(E(\pi|F')) = inf_{i\in I}\ v_i(E(\pi|F')) \geq E\{v(\pi)|F'\}$$

$\blacksquare$

Proposition 1.7 *Under (A1), (A3), (A4) for $v : P(E) \mapsto R$ continuous and concave and $a \in U$, the function*

$$P(E) \ni \nu \to \prod^a(\nu, v) \tag{1.22}$$

is also continuous and concave.

P r o o f. The continuity of the mapping (1.21) follows from the proof of Proposition 1.4. It remains to show concavity.

Let $\nu = \lambda\nu_1 + (1-\lambda)\nu_2, \nu_1, \nu_2 \in P(E), \lambda \in [0,1]$. Define $\bar{\pi}_0$ as a $P(E)$ valued random variable such that

$$P\{\bar{\pi}_0 = \nu_1\} = \lambda \qquad P\{\bar{\pi}_0 = \nu_2\} = 1-\lambda$$

Let (x_n) be a Markov process with transition operator $P^a(x,\cdot)$ and initial law ν and let (y_n) be the observation process satisfying (1.1).

Furthermore let, for $A \in \mathcal{B}(E)$,

$$\begin{aligned} \pi_0(A) &= \nu(A) \\ \pi_n(A) &= P\{x_n \in A|Y^n\} \\ \bar{\pi}_n(A) &= P\{x_n \in A|\bar{Y}^n\} \end{aligned} \tag{1.23}$$

with $Y^n = \sigma\{y_1,\ldots,y_n\}$, $\bar{Y}^n = \sigma\{y_1,\ldots,y_n,\bar{\pi}_0\}$. By the proof of Lemma 1.1 and Lemma 1.3 we see that

$$\bar{\pi}_{n+1}(A) = M^a(y_{n+1},\bar{\pi}_n)(A)$$

P a.e. and $\bar{\pi}_n$ is a Markov process with respect to the σ-field $\bar{Y}^n$, with transition operator $\prod^a$.

Therefore

$$E\{v(\bar{\pi}_1)\} = E\{v(\bar{\pi}_1)|\bar{\pi}_0\} = \lambda \prod^a(\nu_1, v) + (1-\lambda)\prod^a(\nu_2, v) \tag{1.24}$$

By (1.23) we have $E\{\bar{\pi}_1|Y^1\} = \pi_1$, and using Lemma 1.6 we obtain

$$v(\pi_1) \geq E\{v(\bar{\pi}_1)|Y^1\}$$

Therefore by (1.24)

$$\prod^a(\nu, v) = E\{v(\pi_1)\} \geq \lambda \prod^a(\nu_1, v) + (1-\lambda)\prod^a(\nu_2, v)$$

which completes the proof of Proposition.

1.3 A general measure transformation

One of the most efficient methods to perform the approximations leading to the construction of ε-optimal controls, at least for finite horizon problems, is based on measure transformation techniques. The main advantage of these techniques lies in the fact that they allow to have the same admissible controls in the original and the approximating problems; this in turn allows to compare the cost functionals of the original and the approximating problems, evaluated for a same control u.

In this subsection we describe a general measure transformation approach assuming that the observations (y_n) are given by the formula

$$y_n = h(x_n, w_n), \qquad \text{for} \quad n = 1, 2, \ldots \tag{1.25}$$

where w_n are R^d valued i.i.d. with common, strictly positive density g, and, for each n, w_n is independent of x_j for $j \leq n$. Furthermore, for each $x \in E$, $h(x, \cdot)$ is a C^1 diffeomorphism of R^d with inverse function $k(x, \cdot)$ and Jacobian of $k(x, y)$ denoted by $\Delta(x, y)$.

Given $A \in \mathcal{B}(R^d)$ we have for $n = 0, 1, 2, \ldots$

$$\begin{aligned}
&P\{y_{n+1} \in A|x_0, x_1, \ldots, x_{n+1}, Y^n\} = \\
&P\{w_{n+1} \in k(x_{n+1}, A)|x_0, x_1, \ldots, x_{n+1}, Y^n\} \\
&\int\limits_{k(x_{n+1}, A)} g(y)\,dy = \int\limits_A g(k(x_{n+1}, y))|\Delta(x_{n+1}, y)|\,dy
\end{aligned} \tag{1.26}$$

with $|\Delta(x_{n+1}, y)|$ standing for the determinant of Jacobian $\Delta(x_{n+1}, y)$. Hence (1.1) is satisfied with $r(x, y) = g(k(x, y))|\Delta(x, y)|$.

Let $F_n = \sigma\{(x_i, y_i),\ i \leq n\}$, $\Lambda_0 = 1$ and

$$\Lambda_n = \prod_{i=1}^{n} \frac{g(y_i)}{r(x_i, y_i)} \tag{1.27}$$

We have

$$\begin{aligned} E[\Lambda_{n+1}|F_n] &= \Lambda_n E\Big[\frac{g(y_{n+1})}{r(x_{n+1}, y_{n+1})}|F_n\Big] \\ &= \Lambda_n E[E\Big[\frac{g(h(x_{n+1}, w_{n+1}))}{g(w_{n+1})|\Delta(x_{n+1}, h(x_{n+1}, w_{n+1}))|}|x_{n+1}, F_n\Big]|F_n] \\ &= \Lambda_n E\Big\{\int_{R^d} g(h(x_{n+1}, w))|\Delta(x_{n+1}, h(x_{n+1}, w))|^{-1}\, dw|F_n\Big\} = \Lambda_n \end{aligned}$$

using the fact that $|\Delta(x_{n+1}, h(x_{n+1}, w))|^{-1}$ is the determinant of Jacobian of $h(x_{n+1}, \cdot)$.

Thus Λ_n is an F_n martingale. Furthermore its mean is equal to 1 and so we can define a new probability measure P^0 on Ω such that the restrictions $P_{|n}$, $P^0_{|n}$ of P and P^0 respectively to the σ-field F_n for each $n = 1, 2, \ldots$ satisfy

$$P^0_{|n}(d\omega) = \Lambda_n(\omega) P_{|n}(d\omega) \tag{1.28}$$

Clearly

$$P_{|n}(d\omega) = L_n(\omega) P^0_{|n}(d\omega) \tag{1.29}$$

with

$$\begin{aligned} &L_0(\omega) = 1 \\ &L_n(\omega) = (\Lambda_n(\omega))^{-1} = \prod_{i=1}^{n} \frac{r(x_i, y_i)}{g(y_i)} \end{aligned} \tag{1.30}$$

Moreover we have

Lemma 1.8 *Under the measure P^0, the y_n are i.i.d. with common density g, independent of x_j with $j \leq n$ and (x_n) is a controlled Markov process with the same transition probability $P^{a_n}(x_n, dy)$ at the generic period n as under P and where a_n are adapted to Y^n.*

P r o o f. Let $f_1,\ f_2 \in b\mathcal{B}(R^d),\ f_3 \in b\mathcal{B}(E)$. For $j < n$ we have by (1.26)

$$\begin{aligned}
E^0\{f_1(y_n)f_2(y_j)\} &= E\{\Lambda_n f_1(y_n)f_2(y_j)\} \\
&= E\{\Lambda_{n-1}f_2(y_j)E\{f_1(y_n)\frac{g(y_n)}{r(x_n,y_n)}|x_n, F_{n-1}\}\} \\
&= E\{\Lambda_{n-1}f_2(y_j)\int\limits_{R^d} f_1(y)g(y)\,dy\} \\
&= E\{\Lambda_{n-1}f_2(y_j)\}\int\limits_{R^d} f_1(y)g(y)\,dy \\
&= E^0\{f_2(y_j)\}\int\limits_{R^d} f_1(y)g(y)\,dy
\end{aligned}$$

Since, letting $f_2 \equiv 1$, from the above we obtain $E^0\{f_1(y_j)\} = \int\limits_{R^d} f_1(y)g(y)\,dy$ for any $f_1 \in b\mathcal{B}$, we finally have

$$E^0\{f_1(y_n)f_2(y_j)\} = \int\limits_{R^d} f_1(y)g(y)\,dy \int\limits_{R^d} f_2(y)g(y)\,dy$$

which means that, under P^0, the y_n are i.i.d. with common density g.

Now for $n \geq j > 0$

$$\begin{aligned}
E^0\{f_1(y_n)f_3(x_j)\} &= E\{\Lambda_{n-1}f_3(x_j)E\{f_1(y_n)\frac{g(y_n)}{r(x_n,y_n)}|x_n, F_{n-1}\}\} \\
&= \int\limits_{R^d} f_1(y)g(y)\,dy\ E\{\Lambda_{j-1}E\{f_3(x_j)|F_{j-1}\}\} \\
&= \int\limits_{R^d} f_1(y)g(y)\,dy E\{\Lambda_{j-1}P^{a_{j-1}}(x_{j-1}, f_3)\}
\end{aligned}$$

which gives the second statement of the Lemma for $j = 1$.

Taking into account that $a_{j-1} = d_{j-1}(y_1, \ldots, y_{j-1})$ for some $d_{j-1} \in \mathcal{B}((R^d)^{j-1}, U)$ we have for $j > 1$

$$\begin{aligned}
&E\{\Lambda_{j-1}P^{a_{j-1}}(x_{j-1}, f_3)\} \\
&\quad = E\{\Lambda_{j-2}E\{P^{a_{j-1}}(x_{j-1}, f_3)\frac{g(y_{j-1})}{r(x_{j-1},y_{j-1})}|x_{j-1}, F_{j-2}\}\}
\end{aligned}$$

$$= E\{\Lambda_{j-2} \int\limits_{R^d} G_{j-1}(y_1, \ldots, y_{j-2}, y, x_{j-1}) g(y)\, dy\}$$

$$= E\{\Lambda_{j-2} \int\limits_E \int\limits_{R^d} G_{j-1}(y_1, \ldots, y_{j-2}, y, z) g(y) dy P^{a_{j-2}}(x_{j-2}, dz)\}$$

with $G_{j-1}(y_1, \ldots, y_{j-1}, x) = P^{d_{j-1}(y_1, \ldots, y_{j-1})}(x, f_3)$

We can iterate the last equality which leads to the conclusion that, under P^0 , y_n are independent of x_j with $j \leq n$ and (x_n) is a controlled Markov process with transition operator $P^{a_n}(x_n, \cdot)$.

■

For later use it will be convenient to define an unnormalized filtering process σ_n^u as follows, where $A \in \mathcal{B}(E)$

$$\begin{aligned} \sigma_0^u(A) &= \mu(A) \quad \text{the initial law of } (x_n) \\ \sigma_{n+1}^u(A) &= \int\limits_A \frac{r(z, y_{n+1})}{g(y_{n+1})} P^{a_n}(\sigma_n^u, dz) \end{aligned} \tag{1.31}$$

Clearly

$$\pi_n^u(A) = \frac{\sigma_n^u(A)}{\sigma_n^u(E)} \qquad P_\mu^0 \text{ a.e.} \tag{1.32}$$

Moreover

Lemma 1.9 *For $f \in b\mathcal{B}(E)$, $n = 0, 1, \ldots,$ we have under (A3)*

$$\sigma_n^u(f) = E^0\{L_n f(x_n) | Y^n\} \qquad P_\mu^0 \text{ a.e.} \tag{1.33}$$

P r o o f. We use induction. Since $Y^0 = \{\emptyset, \Omega\}$ and $L_0 = 1$ we have $\sigma_0^u(f) = \mu(f)$ and (1.33) holds for $n = 0$.

Assume (1.33) is satisfied for n. Then for $n+1$ we have

$$\begin{aligned} & E^0\{L_{n+1} f(x_{n+1}) | Y^{n+1}\} \\ &= E^0\{L_n E^0\Big\{\frac{r(x_{n+1}, y_{n+1})}{g(y_{n+1})} f(x_{n+1}) | F_n, y_{n+1}\Big\} | Y^{n+1}\} \\ &= E^0\{L_n \int\limits_E \frac{r(z, y_{n+1})}{g(y_{n+1})} f(z) P^{a_n}(x_n, dz) | Y^{n+1}\} \\ &= \int\limits_E \frac{r(z, y_{n+1})}{g(y_{n+1})} f(z) P^{a_n}(\sigma_n^u, dz) \end{aligned}$$

by the independence of y_{n+1} and induction hypothesis on n. Taking into account (1.31) this means that we have obtained

$$E^0\{L_{n+1}f(x_{n+1})|Y^{n+1}\} = \sigma^u_{n+1}(f)$$

Thus by induction (1.33) holds.

■

From the representation (1.33) we immediately have

Corollary 1.10 *The process $\sigma^u_n(E)$ is a (Y^n, P^0) martingale with mean 1.*

With the use of the measure transformation just introduced, Lemma 1.8 and the fact that a_n is Y^n measurable we can rewrite the cost functions for the problems I–III (see (1.4)–(1.6)) as follows, assuming that $c_n,\ c \in b\mathcal{B}(E \times U)$, $b \in b\mathcal{B}(E)$

I)

$$\begin{aligned} J^T_\mu(u) &= E^{0u}_\mu\{L_T\Big(\sum_{n=0}^{T-1} c_n(x_n, a_n) + b(x_T)\Big)\} \\ &= \sum_{n=0}^{T-1} E^{0u}_\mu\{L_n c_n(x_n, a_n)\} + E^{0u}_\mu\{L_T b(x_T)\} \\ &= \sum_{n=0}^{T-1} E^{0u}_\mu\{E^{0u}_\mu\{L_n c_n(x_n, a_n)|Y^n\}\} \\ &\quad + E^{0u}_\mu\{E^{0u}_\mu\{L_T b(x_T)|Y^T\}\} \\ &= \sum_{n=0}^{T-1} E^{0u}_\mu\{\int_E c_n(x_n, a_n)\sigma^u_n(dx)\} \\ &\quad + E^{0u}_\mu\{\int_E b(x)\sigma^u_T(dx)\} \\ &= E^{0u}_\mu\{\sum_{n=0}^{T-1}\int_E c_n(x, a_n)\sigma^u_n(dx) + \int_E b(x)\sigma^u_T(dx)\} \end{aligned} \tag{1.34}$$

II)

$$J^\beta_\mu(u) = \sum_{n=0}^{\infty} \beta^n E^{0u}_\mu\{\int_E c(x, a_n)\sigma^u_n(dx)\} \tag{1.35}$$

III)

$$J_\mu(u) = \limsup_{T\to\infty} T^{-1} E_\mu^{0u}\{\int_E c(x,a_n)\sigma_n^u(dx)\} \tag{1.36}$$

The original control problems with partial observation have thus been reformulated as control problems with complete observation where analogously to (1.10)–(1.12) the new state is the unnormalized filter process σ_n that satisfies the linear relation (1.31) and the cost functionals are given by (1.34)–(1.36).

Notice also that the cost functionals (1.34)–(1.36) are expressed as expectations with respect to the reference probability measure P^0, with respect to which y_n are i.i.d. with common density $g(y)$.

In the particular case when the state and observation spaces are finite sets we can also construct a measure transformation. For this purpose we assume that

$$y_n = h(x_n, w_n)$$

with (w_n) being i.i.d.D-valued random variables, $P\{w_n = d_i\} = g(d_i) > 0$ for $1 \leq i \leq s$ and for each $x \in E$, $h(x,\cdot)$ is a $1-1$ transformation of D with inverse function $k(x,\cdot)$.

Then (1.3) holds with $r(x,y) = g(k(x,y))$, Λ_n defined by (1.27) with $r(x_i,y_i) = g(k(x_i,y_i))$ is a P martingale and Lemma 1.8 holds.

Moreover, by analogy to (1.31), one can define an unnormalized filtering process as

$$\begin{aligned} &\sigma_0^u(j) = \mu(j) \quad \text{the initial law of } (x_n) \\ &\sigma_{n+1}^u(j) = \tfrac{r(j,y_{n+1})}{g(y_{n+1})} P^{a_n}(\sigma_n^u, j) \end{aligned} \tag{1.37}$$

$j = 1, 2, \ldots, m$ for which (1.32), (1.33) and Corollary 1.10 hold.

Furthermore, replacing the integrals over E by suitable summations in (1.34)-(1.36) we obtain the corresponding representations of the cost functionals.

2 Finite horizon problem

2.1 The idea of the approximation approach

Although the basic idea underlying our approximation approach that leads to the construction of nearly optimal controls is the same throughout, in this section we shall present it in a context based on measure transformation for which the original finite horizon partially observed problem can equivalently be represented in the complete observation form (1.31), (1.34). As already mentioned at the beginning of section 1.3, the reason for this is that the measure transformation allows our approximation approach to be performed in the most efficient way.

On the other hand, to perform the measure transformation, we need some regularity of the observation function as described below (1.25); furthermore, see section 2.2, under the measure transformation we need a strong approximation for transition operators. If the conditions for the applicability of the measure transformation do not hold, we have to use the normalized filter (π_n^u) defined as in (1.7) and consider for the original problem the equivalent complete observation form (1.8), (1.10). In the next chapter 3, in the context of the infinite horizon problem with discounting, we shall discuss our approximation approach for this latter situation. We remark here that, although in chapter 3 the approach is worked out for the infinite horizon case with discounting, it can be easily adapted also to the present finite horizon case. Consider the original finite horizon control problem, which in its partial observation form is characterized by the observation equation (1.25) and cost (1.4) and which has the equivalent complete observation representation (1.31), (1.34). We shall now associate with it a sequence of approximating problems such that

a) each approximating problem admits an optimal, or nearly optimal, solution that can be explicitly computed,

b) given $\varepsilon > 0$, there exists an approximating problem such that the optimal, or nearly optimal, solution for the latter is an ε-optimal solution for the original problem

The approximating problems will be obtained by suitably approximating the original transition operator $P^a(x, dz)$ and observation function $h(x, w)$

in (1.25) by a sequence of operators $P_m^a(x, dz)$ and functions $h_m(x, w)$, that induce a sequence of approximating processes (x_n^m) with observations (y_n^m). For fixed x, $h_m(x, w)$ will still be a C^1 diffeomorphism of R^d. Assume for a moment that this has been done; given probability spaces (Ω, F, P^m) on which the approximating processes (x_n^m) and (y_n^m) are defined, following the general measure transformation approach of Section 1.3, we construct on (Ω, F) the reference probability measure P^0 such that

$$P^0(d\omega) = \Lambda_T^m(\omega) P^m(d\omega) \tag{2.1}$$

or equivalently

$$P^m(d\omega) = L_T^m(\omega) P^0(d\omega) \tag{2.2}$$

where

$$\Lambda_T^m = \prod_{i=1}^{T} \frac{g(y_i^m)}{r_m(x_i^m, y_i^m)} \quad \text{and} \quad L_T^m = (\Lambda_T^m)^{-1} \tag{2.3}$$

with

$$r_m(x, y) = g(k_m(x, y))|\Delta_m(x, y)| \tag{2.4}$$

$k_m(x, \cdot)$ being the inverse of $h_m(x, \cdot)$ and $|\Delta_m(x, y)|$ standing for the determinant of the Jacobian of $k_m(x, \cdot)$. It then follows from Section 1.3 that the unobserved processes (x_n^m) have the same distribution under the measures P^m and P^0. The observation processes y_n^m, which under P^m are defined by the relation

$$y_n^m = h_m(x_n^m, w_n) \tag{2.5}$$

with w_n i.i.d. independent of x_i, $(i \le n)$, and having common density $g(\cdot)$, under the measure P^0 form sequences of i.i.d. random variables, independent of x_i $(i \le n)$, with common density $g(\cdot)$.

Therefore, under P^0 we can identify (y_n^m) with (y_n). Thus the measure transformation approach allows us to obtain the same observations (y_n), both as perturbed functions of the original process (x_n) as well as of the approximating processes (x_n^m).

Since the admissible controls are adapted to the σ-algebra generated by the actual observations (y_n), the same controls will therefore be admissible in the original as well as in all approximating problems. For a given control law u and corresponding to (1.31), (1.34) we furthermore have the approximating unnormalized filter processes

$$\begin{aligned} \sigma_0^{m,u}(A) &= \mu(A) \\ \sigma_{n+1}^{m,u}(A) &= \int_A \frac{r_m(z, y_{n+1})}{g(y_{n+1})} P_m^{a_n}(\sigma_n^{m,u}, dz) \end{aligned} \tag{2.6}$$

and the approximating cost functionals

$$J_\mu^{T,m}(u) = E_\mu^{0u}\Big\{ \sum_{n=0}^{T-1} \int_E c_n^m(x, a_n)\sigma_n^{m,u}(dx) + \int_E b^m(x)\sigma_T^{m,u}(dx) \Big\} \tag{2.7}$$

with c_n^m, b^m being approximations of c_n, b respectively.

With the same admissible controls u in the original and approximating problems, and with the cost functionals in (1.34) and (2.7) expressed as expectations with respect to the same reference probability measure P^0, under which y_n are i.i.d. with common density $g(\cdot)$, we can compare the cost functionals of the original and approximating problems, evaluated for a same control law u.

We shall choose the approximating transition operators $P_m^a(x, dz)$, observation functions $h_m(x, w)$ and cost functions c_n^m, b^m in a way to obtain the following properties

(P1) $$\lim_{m\to\infty} \sup_u |J_\mu^{T,m}(u) - J_\mu^T(u)| = 0$$
where the sup over u is for all admissible control laws

(P2) For each given m, the approximating control problem expressed in its equivalent complete observation form by (2.6), (2.7), can be explicitly solved to obtain an optimal or nearly optimal control law u

The meaning of (P1), (P2) is explained by the following

Lemma 2.1 *Assume that for given $\mu \in P(E)$ and $\varepsilon > 0$ there exists m_0 such that for $m > m_0$*

$$\sup_u |J_\mu^{T,m}(u) - J_\mu^T(u)| < \varepsilon \tag{2.8}$$

Then an ε-optimal control $\overline{u}$ for the cost functional $J_\mu^{T,m}$ with $m > m_0$ is 3 ε-optimal for the cost functional J_μ^T.

P r o o f. Notice first that, for $m > m_0$,

$$|\inf_u J_\mu^{T,m}(u) - \inf_u J_\mu^T(u)| \leq \sup_u |J_\mu^{T,m}(u) - J_\mu^T(u)| \leq \varepsilon$$

We then have for $m > m_0$

$$J_\mu^T(\overline{u}) \leq J_\mu^{T,m}(\overline{u}) + \varepsilon \leq \inf_u J_\mu^{T,m}(u) + 2\varepsilon \leq \inf_u J_\mu^T(u) + 3\varepsilon$$

and the Lemma follows.

■

In other words, we can say that a nearly optimal control for m-th approximating problem is, for m sufficiently large, nearly optimal in the original problem.

The uniform in the control u convergence in (P1) will be the subject of next Section 2.2, while the solutions of the approximating problems will be discussed in Section 2.3.

2.2 Convergence

In this section we shall prove, under various assumptions, the uniform in admissible control laws convergence of the cost functionals as expressed in property (P1) above. We shall do this for two cases, namely when the cost functions $c_n(x, a)$, $b(x)$ in (1.4) are bounded and when they have polynomial growth in x. In the latter case we shall require the transition operators of the processes x_n and x_n^m as well as the initial measure μ to admit a density (with respect to Lebesgue measure).

2.2.1 Bounded cost functions

In this section we shall assume that

(B1) $c_n \in b\mathcal{B}(E \times U)$ and $b \in b\mathcal{B}(E)$,

(B2) for $\varepsilon > 0$ and a compact set $L \subset E$ there exists a compact set $K \subset E$ such that

$$\sup_{x \in L} \sup_{a \in U} P^a(x, K^c) < \varepsilon$$

In addition, we assume that the state process (x_n) with initial law μ and transition operator $P^a(x, dz)$ is approximated by (x_n^m) with initial law μ_m and transition operators $P_m^a(x, dz)$, satisfying the following conditions

(C1) $\|\mu_m - \mu\|_{\text{var}} \to 0$ as $m \to \infty$ with $\| \ \|_{\text{var}}$ standing for variation norm

(C2) for each compact set $K \subset E$

$$\sup_{x \in K} \sup_{a \in U} \|P_m^a(x, \cdot) - P^a(x, \cdot)\|_{\text{var}} \to 0$$

as $m \to \infty$

Furthermore the observation function $h(x, w)$ is approximated by functions $h_m(x, w)$ satisfying

(C3) for each compact set $K \subset E$

$$\sup_{x \in K} \int_{R^d} |r(x, y) - r_m(x, y)| \, dy \to 0$$

as $m \to \infty$.

We have

Proposition 2.2 *Let $(\sigma_n^{m,u})$ be the unnormalized filter process defined by (2.6) with initial law μ_m and admissible control u, and let (B1)–(B2), (C1)–(C3) be satisfied.*
Then for $n = 0, 1, 2, \ldots$ we have

(i) *for each $\varepsilon > 0$ there exists a compact set $K \subset E$ such that*

$$\sup_u E^0\{\sigma_n^u(K^c)\} < \varepsilon \tag{2.9}$$

(ii)

$$\sup_u E^0\{\|\sigma_n^{m,u} - \sigma_n^u\|_{\text{var}}\} \to 0 \tag{2.10}$$

as $m \to \infty$

where sup *in both cases is taken over all admissible controls.*

P r o o f. We use induction. By the tightness of μ, (2.9) holds for $n = 0$. Also, by (C1), (2.10) is satisfied for $n = 0$.

Assume then that (2.9)–(2.10) are satisfied for n. We have for $n+1$

$$\begin{aligned} E^0\{\sigma^u_{n+1}(K^c)\} &= E^0\Big\{\int\limits_{K^c} \frac{r(z,y_{n+1})}{g(y_{n+1})} P^{a_n}(\sigma^u_n, dz)\Big\} \\ &= E^0\Big\{\int\limits_{R^d}\int\limits_{K^c} r(z,y) P^{a_n}(\sigma^u_n, dz)\, dy\Big\} \\ &= E^0\{P^{a_n}(\sigma^u_n, K^c)\} \\ &= E^0\Big\{\int\limits_{L^c} P^{a_n}(x,K^c)\sigma^u_n(dx) + \int\limits_{L} P^{a_n}(x,K^c)\sigma^u_n(dx)\Big\} \\ &\le E^0\{\sigma^u_n(L^c)\} + \sup_{x\in L}\sup_{a\in U} P^a(x,K^c) E^0\{\sigma^u_n(E)\} \end{aligned}$$

By the induction hypothesis on n, for given $\varepsilon > 0$ one can find a compact set L such that, uniformly in u, $E^0\{\sigma^u_n(L^c)\} < \frac{\varepsilon}{2}$. Furthermore by (B2) we can find a compact set K for which

$$\sup_{x\in L}\sup_{a\in U} P^a(x,K^c) < \frac{\varepsilon}{2}$$

Using also Corollary 1.10, we finally have

$$\sup_u E^0\{\sigma^u_{n+1}(K^c)\} < \varepsilon$$

which is (2.9) for $n+1$.

Now,

$$\begin{aligned} &E^0\{\sup_A |\sigma^{m,u}_{n+1}(A) - \sigma^u_{n+1}(A)|\} \\ &\le E^0\Big\{\int\limits_E \frac{|r_m(z,y_{n+1}) - r(z,y_{n+1})|}{g(y_{n+1})} P^{a_n}_m(\sigma^{m,u}_n, dz)\Big\} \\ &+ E^0\Big\{\sup_A\Big|\int\limits_A \frac{r(z,y_{n+1})}{g(y_{n+1})}(P^{a_n}_m(\sigma^{m,u}_n, dz) - P^{a_n}(\sigma^{m,u}_n, dz))\Big| \\ &+ \sup_A\Big|\int\limits_A \frac{r(z,y_{n+1})}{g(y_{n+1})}\int\limits_E P^{a_n}(x,dz)(\sigma^{m,u}_n(dx) - \sigma^u_n(dx))\Big|\Big\} \end{aligned}$$

$$
\begin{aligned}
&\leq E^0\Big\{\int_E\int_{R^d} |r_m(z,y)-r(z,y)|\,dyP_m^{a_n}(\sigma_n^{m,u},dz)\}\\
&+E^0\Big\{\int_E \frac{r(z,y_{n+1})}{g(y_{n+1})}|P_m^{a_n}(\sigma_n^{m,u},dz)-P^{a_n}(\sigma_n^{m,u},dz)|\Big\}\\
&+E^0\Big\{\int_E \frac{r(z,y_{n+1})}{g(y_{n+1})}\int_E P^{a_n}(x,dz)|\sigma_n^{m,u}(dx)-\sigma_n^u(dx)|\Big\}\\
&\leq E^0\Big\{\int_E\int_{R^d} |r_m(z,y)-r(z,y)|\,dyP_m^{a_n}(\sigma_n^{n,u},dz)\Big\}\\
&+E^0\{|P_m^{a_n}(\sigma_n^{m,u},E)-P^{a_n}(\sigma_n^{m,u},E)|\}\\
&+E^0\{|\sigma_n^{m,u}(E)-\sigma_n^u(E)|\} = I_m + II_m + III_m
\end{aligned}
$$

where we used the fact that, under P^0, y_{n+1} is independent of Y^n.

Clearly, by the induction hypothesis, $III_m \to 0$, uniformly in the admissible controls u.

Moreover we have

$$
\begin{aligned}
I_m &\leq E^0\Big\{\int_L\int_E\int_{R^d} |r_m(z,y)-r(z,y)|\,dyP_m^{a_n}(x,dz)\sigma_n^{m,u}(dx)+2\sigma_n^{m,u}(L^c)\Big\}\\
&\leq E^0\Big\{\int_L\int_K\int_{R^d} |r_m(z,y)-r(z,y)|\,dyP_m^{a_n}(x,dz)\sigma_n^{m,u}(dx)\\
&\quad+\int_L\int_{K^c}\int_{R^d} |r_m(z,y)-r(z,y)|\,dyP_m^{a_n}(x,dz)\sigma_n^{m,u}(dx)+2\sigma_n^{m,u}(L^c)\Big\}\\
&\leq \sup_{x\in K}\int_{R^d} |r_m(z,y)-r(z,y)|\,dyE^0\{\sigma_n^{m,u}(L)\}\\
&\quad+2\sup_{x\in L}\sup_{a\in U} P_m^a(x,K^c)E^0\{\sigma_n^{m,u}(L)\}+2E^0\{\sigma_n^{m,u}(L^c)\}\\
&\leq \sup_{x\in K}\int_{R^d} |r_m(z,y)-r(z,y)|\,dy + 2\sup_{x\in L}\sup_{a\in U} P^a(x,K^c)\\
&\quad+2\sup_{x\in L}\sup_{a\in U}\|P^a(x,\cdot)-P_m^a(x,\cdot)\|_{\mathrm{var}}+2E^0\{\sigma_n^u(L^c)\}\\
&\quad+2E^0\{\|\sigma_n^{m,u}-\sigma_n^u\|_{\mathrm{var}}\}
\end{aligned}
$$

where we have used the fact that $E^0\{\sigma_n^{m,u}(L)\}\leq 1$ due to the martingale property of $\sigma_n^{m,u}$ (see Corollary 1.10).

Given $\varepsilon > 0$, by the induction hypothesis, we can find a compact set L such that $\sup_u E^0\{\sigma_n^u(L^c)\} < \frac{\varepsilon}{4}$. Furthermore by (B2) we can choose a compact set K for which

$$\sup_{x\in L}\sup_{a\in U} P^a(x, K^c) < \frac{\varepsilon}{4}$$

Letting $m \to \infty$, by (C3), (C2) and the induction hypothesis we obtain

$$\limsup_{m\to\infty} I_m \leq \varepsilon$$

Since ε can be chosen arbitrarily small we have $I_m \to 0$, uniformly in u.

It remains to show that $II_m \to 0$, as $m \to \infty$.

We have

$$\begin{aligned} II_m &\leq \sup_{x\in K}\sup_{a\in U} |P_m^a(x,E) - P^a(x,E)| E^0\{\sigma_n^{m,u}(K)\} \\ &\quad + 2E^0\{\sigma_n^{m,u}(K^c)\} \\ &\leq \sup_{x\in K}\sup_{a\in U} |P_m^a(x,E) - P^a(x,E)| \\ &\quad + 2E^0\{\sigma_n^u(K^c)\} + 2E^0\{\|\sigma_n^{m,u} - \sigma_n^u\|_{\mathrm{var}}\} \end{aligned}$$

By the induction hypothesis for given $\varepsilon > 0$ one can find a compact set K such that

$$\sup_u 2E^0\{\sigma_n^u(K^c)\} < \varepsilon$$

Letting $m \to \infty$, by (C2) and the induction hypothesis we then obtain $\limsup_{m\to\infty} II_m \leq \varepsilon$, which in view of arbitrariness of $\varepsilon > 0$ completes the proof of (2.9)–(2.10) for $n+1$.

Thus by induction (2.9)–(2.10) holds for any $n = 0, 1, 2, \ldots$

$\blacksquare$

To obtain the desired property (P1) we make further assumptions on the cost functions in (2.7).

(C4) $c_n^m \in b\mathcal{B}(E \times U)$, $b^m \in b\mathcal{B}(E)$, $n = 1, 2, \ldots$, c_n^m and b^m are uniformly bounded in m and for any compact set $K \subset E$

$$\sup_{x\in K}\sup_{a\in U} |c_n^m(x,a) - c_n(x,a)| \to 0, \quad \sup_{x\in K} |b^m(x) - b(x)| \to 0$$

From Proposition 2.2 we obtain

Corollary 2.3 *Under (B1)–(B2), (C1)–(C4)*

$$\lim_{m\to\infty} \sup_u |J^{T,m}_{\mu_m}(u) - J^T_\mu(u)| = 0 \tag{2.11}$$

P r o o f. Given $\varepsilon > 0$ we can find a compact set K such that (2.9) holds for $n = 0, 1, 2, \ldots, T$. Then we have

$$\begin{aligned}
|J^{T,m}_{\mu_m}(u) - J^T_\mu(u)| \leq E^{0u}\Big\{ & \sum_{n=0}^{T-1} \int_E |c^m_n(x, a_n)|\ |\sigma^{m,u}_n(dx) - \sigma^u_n(dx)| \\
&+ \int_E |b^m(x)|\ |\sigma^{m,u}_T(dx) - \sigma^u_T(dx)|\Big\} \\
&+ E^{0u}\Big\{ \sum_{n=0}^{T-1} \int_K |c^m_n(x, a_n) - c_n(x, a_n)|\sigma^u_n(dx) \\
&+ \sum_{n=0}^{T-1} \int_{K^c} |c^m_n(x, a_n) - c_n(x, a_n)|\sigma^u_n(dx) \\
&+ \int_K |b^m(x) - b(x)|\sigma^u_T(dx) + \int_{K^c} |b^m(x) - b(x)|\sigma^u_T(dx)\} \\
&\leq \sum_{n=0}^{T-1} \|c^m_n\| E^{0u}\{\|\sigma^{m,u}_n - \sigma^u_n\|_{\text{var}}\} \\
&+ \|b^m\| E^{0u}\{\|\sigma^{m,u}_T - \sigma^u_T\|_{\text{var}}\} \\
&+ \sum_{n=0}^{T-1} \sup_{x\in K} \sup_{a\in U} |c^m_n(x, a) - c_n(x, a)| E^{0u}\{\sigma^u_n(K)\} \\
&+ \sum_{n=0}^{T-1} (\|c^m_n\| + \|c_n\|) E^{0u}\{\sigma^u_n(K^c)\} \\
&+ \sup_{x\in K} |b^m(x) - b(x)| E^{0u}\{\sigma^u_T(K)\} + (\|b^m\| + \|b\|) E^{0u}\{\sigma^u_T(K^c)\} \\
&\leq \sum_{n=0}^{T-1} \|c^m_n\| E^{0u}\{\|\sigma^{m,u}_n - \sigma^u_n\|_{\text{var}} + \|b^m\| E^{0u}\{\|\sigma^{m,u}_T - \sigma^u_T\|_{\text{var}}\} \\
&+ \sum_{n=0}^{T-1} \sup_{x\in K} \sup_{a\in U} |c^m_n(x, a) - c_n(x, a)| + \varepsilon \sum_{n=0}^{T-1} (\|c^m_n\| + \|c_n\|) \\
&+ \sup_{x\in K} |b^m(x) - b(x)| + (\|b^m\| + \|b\|)\varepsilon
\end{aligned}$$

By (2.10) and (C4) letting $m \to \infty$ we obtain that the right hand side does not exceed

$$\varepsilon\Big(\limsup_{m\to\infty}\Big[\sum_{n=0}^{T-1}(\|c_n^m\| + \|c_n\|) + \|b^m\| + \|b\|\Big]\Big)$$

Since we can choose ε arbitrarily small, and our calculation was uniform with respect to admissible controls u we conclude that (2.11) holds.

■

2.2.2 Cost functions with polynomial growth

In this subsection we make the standing assumption that $E = R^k$ and that the transition operator of the process x_n, as well as the initial law μ possess a density with respect to the k-dimensional Lebesgue measure i.e.

(B3) there exist measurable functions

$$p: E \times E \times U \to R \quad \text{and} \quad \mu: E \to R$$

such that

$$P^a(x, dz) = p(x, z, a)\, dz$$

and

$$\mu(dx) = \mu(x)\, dx$$

In addition we shall assume the existence of moment generating functions for p and μ, namely

(B4) for any positive constant K, there exist $C,\ H > 0$ such that

$$\int_E \mu(x) e^{K|x|}\, dx \le C \tag{2.12}$$

and for each $x \in E$

$$\sup_{a\in U}\int_E p(x, z, a) e^{K|z|}\, dz \le C e^{H|x|} \tag{2.13}$$

with $|x| = |x^1| + \ldots + |x^k|$ for $x = (x^1, \ldots, x^k) \in R^k$.

Remark 2.4 *If $\mu(x)$ is a Gaussian density, (2.12) clearly holds. Furthermore, letting*

$$x_{n+1} = f(x_n, a_n) + \sigma w_n \tag{2.14}$$

with (w_n) i.i.d. Gaussian and f such that

$$|f(x,a)| \leq K_1 |x| \tag{2.15}$$

also (2.13) holds with $H = KK_1$.

■

Under (B3) we easily see that the unnormalized filter process (σ_n^u) defined in (1.31) admits densities $\sigma_n^u(x)$ which by (1.31) satisfy the following recursive formula

$$\begin{aligned} \sigma_0^u(x) &= \mu(x) \\ \sigma_{n+1}^u(x) &= \int\limits_E \frac{r(x, y_{n+1})}{g(y_{n+1})} p(z, x, a_n) \sigma_n^u(z)\, dz \end{aligned} \tag{2.16}$$

Moreover under (B4) (σ_n^u) admit moment generating functions. More precisely, we have

Lemma 2.5 *Under (B3)–(B4) for any $K > 0$, and $n \leq T$*

$$\sup_u E_\mu^{0u} \Big\{ \int\limits_E e^{K|x|} \sigma_n^u(x)\, dx \Big\} < \infty \tag{2.17}$$

with supremum over all admissible controls.

P r o o f. For $n = 0$, (2.17) holds by (2.12). If $n > 0$, we have by (2.16) and (2.13)

$$\begin{aligned} &E_\mu^{0u} \Big\{ \int\limits_E e^{K|x|} \sigma_n^u(x)\, dx \Big\} = \\ &= E_\mu^{0u} \Big\{ \int\limits_E \int\limits_E e^{K|x|} \frac{r(x, y_n)}{g(y_n)} p(z, x, a_{n-1}) \sigma_{n-1}^u(z)\, dz\, dx \Big\} \\ &= E_\mu^{0u} \Big\{ \int\limits_E \Big(\int\limits_E e^{K|x|} p(z, x, a_{n-1}) dx \Big) \sigma_{n-1}^u(z)\, dz \Big\} \\ &\leq C E_\mu^{0u} \Big\{ \int\limits_E e^{H|z|} \sigma_{n-1}^u(z)\, dz \Big\} \end{aligned}$$

and by iteration we obtain (2.17).

■

As mentioned above, in this section we allow the cost functions c_n, b to be of polynomial growth. Therefore we assume

(B5) $c_n: E \times U \to R$ and $b: E \to R$ are measurable for $n = 1, 2, \ldots, T$ and there exist constants $C,\ K > 0$ such that

$$|c_n(x,a)| \le Ce^{K|x|}, \qquad |b(x)| \le Ce^{K|x|} \tag{2.18}$$

for $x \in E$, $a \in U$, $n = 1, 2, \ldots, T$.

Considering the cost functional J_μ^T (see (1.4)), namely

$$J_\mu^T(u) = E_\mu^u\Big\{\sum_{n=0}^{T-1} c_n(x_n, a_n) + b(x_T)\Big\} \tag{2.19}$$

under (B5) it is not clear that (2.19) is well defined, namely that the expectation exists and is finite. However, by Lemma 2.5 we obtain

Corollary 2.6 *Under (B3)–(B5) we have*

$$J_\mu^T(u) = E_\mu^u\Big\{\sum_{n=0}^{T-1} \int_E c_n(x, a_n)\sigma_n^u(x)\,dx + \int_E b(x)\sigma_T^u(x)\,dx\Big\} \tag{2.20}$$

and

$$\sup_u J_\mu^T(u) < \infty \tag{2.21}$$

P r o o f. Clearly $\sum\limits_{n=0}^{T-1} c_n(x_n, a_n) + b(x_T)$ is P_μ^u integrable if and only if $L_T\Big(\sum\limits_{n=0}^{T-1} c_n(x_n, a_n) + b(x_T)\Big)$ is P_μ^{0u} integrable. The latter random variable is P_μ^{0u} integrable if

$$\sum_{n=0}^{T-1} E_\mu^{0u}\{L_n|c_n(x_n, a_n)|\ |Y^n\} + E_\mu^{0u}\{L_T|b(x_T)|\ |Y^T\}$$

is P^{0u}_μ integrable. By (B5), Lemma 1.8 and Lemma 2.5 we have

$$\begin{aligned}
&E^{0u}_\mu\Big\{\sum_{n=0}^{T-1} E^{0u}_\mu\{L_n|c_n(x_n,a_n)|\ |Y^n\} + E^{0u}_\mu\{L_T|b(x_T)|\ |Y^T\}\Big\} \\
&\quad\le E^{0u}_\mu\Big\{\sum_{n=0}^{T-1} E^{0u}_\mu\{L_n C e^{K|x_n|}|Y^n\} + E^{0u}_\mu\{L_T C e^{K|x_T|}|Y^T\}\Big\} \\
&\quad= E^{0u}_\mu\Big\{\sum_{n=0}^{T-1} C\int_E e^{K|x|}\sigma^u_n(x)\,dx + C\int_E e^{K|x|}\sigma^u_T(x)\,dx\Big\} < \infty
\end{aligned}$$

Therefore $J^T_\mu(u)$ in (2.19) is well defined and consequently we have (2.20) and (2.21).

■

Notice now that, contrary to the previous subsection, we neither have to approximate the observation function nor the cost functions. We therefore impose only conditions on initial laws μ_m and transition operators $P^a_m(x,dz)$ of the approximating processes (x^m_n). More precisely, we assume that

(C5) there exist Borel measurable functions

$$p_m\colon E\times E\times U\to R \quad\text{and}\quad \mu_m\colon E\to R$$

such that

$$P^a_m(x,dz) = p_m(x,z,a)\,dz, \qquad \mu_m(dz) = \mu_m(z)\,dz$$

(C6) for any positive constant K, there exist $H>0$ and $\Delta_m\to 0$ such that

$$\sup_{a\in U}\int_E |p(x,z,a)-p_m(x,z,a)|e^{K|z|}\,dz \le \Delta_m e^{H|x|} \tag{2.22}$$

for $x\in E$, and

$$\int_E |\mu_m(z)-\mu(z)|e^{K|z|}\,dz \le \Delta_m \tag{2.23}$$

By analogy to (2.16) define approximating unnormalized filter densities as follows

$$\begin{aligned} \sigma_0^{m,u}(x) &= \mu_m(x) \\ \sigma_{n+1}^{m,u}(x) &= \int\limits_E \frac{r(x,y_{n+1})}{g(y_{n+1})} p_m(z,x,a_n)\sigma_n^{m,u}(z)\,dz \end{aligned} \tag{2.24}$$

We now have the following

Proposition 2.7 *Under (B3), (B4), (C5), (C6), for all $K>0$, $n=0,\ldots,T$*

$$\lim_{m\to\infty}\sup_u E^{0,u}\Big\{\int\limits_E |\sigma_n^{m,u}(x)-\sigma_n^u(x)|e^{K|x|}\,dx\Big\}=0 \tag{2.25}$$

P r o o f. For $n=0$, (2.25) clearly holds by (2.23). For $n>0$ by (2.22), (2.13) we have

$$\begin{aligned} &E^{0,u}\Big\{\int\limits_E |\sigma_n^{m,u}(x)-\sigma_n^u(x)|e^{K|x|}\,dx\Big\} \le \\ &\le E^{0,u}\Big\{\int\limits_E\int\limits_E \frac{r(x,y_n)}{g(y_n)}|p_m(z,x,a_{n-1})-p(z,x,a_{n-1})|\sigma_{n-1}^{m,u}(z)e^{K|x|}\,dz\,dx \\ &+\int\limits_E\int\limits_E \frac{r(x,y_n)}{g(y_n)}p(z,x,a_{n-1})|\sigma_{n-1}^{m,u}(z)-\sigma_{n-1}^u(z)|e^{K|x|}\,dz\,dx\Big\} \\ &= E^{0,u}\Big\{\int\limits_E\int\limits_E |p_m(z,x,a_{n-1})-p(z,x,a_{n-1})|\sigma_{n-1}^{m,u}(z)e^{K|x|}\,dz\,dx \\ &+\int\limits_E\int\limits_E p(z,x,a_{n-1})|\sigma_{n-1}^{m,u}(z)-\sigma_{n-1}^u(z)|e^{K|x|}\,dz\,dx\Big\} \\ &\le \Delta_m E^{0,u}\Big\{\int\limits_E e^{H|z|}\sigma_{n-1}^{m,u}(z)\,dz\Big\}+CE^{0,u}\Big\{\int\limits_E e^{H|z|}|\sigma_{n-1}^{m,u}(z)-\sigma_{n-1}^u(z)|\,dz\Big\} \\ &\le \Delta_m E^{0,u}\Big\{\int\limits_E e^{H|z|}\sigma_{n-1}^u(z)\,dz\Big\} \\ &+(C+\Delta_m)E^{0,u}\Big\{\int\limits_E e^{H|z|}|\sigma_{n-1}^{m,u}(z)-\sigma_{n-1}^u(z)|\,dz\Big\} \end{aligned}$$

assuming, as we can, that H from (B4) and (C6) are the same.

Now, by Lemma 2.5 the first term on the right hand side converges to 0 uniformly with respect to admissible controls.

Therefore, if $n = 1$ we have the desired conclusion. In the case when $n > 1$, just iterate the last term in the above inequality. Since at most T iterations are required, (2.25) follows.

■

Let

$$J_\mu^{T,m}(u) = E_\mu^{0,u}\Big\{ \sum_{n=0}^{T-1} \int_E c_n(x, a_n)\sigma_n^{m,u}(x)\, dx + \int_E b(x)\sigma_T^{m,u}(x)\, dx \Big\} \tag{2.26}$$

Using Proposition 2.7 we can apply Corollary 2.6 to obtain that $J_\mu^{T,m}$ is well defined.

Moreover by Proposition 2.7 and (B5), we almost immediately have

Corollary 2.8 *Under (B3)–(B5) and (C5), (C6) property (P1) holds i.e.*

$$\lim_{m\to\infty} \sup_u |J_\mu^T(u) - J_{\mu_m}^{T,m}(u)| = 0 \tag{2.27}$$

P r o o f. By (B5) we have

$$\begin{aligned} &|J_\mu^T(u) - J_{\mu_m}^{T,m}(u)| \\ &\leq C \sum_{n=0}^{T-1} E^{0,u}\Big\{ \int_E e^{K|x|} |\sigma_n^u(x) - \sigma_n^{m,u}(x)|\, dx \Big\} \\ &+ C E^{0,u}\Big\{ \int_E e^{K|x|} |\sigma_T^u(x) - \sigma_T^{m,u}(x)|\, dx \Big\} \end{aligned}$$

from which the conclusion follows by virtue of Proposition 2.7.

■

2.3 Study of the approximating problems

In the previous section we have shown how in the cases of bounded cost functions and of cost functions with polynomial growth we can, under appropriate assumptions, construct a sequence of approximating control problems with cost functionals $J^{T,m}_{\mu_m}(u)$ such that, uniformly with respect to all admissible controls,

$$\lim_{m\to\infty} |J^{T,m}_{\mu_m}(u) - J^T_\mu(u)| = 0$$

It then follows from Lemma 2.1 that an optimal or even nearly optimal control for the m-th approximating problem is, for sufficiently large m, nearly optimal in the original problem.

In this section we shall study, for two specific approximation procedures, the resulting approximating problems. To this effect notice first that the unnormalized filtering processes $\sigma^{m,n}_n$ given in (2.6) take generically their values in the infinite-dimensional space of finite measures on E so that the determination of an optimal or nearly optimal control for the cost function (2.7) is computationally infeasible. The specific approximation procedures are designed to lead either to measures taking values in a finite-dimensional space or to measures that admit a finite dimensional representation. These procedures are more precisely obtained by

2.3.a Assuming that the approximating state transition operator P^a_m corresponds to a finite state Markov chain

2.3.b Assuming that the approximating state transition operator P^a_m is separated in the variables i.e.

$$P^a_m(x, dz) = \sum_{i=1}^{m} \varphi_i(x)\gamma_i(a, dz) \tag{2.28}$$

where $\gamma_i(a, dz)$ are finite measures on E, such that for $B \in \mathcal{B}(\mathcal{E})$ the mappings $U \ni a \to \gamma_i(a, B)$ are Borel measurable and $\varphi_i \in b\mathcal{B}(\mathcal{E})$ with $\varphi_i(x) \geq 0$.

Clearly,

$$\sum_{i=1}^{m} \varphi_i(x)\gamma_i(a, E) = 1 \quad \text{for} \quad x \in E,\ a \in U \tag{2.29}$$

2.3.1 Approximating finite state Markov chain

In this subsection we construct approximating processes x^m_n and $\overline{x}^m_n$ where the latter form finite state Markov chains.

For this purpose let $B_k^m,\ k = 1,2,\ldots,k_m$ be a finite partition of the original state space E and let $e_k^m,\ k = 1,2,\ldots,k_m$, $e_k^m \in B_k^m$ be representative elements of the sets $B_k^m,\ k = 1,2,\ldots,k_m$.

Assume furthermore that

$$\sup_{k<k_m} \operatorname{diam}(B_k^m) \to 0 \quad \text{as } m \to \infty \tag{2.30}$$

$$B_{k_m}^m \supset B_{k_{m+1}}^{m+1} \quad \text{and} \quad \bigcap_{m=1}^{\infty} B_{k_m}^m = \emptyset \tag{2.31}$$

and that for $k \le k_m$ there are indices $r_1,\ldots,r_{i(k)} \in \{1,2,\ldots,k_{m+1}\}$ such that

$$B_k^m = \bigcup_{p=1}^{i(k)} B_{r_p}^{m+1} \tag{2.32}$$

where the last property means that the $(m+1)$-st partition of E should be a subpartition of the m-th partition.

Let now

$$P_m^a(x,dz) = \sum_{k=1}^{k_m} \chi_{B_k^m}(x) P^a(e_k^m, dz) \tag{2.33}$$

define an approximating Markov process (x_n^m) with the same initial law as the original process (x_n).

Define furthermore $(\overline{x}_n^m)$ as the embedded Markov chain with space $E_m = \{1,\ldots,k_m\}$ and transition matrix

$$\overline{P}_m^a(i,j) = P^a(e_i^m, B_j^m) \quad \text{for} \quad i,\ j = 1,2,\ldots,k_m \tag{2.34}$$

Assume that

(B6) the mapping

$$E \times U \ni (x,a) \mapsto P^a(x,A) \in [0,1] \tag{2.35}$$

is uniformly continuous in $A \in \mathcal{B}(E)$, or equivalently the mapping

$$E \times U \ni (x,a) \mapsto P^a(x,\cdot) \in \mathcal{M}(E)$$

is continuous, where $\mathcal{M}(E)$ stands for the set of probability measures on E with the variation norm metric.

Clearly, under (B6) for P_m^a defined in (2.33) the condition (C2) holds.

Define now an approximating observation function h^m by

$$h^m(x,w) = h(e_k^m, w) \quad \text{for} \quad x \in B_k^m \tag{2.36}$$

which for each $x \in E$ is still a C^1 diffeomorphism so that the measure transition approach of Section 1.3 applies also to the approximating problem.

By analogy to (2.4) let

$$r_m(x,y) = g(k^m(x,y))|\Delta^m(x,y)| \tag{2.37}$$

where $k^m(x,\cdot)$ is the inverse function of $h^m(x,\cdot)$ and Δ^m its Jacobian. Notice also that $r_m(x,y) = r(e_k^m, y)$ for $x \in B_k^m$.

Remark 2.9 *Under (A1) and (A4), for r_m defined by (2.36)–(2.37) the condition (C3) is satisfied. In fact, by (A4), the family of measures $\{R(x,\cdot),\ x \in K\}$ is tight in R^d. Therefore it remains to show that for each compact $L \subset R^d$*

$$\sup_{x\in K} \int_L |r(x,y) - r_m(x,y)|\, dy \to 0, \quad \textit{as} \quad m \to \infty$$

which in turn is obvious in view of uniform continuity of $r(x,y)$ for $(x,y) \in K \times L$ and (2.30).

■

Let, by analogy to (2.33) and (2.36),

$$c_n^m(x,a) = \sum_{k=1}^{k_m} \chi_{B_k^m}(x) c_n(e_k^m, a) \tag{2.38}$$

and

$$b^m(x) = \sum_{k=1}^{k_m} \chi_{B_k^m}(x) b(e_k^m) \tag{2.39}$$

Imposing a stronger assumption than (B1), namely that

(B1')

$$c_n \in C(E \times U) \quad \text{and} \quad b \in C(E) \tag{2.40}$$

we easily obtain that (C4) is satisfied for c_n^m and b^m as in (2.38) and (2.39).

Given an admissible control u, let $(\sigma_n^{m,u})$ be the approximating unnormalized filter process defined by (2.6) with r_m, P_m^a of the form (2.37), (2.33).

By Corollary 2.3 and the preceding discussion we immediately have

Corollary 2.10 *Under (B1'), (B2) and (B6) the property (P1) holds i.e.*

$$\lim_{m\to\infty} \sup_u |J_\mu^{T,m}(u) - J_\mu^T(u)| = 0 \tag{2.41}$$

■

Define now the R^{k_m}-valued process $\rho_n^u = (\rho_n^u(1), \ldots, \rho_n^u(k_m))$ where $\rho_n^u(k) = \sigma_n^{m,u}(B_k^m)$. By the recursive formula (2.6) we then obtain

$$\begin{aligned} \rho_0^u &= (\mu(B_1^m), \ldots, \mu(B_{k_m}^m)) \\ \rho_{n+1}^u(k) &= \sum_{j=1}^{k_m} \frac{r(e_k^m, y_{n+1})}{g(y_{n+1})} \overline{P}_m^{a_n}(j,k)\rho_n^u(j) \end{aligned} \tag{2.42}$$

Notice also, that (ρ_n^u) is the approximating unnormalized filtering process that corresponds to the state process $(\overline{x}_n^m)$ with the observation density $r(e_k, y)$. Moreover we can rewrite the cost functional $J_\mu^{T,m}(u)$ in (2.7) in terms of (ρ_n^u) and we have

$$J_\mu^{T,m}(u) = E_\mu^{0u}\Big\{ \sum_{n=0}^{T-1} \sum_{k=1}^{k_m} c_n(e_k^m, a_n)\rho_n^u(k) + \sum_{k=1}^{k_m} b(e_k^m)\rho_T^u(k) \Big\} \tag{2.43}$$

Having obtained in Corollary 2.10 the property (P1), we have now to solve problem (P2) that is, for a given m and cost functional $J_\mu^{T,m}(u)$, find an optimal or nearly optimal control law u. By (2.42) and (2.43) the latter problem is reduced to the control of the finite dimensional process ρ_n^u given by (2.42) with the cost functional (2.43). Under our assumptions the approximating control problem (2.42)–(2.43) admits now an optimal control that can in principle be computed by the following backwards dynamic programming relations where $\rho \in (R^+)^{k_m}$ with $R^+ = [0, \infty)$.

$$\begin{aligned} V_T^m(\rho) &= \sum_{k=1}^{k_m} b(e_k^m)\rho(k) \\ V_n^m(\rho) &= \min_{a\in U} \Big[\sum_{k=1}^{k_m} c_n(e_k^m, a)\rho(k) + \int_{R^d} V_{n+1}^m\Big(\frac{r(e^m, y)}{g(y)} \overline{P}_m^a(\rho, \cdot)\Big) g(y)\, dy \Big] \end{aligned} \tag{2.44}$$

where, to simplify, we have written

$$\frac{r(e^m, y)}{g(y)} \overline{P}^a_m(\rho, \cdot)$$

for the vector

$$\Big(\sum_{j=1}^{k_m} \frac{r(e^m_1, y)}{g(y)} \overline{P}^a_m(j, 1)\rho(j), \ldots, \sum_{j=1}^{k_m} \frac{r(e^m_{k_m}, y)}{g(y)} \overline{P}^a_m(j, k_m)\rho(j)\Big) \tag{2.45}$$

We furthermore have

$$\inf_u J^{T,m}_\mu(u) = V^m_0(\mu(B^m_1), \ldots, \mu(B^m_{k_m})) = J^{T,m}_\mu(\overline{u}) \tag{2.46}$$

with

$$\overline{u} = (u_0(\rho^{\overline{u}}_0), \ldots, u_{T-1}(\rho^{\overline{u}}_{T-1})) \tag{2.47}$$

where u_k are selectors i.e. Borel measurable mappings from $(R^+)^{k_m}$ into U for which the minima in (2.44) are attained.

Although the dynamic programming equations (2.44) are based on the finite dimensional process (ρ^u_n), this latter process still takes an infinite number of values since (see (2.42)) the observations (y_n) as well as the controls (a_n) do. To make these dynamic programming relations computationally feasible, we therefore have to introduce an additional approximation leading to a finite number of possible values of the observations and controls.

For this purpose, given a positive integer H, let R^d be partitioned into $\{D^H_s\}_{s=1,\ldots,s(H)+1}$, with $s(H) = 2^d H^{2d}$, where, for $s \leq s(H)$, D^H_s are hypercubes with sides of length $\frac{1}{H}$, while

$$D^H_{s(H)+1} = \{y \in R^d \colon \|y\|_m > H\}$$

with $\|y\|_m = \max\{|y^1|, \ldots, |y^d|\}$.

For each D^H_s with $s \leq s(H)$ choose then a representative element $d^H_s \in D^H_s$ and take $d^H_{s(H)+1}$ such that $\|d^H_{s(H)+1}\|_m = H + 1$.

Define finally the observation projection operator W_H as

$$W_H \colon R^d \ni y \mapsto \sum_{s=1}^{s(H)+1} \chi_{D^H_s}(y)\, d^H_s \tag{2.48}$$

and the discretized observation process (z_n)

$$z_n = W_H(y_n) \tag{2.49}$$

Analogously, let $(U_k^H)_{k=1,2,\dots,H}$ be a partition of the compact control set U, such that for $H < H'$, the partition $(U_k^{H'})_{k=1,2,\dots,H'}$ is a subpartition of $(U_k^H)_{k=1,2,\dots,H}$ and the diameters of sets U_k^H converge to 0 as $H \to \infty$. Denote by $(\alpha_k^H)_{k=1,2,\dots,H}$ representative control values for the sets U_k^H, and let

$$Z_H \colon U \ni a \mapsto \sum_{k=1}^{H} \chi_{U_k^H}(a)\alpha_k^H \tag{2.50}$$

be the control projection operator onto $U^H = \{\alpha_k^H,\ k = 1, 2, \dots, H\}$.

Recalling the definition of $\overline{P}_m^a(i,j)$ let

$$\overline{P}_{m,H}^a(i,j) = \overline{P}_m^{Z_H a}(i,j) \tag{2.51}$$

From assumption (B6) it then follows that for all $i,\ j \in E_m$

$$\lim_{H\to\infty} \sup_{a\in U} |\overline{P}_{m,H}^a(i,j) - \overline{P}_m^a(i,j)| = 0 \tag{2.52}$$

With the discretized observations and controls consider then (see (2.42))

$$\begin{aligned} \rho_{0,H}^u &= (\mu(B_1^m), \dots, \mu(B_{k_m}^m)) \\ \rho_{n+1,H}^u(k) &= \sum_{j=1}^{k_m} \frac{r(e_k^m, z_{n+1})}{g(z_{n+1})} \overline{P}_m^{Z_H a_n}(j,k)\rho_{n,H}^u(j) \end{aligned} \tag{2.53}$$

as well as the dynamic programming equations

$$\begin{aligned} V_T^{m,H}(\rho) &= \sum_{k=1}^{k_m} b(e_k^m)\rho(k) \\ V_n^{m,H}(\rho) &= \min_{a\in U^H} \Big[\sum_{k=1}^{k_m} c_n(e_k^m, a)\rho(k) + \sum_{s=1}^{s(H)+1} V_{n+1}^{m,H}\Big(\frac{r(e^m, d_s^H)}{g(d_s^H)} \overline{P}_{m,H}^a(\rho,\cdot)\Big)\beta_s^H \Big] \end{aligned} \tag{2.54}$$

with $\beta_s^H = \int_{D_s^H} g(y)\,dy$ where, to simplify, by analogy to (2.45) we have written

$$\frac{r(e^m, d_s^H)}{g(d_s^H)} \overline{P}_{m,H}^a(\rho,\cdot)$$

for the vector

$$\Big(\sum_{j=1}^{k_m}\frac{r(e_1^m,d_s^H)}{g(d_s^H)}\overline{P}^a_{m,H}(j,1)\rho(j),\ldots,\sum_{j=1}^{k_m}\frac{r(e_{k_m}^m,d_s^H)}{g(d_s^H)}\overline{P}^a_m(j,k_m)\rho(j)\Big)$$

The dynamic programming equations (2.54) can now indeed be computed to yield an optimal control $u^*=(a_0^*,\ldots,a_{T-1}^*)$ for the last stage approximating problem, which, in the complete observation representation has the state given by the sequence of finite dimensional and finite valued random variables $(\rho_{n,H}^u)$ satisfying (2.53) while the cost functional is given by

$$J_\mu^{T,m,H}(u)=E_\mu^{0u}\Big\{\sum_{n=0}^{T-1}\sum_{k=1}^{k_m}c_n^m(e_k^m,Z_Ha_n)\rho_{n,H}^u(k)+\sum_{k=1}^{k_m}b^m(e_k^m)\rho_{T,H}^u(k)\Big\} \quad (2.55)$$

The generic term a_n^* of this optimal control is of the feedback type $a_n^*=u^*(\rho_{n,H})$ and therefore a function of the discretized observations i.e.

$$a_n^*=\alpha_n^*(z_1,\ldots,z_n) \quad (2.56)$$

It can be extended to a function of the original obsevations (y_n) putting

$$a_n^*=\overline{\alpha}_n^*(y_1,\ldots,y_n)=\alpha_n^*(W_H(y_1),\ldots,W_H(y_n)) \quad (2.57)$$

where the projection operator W_H is defined in (2.48). The optimal control u^* for the approximating problem with cost functional $J_\mu^{T,m,H}$ is therefore admissible also in the problem with cost functional $J_\mu^{T,m}$ and it remains to show that, for sufficiently large H, it is nearly optimal for the latter.

Notice also that any admissible control u for the problem with cost function $J_\mu^{T,m}$ can obviously be used also with the cost function $J_\mu^{T,m,H}$ in (2.55).

To complete this subsection we need an auxiliary result, which is formulated in an independent way because of its use also in the next subsection.

Proposition 2.11 *Assume (y_n) are under P^0 i.i.d. R^d-valued with common density g, $u=(a_0,a_1,\ldots)$ is a sequence of U-valued, $Y^n=\sigma\{y_1,\ldots,y_n\}$ adapted random variables, and*

$$\begin{cases}\eta_0^u(k)=\alpha(k)\\ \eta_{n+1}^u(k)=\sum_{j=1}^{p}G_j(y_{n+1},a_n)(k)\eta_n^u(j)\end{cases} \quad (2.58)$$

$$\left\{ \begin{array}{l} \eta_0^{u,H}(k) = \alpha(k) \\ \eta_{n+1}^{u,H}(k) = \sum_{j=1}^{p} G_j(W_H y_{n+1}, Z_H a_n)(k)\eta_n^{u,H}(j) \end{array} \right. \tag{2.59}$$

for $k = 1, 2, \ldots, p$, H being a positive integer, α a deterministic vector in R^d and with $G_j(y, a)(k)$ being for $j, k = 1, 2, \ldots, p$, functions in $(y, a) \in R^d \times U$ such that

$$\sup_a E^0\{|G_j(y_1, a)(k)|\} < \infty \tag{2.60}$$

and

$$\lim_{H\to\infty} \sup_{a\in U} E^0\{|G_j(y_1, a)(k) - G_j(W_H y_1, Z_H a)(k)|\} = 0 \tag{2.61}$$

for $j, k = 1, 2, \ldots, p$.

Then for $n = 0, 1, 2, \ldots$

$$\sup_u E^0\{|\eta_n^u(k)|\} < \infty \tag{2.62}$$

and

$$\lim_{H\to\infty} \sup_u E^0\{|\eta_n^u(k) - \eta_n^{u,H}(k)|\} = 0 \tag{2.63}$$

with $k = 1, 2, \ldots, p$.

P r o o f. We show (2.62) and (2.63) by induction on n. For $n = 0$, (2.62) and (2.63) are clearly satisfied. Assume (2.62) and (2.63) are true for n. Then for $n + 1$ we have

$$\begin{aligned} \sup_u E^0\{|\eta_{n+1}^u(k)| &\le \sup_u \sum_{j=1}^{p} E^0\{E^0\{|G_j(y_{n+1}, a_n)(k)|Y^n\}|\eta_n^u(j)|\} \\ &\le \sum_{j=1}^{p} \sup_a E^0\{|G_j(y_1, a)(k)|\} \sup_u E^0\{|\eta_n^u(j)|\} < \infty \end{aligned}$$

and

$$\begin{aligned} &E\{|\eta_{n+1}^u(k) - \eta_{n+1}^{u,H}(k)|\} \le \\ &\quad \le E^0\Big\{ \sum_{j=1}^{p} |G_j(y_{n+1}, a_n)(k)| \; |\eta_n^u(j) - \eta_n^{u,H}(j)| \\ &\quad + \sum_{j=1}^{p} |G_j(y_{n+1}, a_n)(k) - G_j(W_H y_{n+1}, Z_H a_n)(k)| \; |\eta_n^{u,H}(j)|\Big\} \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{p} E^0\{|\eta_n^u(j) - \eta_n^{u,H}(j)|E^0\{|G_j(y_{n+1}, a_n)(k)|Y^n\}\Big\} \\
&+ \sum_{j=1}^{p} E^0\{|\eta_n^{u,H}(j)|E^0\{|G_j(y_{n+1}, a_n)(k) - G_j(W_H y_{n+1}, Z_H a_n)(k)|Y^n\}\} \\
&\le \sum_{j=1}^{p} E^0\{|\eta_n^u(j) - \eta_n^{u,H}(j)|\} \sup_a E^0\{|G_j(y_1, a)(k)|\} \\
&+ \sum_{j=1}^{p} E^0\{|\eta_n^{u,H}(j)|\} \sup_a E^0\{|G_j(y_1, a)(k) - G_j(W_H y_1, Z_H a)(k)|\} \\
&\to 0 \quad \text{as} \quad H \to \infty
\end{aligned}
$$

by (2.60), (2.61) and the induction hypothesis.

Therefore we have obtained (2.62), (2.63) for $n+1$, which by induction completes the proof of Proposition.

■

We apply now Proposition 2.11 to the processes (ρ_n^u) and $(\rho_{n,H}^u)$.

Corollary 2.12 *Assume (B6) and*

(B7) *g is a continuous function*

(B8) *the random variables*

$$
\frac{r(e_k^m, W_H y_1)}{g(W_H y_1)} \qquad k = 1, 2, \ldots, p, \;\; H = 1, 2, \ldots,
$$

are uniformly integrable i.e.

$$
\lim_{M\to\infty} \sup_H \int_{R^d} \chi_{|y|\ge M} \frac{r(e_k^m, W_H y)}{g(W_H y)} g(y)\, dy = 0 \tag{2.64}
$$

Then

$$
\lim_{H\to\infty} \sup_u E^0\{|\rho_n^u(k) - \rho_{n,H}^u(k)|\} = 0 \tag{2.65}
$$

for $k = 1, 2, \ldots, k_m$ and $n = 1, 2, \ldots$

P r o o f. We use Proposition 2.11 with $p = k_m$, $\alpha(k) = \mu(B_k^m)$, $\eta_n^u = \rho_n^u$, $\eta_{n,H}^u = \rho_{n,H}^u$ and

$$G_j(y,a)(k) = \frac{r(e_k^m, y)}{g(y)} \overline{P}_m^a(j,k)$$

Clearly (2.60) is satisfied.

By (B6) and (B7) the mapping

$$R^d \times U \ni (y,a) \mapsto G_j(y,a)(k)$$

is continuous.

Therefore to obtain (2.61) it is sufficient to show that given $\varepsilon > 0$, there is M such that for any $k = 1, 2, \ldots$

$$\sup_{a \in U} E^0\{\chi_{|y_1| \geq M} |G_j(y_1,a)(k) - G_j(W_H y_1, Z_H a)(k)|\} < \varepsilon$$

for which in turn it suffices to prove that

$$E^0\{\chi_{|y_1| \geq M} \Big(\frac{r(e_k^m, y_1)}{g(y_1)} + \frac{r(e_k^m, W_H y_1)}{g(W_H y_1)} \Big)\} < \varepsilon$$

Since the last inequality can be achieved by the integrability of $\frac{r(e_k^m, y_1)}{g(y_1)}$ and (2.64), we obtain (2.61) which allows us to use Proposition 2.11.

$\blacksquare$

We now come to our final conclusion

Corollary 2.13 *Under (B1'), (B6)–(B8) we have*

$$\lim_{H \to \infty} \sup_u |J_\mu^{T,m}(u) - J_\mu^{T,m,H}(u)| = 0 \tag{2.66}$$

Moreover, if for $H > H_0$

$$\sup_u |J_\mu^{T,m}(u) - J_\mu^{T,m,H}(u)| < \varepsilon \tag{2.67}$$

then any control $\overline{u}$ that is ε-optimal for the cost functional $J_\mu^{T,m,H}$ with $H > H_0$, is 3ε optimal for the cost functional $J_\mu^{T,m}$. If in particular $\overline{u}$ is optimal for $J_\mu^{T,m,H}$ with $H > H_0$, then it is 2ε optimal for $J_\mu^{T,m}$.

P r o o f. We have

$$|J_\mu^{T,m}(u) - J_\mu^{T,m,H}(u)| \leq \sum_{n=0}^{T-1}\sum_{k=1}^{k_m} E_\mu^{0u}\{c_n^m(e_k^m, a_n)$$
$$-c_n^m(e_k^m, Z_H a_n)|\rho_n^u(k) + c_n^m(e_k^m, Z_H a_n)|\rho_n^u(k) - \rho_{n,H}^u(k)|\}$$
$$+\sum_{k=1}^{k_m} E_\mu^{0u}\{b^m(e_k^m)|\rho_T^u(k) - \rho_{T,H}^u(k)|\}$$

and (2.66) follows from (B1') and Corollary 2.12.

The proof of the second part of Corollary 2.13 is analogous to that of Lemma 2.1.

■

Concluding this subsection 2.3.1 we have that, if all assumptions are satisfied, a nearly optimal control for $J_\mu^{T,m}$ can be obtained as follows: For a sufficiently large value of H compute the dynamic programming relations (2.54) for each of the finite number of possible values of $\rho_{n,H}^u$. The control functions thereby obtained, lead (see Corollary 2.13) to nearly optimal controls for $J_\mu^{T,m}$. If furthermore also m is sufficiently large, by Lemma 2.1 these controls are nearly optimal also for J_μ^T.

2.3.2 Approximating operators separated in the variables

We consider now the case when $P_m^a(x, dz)$ has the form (2.28) with the condition (2.29) which includes e.g. the case of (2.33) by putting $m = k_m$, $\varphi_i(x) = \chi_{B_i^m}(x)$ and $\gamma_i(a, dz) = P^a(e_i^m, dz)$; the condition (2.29) then becomes in fact

$$\sum_{i=1}^{k_m} \chi_{B_i^m}(x) P^a(e_i^m, E) = 1$$

which is true here by definition.

Notice however, that there are situations when the approximating transition kernels are of the form (2.28) without being of type (2.33). We now show two examples for such situations.

Example 1. Assume the state process (x_n) is 1-dimensional and is given by the recursive formula

$$x_{n+1} = b(x_n) + d(a_n) + v_n$$

with v_n i.i.d. $N(0,1)$ random variables.

Then for $A \in B(E)$

$$P^a(x,A) = \frac{1}{\sqrt{2\pi}} \int\limits_A e^{-\frac{1}{2}(z-d(a))^2} e^{(z-d(a))(b(x))} e^{-\frac{1}{2}(b(x))^2}\, dz$$

We approximate the term $e^{-(z-d(a))(b(x))}$ by its $(m+1)$-st Taylor expansion

$$\sum_{i=0}^{m} \frac{1}{i!}(z-d(a))^i (b(x))^i$$

and construct approximating kernels as follows

$$P_m^a(x,A) = \frac{1}{\sqrt{2\pi}} \int\limits_A e^{-\frac{1}{2}(z-d(a))^2} \sum_{i=0}^{m} \frac{1}{i!}(z-d(a))^i (b(x))^i$$
$$e^{-\frac{1}{2}(b(x))^2}\, dz \Big(\frac{1}{\sqrt{2\pi}} \int\limits_E e^{-\frac{1}{2}(z-d(a))^2} \sum_{i=0}^{m} \frac{1}{i!}(z-d(a))^i (b(x))^i e^{-\frac{1}{2}(b(x))^2}\, dz \Big)^{-1}$$

By an easy calculation we obtain

$$P_m^a(x,A) = \sum_{i=0}^{m} \frac{b^i(x)\frac{1}{i!}}{\sum\limits_{j=0}^{n} b^j(x)\frac{j!!}{j!}} \frac{1}{\sqrt{2\pi}} \int\limits_A (z-d(a))^i e^{-\frac{(z-d(a))^2}{2}}\, dz$$

with $j!! = 1 \cdot 3 \cdot 5 \cdot (j-3) \cdot (j-1)$ for j-even and $j!! = 0$ for j-odd. Clearly P_m^a is of the form (2.28).

Example 2. Assume we are given a set $E_m = \{e_1^m, \ldots, e_{k_m}^m\} \subset E$ such that the points e_i^m, $i < k_m$ form a δ-net of a ball $B \subset E$, $e_{k_m}^m \in E \setminus B$, and $\rho_E(e_i^m, e_k^m) \geq \delta$ for $i, k \leq k_m$, with ρ_E standing for a metric on E.

Let

$$\overline{\varphi}_j(x) = \begin{cases} \delta - \rho_E(x, e_j^m) & \text{if} \quad \rho_E(x, e_j^m) \leq \delta \\ 0 \quad \text{elsewhere} \end{cases} \qquad \text{for } j < k_m$$

and

$$\overline{\varphi}_{k_m}(x) = \min\{1, \rho_E(x, \{e_1^m, \ldots, e_{k_m-1}^m\})\}$$

where $\rho_E(x, \{e_1^m, \ldots, e_{k_m-1}^m\}) = \min\{\rho_E(x, e_i^m);\ 1 \leq i \leq k_m - 1\}$.

Define now,

$$\varphi_j(x) = \frac{\overline{\varphi}_j(x)}{\sum_{i=1}^{k_m} \overline{\varphi}_i(x)}$$

Clearly $\varphi_j(e_j^m) = 1$ for $j = 1, 2, \ldots, k_m$, φ_j are continuous and form a partition of unity of E i.e. $\sum_{i=1}^{k_m} \varphi_i(x) \equiv 1$.

Given any transition kernel $P^a(x, dz)$ and letting $\gamma_i(a, dz) = P^a(e_i^m, dz)$, we may define an approximating transition operator P_m^a as $P_m^a(x, dz) = \sum_{i=1}^{k_m} \varphi_i(x)\gamma_i(a, dz)$ i.e. in the form (2.28). Notice that, comparing the just defined transition operator with (2.33), we see that we replaced the characteristic functions $\chi_{B_i^m}(x)$ by a suitable partition (φ_i) of unity of E. The use of an approximated kernel constructed in this way allows us to relax some of the assumptions imposed on the original kernel P^a, for example (B9) below can be avoided.

Although this case includes that of (2.33), here the cost functions need not necessarily be bounded nor do we have to approximate them. Moreover, in this subsection we do not approximate the observation function $h(x, w)$, and consequently neither the function $r(x, y)$. On the other hand, the dimensionality of the approximating problem will be larger than in the case of (2.33) as can be seen by comparing the dynamic programming relations (2.54) and (2.73) below. In what follows let either the assumptions (B1), (B2), (C1), (C2) of Section 2.2.1 or (B3)–(B5), (C5), (C6) of Section 2.2.2 be satisfied. Consequently Corollaries 2.3 and 2.8 hold, so we confine ourselves to the study of the approximating problem with the cost functional $J_{\mu_m}^{T,m}$ given by (2.26) where the approximating unnormalized filter process $(\sigma_n^{m,u})$ satisfies

$$\begin{aligned} \sigma_0^{m,u}(A) &= \mu_m(A) \\ \sigma_{n+1}^{m,u}(A) &= \sum_{i=1}^{m} \int_A \frac{r(z, y_{n+1})}{g(y_{n+1})} \gamma_i(a_n, dz) \sigma_n^{m,u}(\varphi_i) \end{aligned} \tag{2.68}$$

for $n = 0, 1, 2, \ldots$, and admissible control $u = (a_0, a_1, \ldots)$. Therefore $J_{\mu_m}^{T,m}$

can be rewritten as

$$
\begin{aligned}
J^{T,m}_{\mu_m}(u) = E^{0u}_{\mu_m}\Big\{ &\sum_{n=1}^{T-1}\sum_{i=1}^{m}\int_E c_n(x,a_n)\frac{r(x,y_n)}{g(y_n)}\gamma_i(a_{n-1},dx)\sigma^{m,u}_{n-1}(\varphi_i)+ \\
+&\sum_{i=1}^{m}\int_E b(x)\frac{r(x,y_T)}{g(y_T)}\gamma_i(a_{T-1},dx)\sigma^{m,u}_{T-1}(\varphi_i)+\int_E c_0(x,a_0)\mu_m(dx)\Big\}
\end{aligned}
\tag{2.69}
$$

and we see that to evaluate (2.69), instead of the unnormalized measures $(\sigma^{m,\mu}_n)$, we need only the values $\sigma^{m,\mu}_n(\varphi_i)$, $i=1,2,\dots,m$. Moreover by (2.68) we immediately have a recursive formula for $\sigma^{m,u}_n(\varphi_i)$, namely

$$
\begin{aligned}
\sigma^{m,u}_{n+1}(\varphi_i) &= \sum_{j=1}^{m}\int_E \varphi_i(z)\frac{r(z,y_{n+1})}{g(y_{n+1})}\gamma_j(a_n,dz)\sigma^{m,u}_n(\varphi_j) \\
&:= \sum_{j=1}^{m} G^m_j(y_{n+1},a_n)(i)\sigma^{m,u}_n(\varphi_j)
\end{aligned}
\tag{2.70}
$$

defining implicitly the operators G^m_j.

As will be clear later on, assuming a continuity with respect to a of certain terms, from the backwards dynamic programming equations we can obtain the existence of an optimal control $u^*=(a^*_0,\dots,a^*_{T-1})$, where a^*_0 is a measurable function of $\sigma^{m,u}_0$, and a^*_n for $n=1,2,\dots,T-1$ are measurable functions of $(y_n,\sigma^{m,u}_{n-1}(\varphi_1),\dots,\sigma^{m,u}_{n-1}(\varphi_m),a^*_{n-1})$.

However, similarly as in previous subsection, although (y_n), a^*_{n-1} and $(\sigma^{m,u}_{n-1}(\varphi_1),\dots,\sigma^{m,u}_{n-1}(\varphi_m))$ are finite dimensional, they take an infinite number of values and therefore we cannot calculate the optimal control law in practice. To overcome this difficulty we again discretize the observations and controls. For this purpose we use the projection operators W_H and Z_H defined in (2.48) and (2.50).

Let, for $k=1,2,\dots,m$ and a positive integer H,

$$
\begin{aligned}
\eta^u_0(k) &= \mu_m(\varphi_k) \\
\eta^u_{n+1}(k) &= \sigma^{m,u}_{n+1}(\varphi_k)
\end{aligned}
$$

and

$$
\begin{aligned}
\eta^{u,H}_0(k) &= \mu_m(\varphi_k) \\
\eta^{u,H}_{n+1}(k) &= \sum_{i=1}^{m}\int_E \varphi_k(z)\frac{r(z,W_H y_{n+1})}{g(W_H y_{n+1})}\gamma_i(Z_H a_n,dz)\eta^{u,H}_n(i)
\end{aligned}
\tag{2.71}
$$

By analogy to (2.69) define the cost functional

$$J_{\mu_m}^{T,m,H}(u) = E_{\mu_m}^{0u}\Big\{\sum_{n=1}^{T-1}\sum_{i=1}^{m}\int_E c_n(x, Z_H a_n)\frac{r(x, W_H y_n)}{g(W_H y_n)}\gamma_i(Z_H a_{n-1}, dx)\eta_{n-1}^{u,H}(i)+ \sum_{i=1}^{m}\int_E b(x)\frac{r(x, W_H y_T)}{g(W_H y_T)}\gamma_i(Z_H a_{T-1}, dx)\eta_{T-1}^{u,H}(i) + \int_E c_0(x, Z_H a_0)\mu_m(dx)\Big\} \tag{2.72}$$

Assume

(C7) $\sup_{a\in U}\sup_H E^0\Big[\sup_{a'\in U}\int_E c_n(x,a')\frac{r(x,W_H y_1)}{g(W_H y_1)}\gamma_i(a,dx)\Big] < \infty$

$\sup_{a\in U}\sup_H E^0\Big[\int_E b(x)\frac{r(x,W_H y_1)}{g(W_H y_1)}\gamma_i(a,dx)\Big] < \infty$

for $i = 1, 2, \ldots, m$, $n = 1, 2, \ldots, T-1$, and

(C8) the random variables $\sup_x \frac{r(x,W_H y_1)}{g(W_H y_1)}$, $H = 1, 2, \ldots$, are uniformly integrable.

Then, by (C8), $\eta_n^{u,H}(k)$ are integrable and consequently by (C7) the right hand side of (2.72) is well defined. Moreover we can find an optimal control for (2.72) by the following system of backwards dynamic programming equations

$$V_T^{m,H}(y,\eta,a) = \sum_{i=1}^{m}\int_E b(x)\frac{r(x, W_H y)}{g(W_H y)}\gamma_i(Z_H a, dx)\eta(i)$$

$$V_n^{m,H}(y,\eta,a) = \inf_{a'\in U}\Big\{\sum_{i=1}^{m}\int_E c_n(x, Z_H a')\frac{r(x, W_H y)}{g(W_H y)}\gamma_i(Z_H a, dx)\eta(i) + E^0\Big[V_{n+1}^{m,H}(W_H y_1, \sum_{j=1}^{m} G_j^m(W_H y, Z_H a)\eta(j), Z_H a')\Big]\Big\} \tag{2.73}$$

for $n = 1, 2, \ldots, T-1$

$$V_0^{m,H}(\mu_m) = \inf_{a\in U}\Big\{\int_E c_0(x, Z_H a)\mu_m(dx) + E^0[V_1^{m,H}(W_H y_1, \mu_m(\varphi_1), \ldots, \mu_m(\varphi_m), Z_H a)]\Big\}$$

Since the equations (2.73) depend only on the discretized values of the controls, all infima can be replaced by minima, and there exist measurable selectors $u_n(y,\eta,a)$ for which the infimum in $V_n^{m,H}$ is attained. Notice also that $V_n^{m,H}(y,\eta,a)$ depends on y and a only through the values of $W_H y$ and $Z_H a$.

Lemma 2.14 *Under (C7), (C8) we have*

$$\inf_u J_{\mu_m}^{T,m,H}(u) = V_0^{m,H}(\mu_m) \tag{2.74}$$

and the control $u_H^* = (a_{0,H}^*, \dots, a_{T-1,H}^*)$ is optimal, where

$$\begin{aligned} a_{0,H}^* &= u_0(\mu_m) \\ a_{n,H}^* &= u_n(W_H y_n, \eta_{n-1}^{u,H}, Z_H a_{n-1,H}^*) \end{aligned} \tag{2.75}$$

for $n = 1, 2, \dots, T-1$, with u_0 and u_n being selectors for which the infima in $V_0^{m,H}$ and $V_n^{m,H}$ are attained respectively.

P r o o f. Conditioning succesively the right hand side of (2.72) with respect to Y^n, $n = 0, 1, \dots, T-1$, we obtain

$$\begin{aligned}
&J_{\mu_m}^{T,m,H}(u) = E_{\mu_m}^{0u}\Big\{ \sum_{n=1}^{T-2}\sum_{i=1}^{m} \int_E c_n(x, Z_H a_n)\frac{r(x, W_H y_n)}{g(W_H y_n)}\gamma_i(Z_H a_{n-1}, dx)\eta_{n-1}^{u,H}(i) + \\
&\sum_{i=1}^{m}\int_E c_{T-1}(x, Z_H a_{T-1})\frac{r(x, W_H y_{T-1})}{g(W_H y_{T-1})}\gamma_i(Z_H a_{T-2}, dx)\eta_{T-2}^{u,H}(i) + \\
&E_{\mu_m}^{0u}\{V_T^{m,H}(W_H y_T, \eta_{T-1}^{u,H}, Z_H a_{T-1})|Y^{T-1}\}\Big\} + \int_E c_0(x, Z_H a_0)\mu_m(dx) \geq \\
&\geq E_{\mu_m}^{0u}\Big\{ \sum_{n=1}^{T-2}\sum_{i=1}^{m}\int_E c_n(x, Z_H a_n)\frac{r(x, W_H y_n)}{g(W_H y_n)}\gamma_i(Z_H a_{n-1}, dx)\eta_{n-1}^{u,H}(i) + \\
&V_{T-1}^{m,H}(W_H y_{T-1}, \eta_{T-2}^{u,H}, Z_H a_{T-2})\Big\} + \int_E c_0(x, Z_H a_0)\mu_m(dx) \\
&\geq \dots \geq E_{\mu_m}^{0u}\Big\{ \sum_{i=1}^{m} c_1(x, Z_H a_1)\frac{r(x, W_H y_1)}{g(W_H y_1)}\gamma_i(Z_H a_0, dx)\eta_0^{u,H}(i) + \\
&E_{\mu_m}^{0u}\{V_2^{m,H}(W_H y_2, \eta_1^{u,H}, Z_H a_1)|Y^1\}\Big\} + \int_E c_0(x, Z_H a_0)\mu_m(dx) \\
&\geq \int_E c_0(x, Z_H a_0)\mu_m(dx) + E_{\mu_m}^{0u}\{V_1^{m,H}(W_H y_1, \eta_0^{u,H}, Z_H a_0)\} \geq \\
&\geq V_0^{m,H}(\eta_0^{u,H})
\end{aligned}$$

with equalities corresponding to the u_H^* defined by (2.75).

∎

The vectors $(W_H y_n, \eta_{n-1}^{u,H}, Z_H a_{n-1})$ are now finite valued so that for $H > 0$ the optimal control u_H^* can actually be computed. The generic term of this optimal control is of the feedback type and thus as in (2.56) a function of the discretized observations. Analogously to (2.57) it can be extended to a function of the original observations (y_n) so that u_H^* is admissible also in the original problem. It remains to show that, for large H, u_H^* is nearly optimal for $J_{\mu_m}^{T,m}$. For this purpose we first prove that $\eta_n^{u,H}$ converges to η_n^u in L^1 norm.

Assume

(C9) $\lim\limits_{H\to\infty} \sup\limits_{a\in U} E^0\{|G_j^m(W_H y_1, Z_H a)(k) - G_j^m(y_1, a)(k)|\} = 0$
for $j,\ k = 1, 2, \ldots, m$.

By Proposition 2.11 we immediately have

Corollary 2.15 *Under (C9) we have*

$$\lim_{H\to\infty} \sup_{u} E^0\{|\eta_n^{u,H}(k) - \eta_n^u(k)|\} = 0 \tag{2.76}$$

for $k = 1, 2, \ldots, m$ and $n = 0, 1, 2, \ldots$

∎

Let

(C10)

$$\begin{aligned}
&\sup_{a\in U} E^0[\sup_{a'\in U} |\int_E c_n(x, Z_H a')\frac{r(x, W_H y_1)}{g(W_H y_1)}\gamma_i(Z_H a, dx) \\
&- \int_E c_n(x, a')\frac{r(x, y_1)}{g(y_1)}\gamma_i(a, dx)|] \to 0 \\
&\quad \text{for } n = 1, 2, \ldots, m-1, \text{ as } H \to \infty
\end{aligned}$$

$$\sup_{a\in U} E^0[|\int_E b(x)\frac{r(x,W_H y_1)}{g(W_H y_1)}\gamma_i(Z_H a,dx) \\ -\int_E b(x)\frac{r(x,y_1)}{g(y_1)}\gamma_i(a,dx)|] \to 0 \quad \text{as } H\to\infty$$

$$\sup_{a\in U}\int_E |c_0(x,a)-c_0(x,Z_H a)|\mu_m(dx) \to 0 \quad \text{as } H\to\infty$$

We conclude this section with the following

Corollary 2.16 *Under (C7)–(C10)*

$$\lim_{H\to\infty}\sup_u |J^{T,m,H}_{\mu_m}(u) - J^{T,m}_{\mu_m}(u)| = 0 \tag{2.77}$$

Moreover, if for $H > H_0$

$$\sup_u |J^{T,m,H}_{\mu_m}(u) - J^{T,m}_{\mu_m}(u)| < \varepsilon$$

then any ε-optimal control u^*_H for $J^{T,m,H}_{\mu_m}$ is for $H > H_0$, 3ε-optimal for the cost functional $J^{T,m}_{\mu_m}$. If in particular u^*_H is optimal for $J^{T,m,H}_{\mu_m}$ for $H > H_0$, then it is 2ε-optimal for $J^{T,m}_{\mu_m}$.

P r o o f. We show (2.77) only, since the second assertion can be proved analogously to Lemma 2.1. We have

$$
\begin{aligned}
&|J^{T,m,H}_{\mu_m}(u) - J^{T,m}_{\mu_m}(u)| \leq \\
&E^{0u}_{\mu_m}\Big\{\sum_{n=1}^{T-1}\sum_{i=1}^{m}(E^0\{|\int_E c_n(x, Z_H a_n)\frac{r(x, W_H y_n)}{g(W_H y_n)}\gamma_i(Z_H a_{n-1}, dx) - \\
&\int_E c_n(x, a_n)\frac{r(x, y_n)}{g(y_n)}\gamma_i(a_{n-1}, dx)|Y^{n-1}\}\eta^{u,H}_{n-1}(i) + \\
&E^0\{\int_E c_n(x, a_n)\frac{r(x, y_n)}{g(y_n)}\gamma_i(a_{n-1}, dx)|Y^{n-1}\}|\eta^{u,H}_{n-1}(i) - \eta^u_{n-1}(i)|) + \\
&\sum_{i=1}^{m} E^0\{|\int_E b(x)\frac{r(x, W_H y_T)}{g(W_H y_T)}\gamma_i(Z_H a_{T-1}, dx) - \\
&\int_E b(x)\frac{r(x, y_T)}{g(y_T)}\gamma_i(a_{T-1}, dx)|\,|Y^{T-1}\}\eta^{u,H}_{T-1}(i) + \\
&E^0\Big\{\int_E b(x)\frac{r(x, y_T)}{g(y_T)}\gamma_i(a_{T-1}, dx)|Y^{T-1}\Big\}|\eta^{u,H}_{T-1}(i) - \eta^u_{T-1}(i)|\} + \\
&\int_E |c^m_0(x, Z_H a_0) - c^m_0(x, a_0)|\mu_m(dx)\} = I_H + II_H + III_H + IV_H + V_H
\end{aligned}
$$

By (C8) and (C10), $I_H + III_H \to 0$ as $H \to \infty$. From (C7) and Corollary 2.15, $II_H + IV_H \to 0$ as $H \to \infty$. Since by (C10) also $V_H \to 0$ as $H \to \infty$ and all limits are uniform in u, we obtain (2.77).

■

Concluding this subsection 2.3.2 we have that, if all assumptions are satisfied, a nearly optimal control for $J^{T,m}_{\mu_m}$ can be obtained as follows:

For a sufficiently large value of H compute the dynamic programming relations (2.73) for each of the finite number of values of $(W_H y, \eta, Z_H a)$. The control functions thereby obtained, lead (see Lemma 2.14 and Corollary 2.16) to nearly optimal controls for $J^{T,m}_{\mu_m}$. If furthermore also m is sufficiently large, by Lemma 2.1 these controls are nearly optimal also for J^T_μ.

3 Infinite horizon with discounting

3.1 Introduction

This chapter considers the problem of determining nearly optimal controls for the infinite horizon problem with discounting, where the cost functional $J_\mu^\beta(u)$ is given by (1.5), or equivalently by (1.11). The previous chapter, dealing with the finite horizon problem, was entirely based on the representation (1.34) of the cost function in terms of the unnormalized filter process. The reason for this has already been given at the beginning of Sections 1.3 and 2.1. In this chapter we shall work only with normalized filters, and therefore with the cost functional given in the form (1.11), and this will be useful also in the next chapter concerning the average cost per unit time case. An alternative approach, based as in chapter 2 on measure transformation and using unnormalized filters, can be found in [29].

Similarly to the previous chapter, here too our approach to the construction of nearly optimal controls is based on an approximation approach. The main tool in the previous chapter was the uniform in the control approximation of the cost functional. Here, without the benefits of the measure transformation that made it possible to consider the same observations in the original and the approximating problems, we shall instead have to make use of some compactness arguments which will be achieved by either assuming that the state space is compact, or approximating the class of admissible controls by a compact family of controls. Furthermore, the approach in this (and the following) chapter will be structured into two parts. A first part consists of the construction of nearly optimal control functions which, when applied to the true filter values, lead to nearly optimal controls. Since the true filter process takes its values in an infinite dimensional space of measures, a direct construction of nearly optimal controls is computationally infeasible. The approximation approach for this first part now allows the original problem to be approximated by problems for which the associated filtering process takes its values in a finite dimensional space of measures so that for these problems the construction of nearly optimal control functions becomes computationally feasible. Notice that these approximating problems are auxiliary problems and that the associated filtering processes, based on observations that are not available in practice, are fictitions processes serving only the purpose of allowing a computationally feasible construction of a control function. These functions can however be extended to become functions defined on the infinite-dimensional space of measures, where the true filtering process of the original problem takes its values, and it will be shown that these extended functions are the desired nearly optimal control functions for the original problem. At this point there remains to compute the true filtering process. Again there is the problem of its infinite dimensionality, so that the purpose of the second part of our approach in this chapter is the construction of a computable approximating filter process and the proof that the nearly optimal control functions, provided they are continuous, still lead to nearly optimal controls when applied to the approximating filter.

We finally remark that, under some boundedness assumptions, the approach of the previous chapter can also be used for the construction of nearly optimal controls in infinite horizon problems with discounting. Defining in fact a finite horizon truncation of (1.5) as the finite horizon problem with cost functional

$$J^{\beta}_{\mu,P}(u) = E^{u}_{\mu}\Big\{\sum_{n=0}^{P-1}\beta^{u}c(x_n,a_n)\Big\}$$

where the terminal cost $b(x_P)$ is zero, it is easily seen that for a bounded cost function c, i.e. $|c(x,a)| \leq C$ for $x \in E$, $a \in U$, we have

$$\sup_{\mu\in P(E)}\sup_{u}|J^{\beta}_{\mu,T}(u) - J^{\beta}_{\mu}(u)| \leq \frac{\beta^{T}C}{1-\beta}$$

It follows that, for sufficiently large T, a nearly optimal control for the finite horizon problem, when extended to an infinite horizon control by taking arbitrary values after T, is nearly optimal also in the infinite horizon case with discounting. Vice versa, the methods of this chapter can be easily adapted to the finite horizon case.

In the next subsection 3.2 we recall the Bellman equation (value iteration) for the infinite horizon case with discounting. This equation will be the basis of our approach for the case when the state space E is compact.

The following section 3.3 concerns the construction of nearly optimal control functions. It will be devided into further subsections: In 3.3.1 we give general convergence results related to approximations of the state transition kernel, observation structure and cost function that satisfy suitable assumptions. This will then be particularized both to the case of a compact state space E using the Bellman equation (subsection 3.3.1.a) as well as when the controls are suitably approximated to become a compact class (subsection 3.3.1.b).

In 3.3.2, paralleling section 2.3.1 we present a specific method to obtain approximating transition operators, observation structure and costs that satisfies the required assumptions.

Although the approximation leads to finite dimensional filters, still a nearly optimal control function cannot be computed in practice, since these filters take an infinite number of values. In 3.3.3 we therefore perform additional approximations and show various possibilities to actually compute a nearly optimal control function.

In subsection 3.4, for the case of compact state space E and in connection with a generalized Bellman equation, we consider an additional specific approximation method which parallels that of section 2.3.2.

Finally, in subsection 3.5 we consider the problem of filter approximation.

3.2 The Bellman equation

First consider the additional assumption

(A5) $c\colon E \times U \to [0, \infty)$ is continuous and bounded

We have

Theorem 3.1 *Assume (A1)–(A5). Let*

$$v^\beta(\mu) := \inf_u J_\mu^\beta(u) \tag{3.1}$$

Then $v^\beta \in C(P(E))$ is the unique solution to the following Bellman equation

$$v^\beta(\mu) = \inf_{a \in U} \Big\{ \int_E c(x,a)\mu(dx) + \beta \prod^a (\mu, v^\beta) \Big\} \tag{3.2}$$

for $\mu \in P(E)$.

Moreover there is a Borel measurable function $u^\beta\colon P(E) \mapsto U$ for which

$$v^\beta(\mu) = \int_E c(x, u^\beta(\mu))\mu(dx) + \beta \prod^{u^\beta(\mu)} (\mu, v^\beta) \tag{3.3}$$

for $\mu \in P(E)$ and this $u^\beta(\cdot)$ is an optimal control function of the type considered in (1.13) which, when applied to the normalized filter process (π_n), leads to optimal controls so that we have

$$v^\beta(\mu) = J_\mu^\beta((u^\beta(\pi_n))) \tag{3.4}$$

In addition, v^β can be uniformly approximated by the increasing sequence $v_n^\beta \in C(P(E))$ obtained from the so called value iteration algorithm

$$\begin{aligned} &v_0^\beta(\mu) \equiv 0 \\ &v_{n+1}^\beta(\mu) = \inf_{a\in U} \Big\{ \int_E c(x,a)\mu(dx) + \beta \prod{}^a(\mu, v_n^\beta) \Big\} \end{aligned} \tag{3.5}$$

and we have

$$\|v^\beta - v_{n+1}^\beta\| \leq (1-\beta)^{-1}\beta^n \|c\| \tag{3.6}$$

with $\|v\|$ standing for the supremum of $|v(\nu)|$ over $\nu \in P(E)$.

Furthermore, for each n there exists a Borel measurable function $u^{\beta,n}$: $P(E) \mapsto U$ such that

$$v_{n+1}^\beta(\mu) = \int_E c(x, u^{\beta,n}(\mu))\mu(dx) + \beta \prod{}^{u^{\beta,n}(\mu)}(\mu, v_n^\beta) \tag{3.7}$$

for $\mu \in P(E)$.

Finally, each v_n^β is concave i.e. for $\mu,\ \nu \in P(E)$ and $\alpha \in [0,1]$

$$v_n^\beta(\alpha\mu + (1-\alpha)\nu) \geq \alpha v_n^\beta(\mu) + (1-\alpha)v_n^\beta(\nu) \tag{3.8}$$

P r o o f. Define, for $v \in C(P(E))$

$$Tv(\mu) = \inf_{a\in U} \Big\{ \int_E c(x,a)\mu(dx) + \beta \prod{}^a(\mu, v) \Big\}$$

By Proposition 1.4, T is a contraction on $C(P(E))$. Thus, by the Banach contraction principle there is a unique fixed point $v^\beta \in C(P(E))$ of T, which is the unique solution to the Bellman equation (3.2). Since by (A5) and Proposition 1.4 the mapping

$$U \ni a \mapsto \int_E c(x,a)\mu(dx) + \beta \prod{}^a(\mu, v^\beta)$$

is continuous, there exists a Borel measurable selector u^β for which (3.3) holds. The identity (3.4) is then almost immediate. By similar arguments there exist Borel measurable functions $u^{\beta,n}: P(E) \mapsto U$ satisfying (3.7). Since the operator T is monotonic and contractive, the sequence v_n^β, $n = 0, 1, \ldots$, is increasing and converges to v^β with the rate (3.6). It remains to show the concavity of v_n^β. We prove this by induction.

Clearly $v_0^\beta \equiv 0$ is concave. Provided v_n^β is concave, by Proposition 1.7, for fixed $a \in U$, $\prod^a(\mu, v_n^\beta)$ is concave i.e. for $\mu,\ \nu \in P(E)$, $\alpha \in [0,1]$

$$\prod{}^a(\alpha\mu + (1-\alpha)\nu, v_n^\beta) \geq \alpha \prod{}^a(\mu, v_n^\beta) + (1-\alpha)\prod{}^a(\nu, v_n^\beta)$$

and therefore from the definition of v_{n+1}^β we obtain

$$v_{n+1}^\beta(\alpha\mu + (1-\alpha)\nu) \geq \alpha v_{n+1}^\beta(\mu) + (1-\alpha)v_{n+1}^\beta(\nu)$$

i.e. v_{n+1}^β is concave, and by induction v_n^β is concave for each n. The proof of Theorem 3.1 is complete.

■

Corollary 3.2 *Under the assumptions of Theorem 3.1*

$$v_n^\beta(\mu) = \inf_u J_{\mu,n}^\beta(u) = J_{\mu,n}^\beta((u^{\beta,n-1-i}(\pi_i))) \tag{3.9}$$

with

$$J_{\mu,n}^\beta(u) = E_\mu^u\Big\{\sum_{i=0}^{n-1}\beta^i c(x_i, a_i)\Big\}$$

where $u = (a_0, a_1, \ldots)$. *Moreover, given* $\varepsilon > 0$, *there is* n_0 *such that for* $n \geq n_0$ *the control function* $u^{\beta,n}$ *obtained at stage* n *from the value iteration (3.5) is* ε*-optimal, i.e.*

$$J_\mu^\beta((u^{\beta,n}(\pi_i)) \leq v^\beta(\mu) + \varepsilon \tag{3.10}$$

P r o o f. The equality (3.9) is almost immediate from (3.5) and (3.7). Combining (3.7) with (3.6) we obtain

$$\begin{aligned} &v^\beta(\mu) + (1-\beta)^{-1}\beta^n\|c\| \geq v_{n+1}^\beta(\mu) \geq \\ &\geq \int_E c(x, u^{\beta,n}(\mu))\mu(dx) + \beta\prod{}^{u^{\beta,n}(\mu)}(\mu, v^\beta) - (1-\beta)^{-1}\beta^n\|c\| \end{aligned}$$

and therefore

$$J_\mu^\beta((u^{\beta,n}(\pi_i)) = \sum_{i=0}^{\infty} \beta^i E_\mu \Big\{ \int_E c(x, u^{\beta,n}(\pi_i))\pi_i(dx) \Big\}$$
$$\leq v^\beta(\mu) + 2(1-\beta)^{-2}\beta^n \|c\|$$

Choosing n_0 such that for $n \geq n_0$

$$2(1-\beta)^{-2}\beta^n\|c\| \leq \varepsilon$$

we obtain (3.10).

■

Remark 3.3 *When the cost function c is bounded Borel measurable only, the value function v^β defined by (3.1) may not be Borel measurable. However, since the transition operator $P^a(x,dz)$ is Borel in the sense that, for $B \in \mathcal{B}(E)$, the mapping $U \times E \ni (a,x) \mapsto P^a(x,B)$ is Borel measurable, it can be shown, using the results of Chapters 7-9 of [5], that v^β is a lower semi-analytic solution to the Bellman equation (3.2). Moreover, v^β can be uniformly approximated by the sequence of lower semi-analytic functions v_n^β given by (3.5). In addition for a given $\varepsilon > 0$ we can find (see Prop. 7.50 of [5]) an analytic function $u_\varepsilon^\beta \in A(P(E),U)$ such that*

$$v^\beta(\mu) + \varepsilon \geq \int_E c(x, u_\varepsilon^\beta(\mu))\mu(dx) + \beta \prod^{u_\varepsilon^\beta(\mu)}(\mu, v^\beta) \tag{3.11}$$

for $\mu \in P(E)$

Clearly, u_ε^β will be an $\varepsilon(1-\beta)^{-1}$-optimal control function for J_μ^β, namely $J_\mu^\beta((u_\varepsilon^\beta(\pi_n))) \leq v^\beta(\mu) + \dfrac{\varepsilon}{1-\beta}$.

The value iteration (3.5) as well as (3.11) give a (theoretical) possibility to determine an ε-optimal control function. In practice however these relations cannot be computed since the functions entering (3.5) and (3.11) are defined on the space of measures $P(E)$ that is infinite-dimensional. As a result, an approximation leading to a space of finite dimensional measures is required even if we use the Bellman equation to determine a nearly optimal control function.

3.3 Construction of nearly optimal control functions

3.3.1 Convergence

Assume that the state process (x_n) is approximated by a process (x_n^m) corresponding to a transition kernel P_m^a; assume furthermore that the observation density $r(x,y)$ is approximated by $r_m(x,y)$ and $c(x,a)$ by $c_m(x,a)$ and we have

(D1) if $U \ni a_m \to a$, then for $\varphi \in C(E)$

$$P_m^{a_m}(x,\varphi) \to P^a(x,\varphi), \quad \text{as } m \to \infty,$$

uniformly in x from compact subsets of E,

(D2) $r_m \in b\mathcal{B}(E \times R^d)$ are uniformly in m bounded, $r_m(x,y) \to r(x,y)$ uniformly on compact subsets of $E \times R^d$ and for any compact set $K \subset E$

$$\sup_{x \in K} \int_{R^d} |r_m(x,y) - r(x,y)| dy \to 0, \quad \text{as } m \to \infty$$

(D3) $c_m \in b\mathcal{B}(E \times U)$, are uniformly in m bounded, and

$$c_m(x,a) \to c(x,a), \quad \text{as } m \to \infty,$$

uniformly on compact subsets of $E \times U$.

Given an admissible control $u = (a_0, a_1, \ldots)$ let

$$J_\mu^{\beta,m}(u) = \sum_{n=0}^{\infty} \beta^n E_\mu^u \{c_m(x_n^m, a_n)\} \tag{3.12}$$

and define

$$v^{\beta,m}(\mu) = \inf_u J_\mu^{\beta,m}(u) \tag{3.13}$$

Furthermore given an initial measure μ of x_n^m, by analogy to (1.8) define the approximating filter process $(\pi_n^{m,u})$ taking values in $P(E)$, as

$$\pi_0^{m,u}(A) = \mu(A)$$

$$\pi_{n+1}^{m,u}(A) = \frac{\int_A r_m(z, y_{n+1}) P_m^{a_n}(\pi_n^{m,u}, dz)}{\int_E r_m(z, y_{n+1}) P_m^{a_n}(\pi_n^{m,u}, dz)} := M_m^{a_n}(y_{n+1}, \pi_n^{m,u})(A) \tag{3.14}$$

Notice that, corresponding to controls of the form $a_n = u(\pi_n^m)$, the filter process $(\pi_n^{m,u})$ is clearly Markov with transition operator

$$\prod\nolimits_m^{u(\nu)}(\nu, F) = \int_E \int_{R^d} F(M_m^{u(\nu)}(y,\nu)) r_m(z,y) dy P_m^{u(\nu)}(\nu, dz) \tag{3.15}$$

for $F \in b\mathcal{B}(P(E))$.

In the case when the control is given by the sequence $a_n = u(\pi_n)$, for a Borel measurable function $u\colon P(E) \to U$, below we shall identify the cost functional $J_\mu^\beta((u(\pi_n)))$ with $J_\mu^\beta(u)$; similarly for $J_\mu^{\beta,m}((u(\pi_n^m)))$.

Lemma 3.4 *Under (A1), (A2) and (D1), for each compact set $H \subset P(E)$ and $\varepsilon > 0$, there is a compact set $K \subset E$ and a positive integer m_0 such that*

$$\inf_{a\in U} P_m^a(\nu, K) \geq 1 - \varepsilon \quad \inf_{a \in U} P^a(\nu, K) \geq 1 - \varepsilon \tag{3.16}$$

for $\nu \in H$ and $m \geq m_0$.

P r o o f. By (A2) and (A3) the set of measures $\{P^a(\nu, \cdot),\ \nu \in H,\ a \in U\}$ is compact in $P(E)$. Therefore by Prokhorov's theorem (see Theorem 1.6.2 of [6]) for a given $\varepsilon > 0$, there is a compact set K_1 such that for $\nu \in H$, $a \in U$, $P^a(\nu, K_1) \geq 1 - \frac{\varepsilon}{2}$.

Let

$$\varphi(x) = \begin{cases} 1 - \inf\limits_{z \in K_1} \rho_E(z,x) \text{ if } \inf\limits_{z\in K_1} \rho_E(z,x) \leq 1 \\ 0 \quad elsewhere \end{cases}$$

with ρ_E standing for a metric on E compatible with the topology.

Clearly $\varphi \in C(E)$. Therefore for a sufficiently large m, say $m \geq m_0$

$$\sup_{a\in U} \sup_{\nu \in H} |P_m^a(\nu, \varphi) - P^a(\nu, \varphi)| < \frac{\varepsilon}{2} \tag{3.17}$$

since otherwise we would have for some $a_m \to a$, $\nu_m \Rightarrow \nu$, $\nu_m \in H$

$$|P_m^{a_m}(\nu_m, \varphi) - P^{a_m}(\nu_m, \varphi)| > \frac{\varepsilon}{4}$$

a contradiction to (D1).

Let $K=\{x\in E\colon \inf_{z\in K_1}\rho_E(z,x)\le 1\}$. For $\nu\in H$, $a\in U$, $m\ge m_0$ we then have

$$\begin{aligned}&P^a_m(\nu,K)\ge P^a_m(\nu,\varphi)\ge P^a(\nu,\varphi)-\tfrac{\varepsilon}{2}\ge\\ &\ge P^a(\nu,K_1)-\tfrac{\varepsilon}{2}\ge 1-\varepsilon\end{aligned}$$

Since E is locally compact, the set K is also compact and (3.16) holds.

■

The next two propositions are devoted to the study of the limit properties of the operators M^a_m and $\prod^a_m$ for $m\to\infty$.

Proposition 3.5 *Under (A1)–(A3), (D1)–(D2) for $\varphi\in C(E)$*

$$\sup_{a\in U}|M^a_m(y,\nu)(\varphi)-M^a(y,\nu)(\varphi)|\to 0 \tag{3.18}$$

as $m\to\infty$, uniformly in (y,ν) from compact subsets of $R^d\times P(E)$.

P r o o f. It suffices to show that for any $\varphi\in C(E)$

$$\sup_{a\in U}\Big|\int_E\varphi(z)r_m(z,y)P^a_m(\nu,dz)-\int_E\varphi(z)r(z,y)P^a(\nu,dz)\Big|\to 0\quad\text{as } m\to\infty \tag{3.19}$$

uniformly in (y,ν) from compact subsets of $R^d\times P(E)$.

By Lemma 3.4 and (D2)

$$\sup_{a\in U}\Big|\int_E\varphi(z)(r_m(z,y)-r(z,y))P^a_m(\nu,dz)\Big|\to 0\quad\text{as } m\to\infty \tag{3.20}$$

uniformly in (y,ν) from compact subsets of $R^d\times P(E)$.

In fact, for any $\varepsilon>0$ and a compact set $H\subset P(E)$, by Lemma 3.4 we can find a compact set $K\subset E$ such that for $m\ge m_0$, $\nu\in H$

$$\sup_{a\in U}P^a_m(\nu,K^c)\le\varepsilon$$

Therefore, for $m\ge m_0$

$$\begin{aligned}&\sup_{a\in U}\Big|\int_E\varphi(z)(r_m(z,y)-r(z,y))P^a_m(\nu,dz)\Big|\le\\ &\le[\|\varphi\|(\|r_m\|+\|r\|)\cdot\varepsilon+\|\varphi\|\sup_{z\in K}|r_m(z,y)-r(z,y)|]\end{aligned}$$

Since by (D2) $|r_m(z,y) - r(z,y)| \to 0$ as $m \to \infty$, uniformly on compact subsets of $E \times R^d$, $\|r_m\|$ are bounded and ε could be chosen arbitrarily small, we obtain (3.20).

It remains to show that

$$\sup_{a\in U}\Big|\int\limits_E \varphi(z) r_m(z,y)(P^a_m(\nu,dz) - P^a(\nu,dz))\Big| \to 0 \quad \text{as } m\to\infty \tag{3.21}$$

uniformly in (y,ν) from compact subsets of $R^d \times P(E)$.

By (D1) and (3.17), for any $\psi \in C(E)$

$$\sup_{a\in U}|P^a_m(\nu,\psi) - P^a(\nu,\psi)| \to 0 \quad \text{as } m\to\infty \tag{3.22}$$

uniformly in ν from a compact subset H of $P(E)$.

To show (3.21) we need the following simple lemma, the proof of which is left to the reader.

Lemma 3.6 *Let (M_1,ρ_1), (M_2,ρ_2) be metric spaces. $F\colon (M_1,\rho_1) \to (M_2,\rho_2)$ be a continuous mapping and $K \subset M_1$ be a fixed compact set. Then, for a given $\varepsilon > 0$ there is $\delta > 0$ such that for $x \in K$, $x' \in M_1$, $\rho_1(x,x') < \delta$ implies $\rho_2(F(x),F(x')) < \varepsilon$.*

Let $L \subset R^d$ and $H \subset P(E)$ be compact sets. By (A1) and (A2), the set

$$\widetilde{H} = \{P^a(\nu,\cdot),\ \text{for } a\in U,\ \nu \in H\}$$

is compact in $P(E)$.

Let

$$F\colon R^d \times P(E) \ni (y,\nu) \mapsto \int\limits_E \varphi(z) r(z,y)\nu(dz)$$

and $M_1 = R^d \times P(E)$, $M_2 = R_1$, $K = L \times \widetilde{H}$.

Then by Lemma 3.6, for a given $\varepsilon > 0$, there is $\delta > 0$ such that

> if for some $\nu \in \widetilde{H}$, $\nu' \in P(E)$, $\rho_w(\nu,\nu') < \delta$ then for all $y \in L$, $|\int\limits_E \varphi(z) r(z,y)(\nu(dz) - \nu'(dz))| < \varepsilon$,

with ρ_w standing for a metric compatible with the weak convergence topology of $P(E)$.

By (3.22), for a sufficiently large m

$$\sup_{\nu\in H}\sup_{a\in U}\rho_w(P^a_m(\nu,\cdot),P^a(\nu,\cdot))<\delta$$

and consequently

$$\sup_{\nu\in H}\sup_{a\in U}\sup_{y\in L}\Big|\int_E\varphi(z)r(z,y)(P^a_m(\nu,dz)-P^a(\nu,dz))\Big|<\varepsilon$$

from which (3.21) follows.

The proof of Proposition 3.5 is complete.

■

We use now Proposition 3.5 to obtain the convergence of the approximating transition operators $\prod^a_m$, i.e. of $\prod^{u(\nu)}_m(\nu,\cdot)$ corresponding to a control function $u(\nu)\equiv a\in U$.

Proposition 3.7 *Assume (A1)–(A4) and (D1), (D2). If $b\mathcal{B}(P(E))\ni F_m\to F\in C(P(E))$ uniformly on compact subsets of $P(E)$ and F_m are uniformly bounded, then*

$$\sup_{a\in U}|\prod^a_m(\nu,F_m)-\prod^a(\nu,F)|\to 0 \tag{3.23}$$

as $m\to\infty$, uniformly in ν from compact subsets of $P(E)$.

P r o o f. We have

$$\begin{aligned}
&\sup_{a\in U}|\textstyle\prod^a_m(\nu,F_m)-\prod^a(\nu,F)|\le\\
&\le\sup_{a\in U}\Big|\int_E\int_{R^d}F_m(M^a_m(y,\nu))(r_m(z,y)-r(z,y))dyP^a_m(\nu,dz)\Big|\\
&+\sup_{a\in U}\Big|\int_E\int_{R^d}(F_m(M^a_m(y,\nu))-F(M^a(y,\nu)))r(z,y)dyP^a_m(\nu,dz)\Big|\\
&+\sup_{a\in U}\Big|\int_E\int_{R^d}F(M^a(y,\nu))r(z,y)dy(P^a_m(\nu,dz)-P^a(\nu,dz))\Big|\\
&=I_m+II_m+III_m
\end{aligned}\tag{3.24}$$

Let $H \subset P(E)$ be a compact set. By Lemma 3.4 for a given $\varepsilon > 0$ we can find a compact set $K \subset E$ and a positive integer m_0 such that

$$\sup_{\nu \in H} \sup_{a \in U} P_m^a(\nu, K^c) \leq \varepsilon, \quad \text{for } m \geq m_0 \tag{3.25}$$

Therefore, for $\nu \in H$ and $m \geq m_0$

$$I_m \leq 2\|F_m\|\varepsilon + \sup_{z \in K} \|F_m\| \int_{R^d} |r_m(z, y) - r(z, y)| dy$$

By (D2), and the fact that ε can be chosen arbitrarily small, we obtain that $I_m \to 0$.

By (A4) the family of measures $\{R(z, \cdot),\ z \in K\}$ is compact. Therefore there is a compact set $L \subset R^d$ such that

$$\sup_{z \in K} R(z, L^c) \leq \varepsilon \tag{3.26}$$

Hence, using (3.25), for $\nu \in H$, $m \geq m_0$ we have

$$II_m \leq 2\|F_m\|\varepsilon + 2\|F\|\varepsilon + \sup_{a \in U} \sup_{y \in L} |F_m(M_m^a(y, \nu)) - F(M^a(y, \nu))| \tag{3.27}$$

Notice now, that by Proposition 3.5,

$$M_m^a(y, \nu)(\cdot) \Rightarrow M^a(y, \nu)(\cdot) \quad \text{as } m \to \infty,$$

uniformly in $a \in U$, $y \in L$, $\nu \in H$, and by Proposition 1.4 the set

$$\widetilde{H} = \{M^a(y, \nu),\ a \in U,\ y \in L,\ \nu \in H\}$$

is compact in $P(E)$.

At this stage we need the following, easy to prove, slightly strengthened version of Lemma 3.6.

Lemma 3.8 *Assume (M_1, ρ_1), (M_2, ρ_2) are metric spaces, $F_m \colon (M_1, \rho_1) \to (M_2, \rho_2)$ is a sequence of Borel measurable mappings, $F_m \to F$ uniformly on compact subsets of M_1, as $m \to \infty$, $F \colon (M_1, \rho_1) \to (M_2, \rho_2)$ is continuous and $K \subset M_1$ is compact.*

Then

$$\underset{\varepsilon>0}{\forall} \ \underset{m_1}{\exists} \ \underset{\delta>0}{\exists} \ \underset{m>m_1}{\forall} \ \underset{x \in K, x' \in M_1}{\forall} \rho_1(x, x') < \delta \Rightarrow \rho_2(F_m(x'), F(x)) < \varepsilon$$

Taking now $M_1 = P(E)$, $M_2 = R$ and $K = H$, by Lemma 3.8 we obtain that

$$\sup_{a\in U}\sup_{y\in L}\sup_{\nu\in H}|F_m(M_m^a(y,\nu)) - F(M^a(y,\nu))| \to 0 \quad \text{as } m\to\infty$$

which comletes the proof of $II_m \to 0$.

The proof that $III_m \to 0$ is based on Lemma 3.6. Let

$$G\colon C_c^M(E)\times P(E) \ni (\varphi,\nu) \mapsto \int_E \varphi(z)\nu(dz) \tag{3.28}$$

where C_c^M stands for the space of continuous functions on E that are bounded by a constant M, with a metric generated by the supremum norm on compact sets.

Define

$$\begin{aligned} H_1 &= \{\text{mappings } \ E\ni z\mapsto \int_{R^d} F(M^a(y,\nu))r(z,y)dy,\ a\in U,\ \nu\in H\} \\ H_2 &= \{P^a(\nu,\cdot),\ a\in U,\ \nu\in H\}\end{aligned}$$

By (A1) and (A2), H_2 is compact in $P(E)$.

We shall now show that H_1 is compact in $C_c^{\|F\|}(E)$. By (A4) and Proposition 1.4 we clearly have $H_1\subset C_c^{\|F\|}(E)$.

Consider a sequence of functions

$$h_n(z) := \int_{R^d} F(M^{a_n}(y,\nu_n))r(z,y)dy \quad a_n\in U,\ \nu_n\in H.$$

Since U and H are compact, we can choose subsequences (n_k), for simplicity denoted again by n such that $a_n\to a\in U$, $\nu_n\to\nu\in H$.

If we showed that, as $n\to\infty$,

$$h_n(z)\to h(z) = \int_{R^d} F(M^{a_n}(y,\nu))r(z,y)dy$$

in $C_c^{\|F\|}(E)$ i.e. uniformly on compact subsets of E, we would obtain the compactness of H_1.

Fix a compact set $K \subset E$. Given $\varepsilon > 0$, by (A4) there is a compact set $L \subset R^d$ such that (3.26) holds; consequently we have

$$\sup_{z \in K} |h_n(z) - h(z)| \leq 2\|F\|\varepsilon + \sup_{y \in L} |F(M^{a_n}(y, \nu_n)) - F(M^a(y, \nu))|$$

Since by Proposition 1.4

$$\sup_{\nu \in H} \sup_{y \in L} |F(M^{a_n}(y, \nu)) - F(M^a(y, \nu))| \to 0$$

we obtain the convergence $h_n(z) \to h(z)$ uniformly on compact subsets of E, and therefore the compactness of H_1.

We claim now that the mapping G, defined in (3.28) is continuous i.e.

for $\varphi_n \to \varphi$ in $C_c^{\|F\|}$, and $\nu_n \Rightarrow \nu$ in $P(E)$, we have $G(\varphi_n, \nu_n) \to G(\varphi, \nu)$ as $n \to \infty$.

In fact, the family $\{\nu, \nu_n,\ n = 1, 2, \ldots\}$ is tight. Therefore for a given $\varepsilon > 0$ there is a compact set $K \subset E$ such that $\nu_n(K^c) \leq \varepsilon$, $\nu(K^c) \leq \varepsilon$, $n = 1, 2, \ldots$, and consequently

$$\begin{aligned} &|G(\varphi, \nu) - G(\varphi_n, \nu_n)| \leq \Big| \int_E \varphi(z)(\nu(dz) - \nu_n(dz)) \Big| \\ &+2\|F\|\varepsilon + \sup_{z \in K} |\varphi_n(z) - \varphi(z)| \to 2\|F\|\varepsilon \end{aligned}$$

as $n \to \infty$, and since ε can be chosen arbitrarily small, the mapping G is continuous.

Applying now Lemma 3.6 with $M_1 = C_c^{\|F\|}(E) \times P(E)$, $M_2 = R$, $F = G$, and $K = H_1 \times H_2$, by (3.22) we finally obtain that $III_m \to 0$ as $m \to \infty$, uniformly for $\nu \in H$.

This way we complete the proof of Proposition 3.7.

■

Given $u \in \mathcal{B}(P(E), U)$ define the following iterations of the transition operator $\prod^{u(\nu)}(\nu, \cdot)$,

$$\begin{aligned} &(\textstyle\prod^{u(\nu)})^1(\nu, \cdot) = \prod^{u(\nu)}(\nu, \cdot) \\ &(\textstyle\prod^{u(\nu)})^{n+1}(\nu, \cdot) = \displaystyle\int_{P(E)} (\textstyle\prod^{u(\nu)}(\nu', \cdot)^n \prod^{u(\nu)}(\nu, d\nu') \end{aligned} \tag{3.29}$$

From Proposition 3.7 we obtain the following

Corollary 3.9 *Assume (A1)–(A4), (D1), (D2). Let* $\mathcal{B}(P(E),U) \ni u_m \to u \in C(P(E),U)$, *and* $b\mathcal{B}(P(E)) \ni F_m \to F \in C(P(E))$, *uniformly on compact subsets of* $P(E)$ *as* $m \to \infty$, *and let* F_m *be uniformly in* m *bounded.*

Then, for any compact subset $H \subset P(E)$, *and* $n = 1, 2, \ldots$, *we have*

$$\sup_{\nu \in H} |(\prod_m^{u_m(\nu)})^n(\nu, F_m) - (\prod^{u(\nu)})^n(\nu, F)| \to 0 \quad \text{as } m \to \infty \tag{3.30}$$

P r o o f. We use induction in n. For $n = 1$

$$\begin{aligned}&\sup_{\nu \in H} |\prod_m^{u_m(\nu)}(\nu, F_m) - \prod^{u(\nu)}(\nu, F)| \\ &\leq \sup_{\nu \in H} |\prod_m^{u_m(\nu)}(\nu, F_m) - \prod^{u_m(\nu)}(\nu, F)| \\ &+ \sup_{\nu \in H} |\prod^{u_m(\nu)}(\nu, F) - \prod^{u(\nu)}(\nu, F)| = I_m + II_m\end{aligned}$$

Clearly, $I_m \to 0$ as $m \to \infty$ by Proposition 3.7, and $II_m \to 0$ as $m \to \infty$ by Proposition 1.4. Therefore (3.30) holds for $n = 1$.

Assume that (3.30) holds for n. Then for $n + 1$ we have

$$\begin{aligned}&\sup_{\nu \in H} |(\prod_m^{u_m(\nu)})^{n+1}(\nu, F_m) - (\prod^{u(\nu)})^{n+1}(\nu, F)| = \\ &\sup_{\nu \in H} |\prod_m^{u_m(\nu)}(\nu, \overline{F}_m) - \prod^{u_m(\nu)}(\nu, \overline{F})\end{aligned} \tag{3.31}$$

with

$$\overline{F}_m(\nu) = (\prod_m^{u_m(\nu)})^n(\nu, F_m)$$

and

$$\overline{F}(\nu) = (\prod^{u(\nu)})^n(\nu, F_m)$$

By the induction hypothesis $\overline{F}_m(\nu) \to \overline{F}(\nu)$ as $m \to \infty$ uniformly on compact subsets of $P(E)$.

Therefore by step $n = 1$

$$\sup_{\nu \in H} |\prod_m^{u_m(\nu)}(\nu, \overline{F}_m) - \prod^{u(\nu)}(\nu, \overline{F})| \to 0 \quad \text{as } m \to \infty$$

and by (3.31), the convergence (3.30) holds for $n + 1$. Thus by induction (3.30) holds for $n = 1, 2, 3, \ldots$

∎

For a general approximation scheme satisfying (D1)–(D3) we have so far obtained convergence results for operators related to the approximating filter process. In the following two subsections we shall apply these results for the construction of nearly optimal controls in two specific cases, namely when the state space E is compact and when the admissible control functions are continuous.

3.3.1.a Compact state space E

We first prove the following general result where E need not to be compact

Theorem 3.10 *Assume (A1)–(A5) and (D1)–(D3). Then*

$$v^{\beta,m}(\mu) \to v^{\beta}(\mu) \quad \text{as } m \to \infty \tag{3.32}$$

uniformly on compact subsets of $P(E)$.

P r o o f. By Theorem 3.1 we have

$$\|v^{\beta} - v^{\beta}_{n+1}\| \leq (1-\beta)^{-1}\beta^n\|c\|$$

and, using also Remark 3.3

$$\|v^{\beta,m} - v^{\beta,m}_{n+1}\| \leq (1-\beta)^{-1}\beta^n\|c\|$$

where $v_n^{\beta,m}$ are defined, by analogy to v_n^{β}, by the value iteration algorithm

$$\begin{aligned} &v_0^{\beta,m}(\mu) \equiv 0 \\ &v_{n+1}^{\beta,m}(\mu) = \inf_{a\in U}\Big\{\int_E c_m(x,a)\mu(dx) + \beta\textstyle\prod_m^a(\mu, v_n^{\beta,m})\Big\}. \end{aligned} \tag{3.33}$$

Therefore it suffices to show that for each $n = 1, 2, \ldots$,

$$v_n^{\beta,m}(\mu) \to v_n^{\beta}(\mu) \quad \text{as } m \to \infty \tag{3.34}$$

uniformly on compact subsets of $P(E)$.

We prove the convergence (3.34) by induction in n. For $n = 0$, clearly $v_0^{\beta,m} \equiv v_0^{\beta}$. Given (3.34) true for n, we have for $n+1$

$$\begin{aligned} &|v_{n+1}^{\beta,m}(\mu) - v_{n+1}^{\beta}(\mu)| \leq \sup_{a\in U}\int_E |c_m(x,a) - c(x,a)|\mu(dx) \\ &+\beta\sup_{a\in U}|\textstyle\prod_m^a(\mu, v_n^{\beta,m}) - \prod^a(\mu, v_n^{\beta}) \to 0 \end{aligned}$$

as $m \to \infty$, uniformly on compact subsets of $P(E)$, by (D3) and Proposition 3.7. Thus, (3.34) holds for $n+1$ and consequently by induction for any positive integer n. The proof of Theorem 3.10 is completed.

■

Considering now a compact state space E, we obtain

Corollary 3.11 *Under the assumptions of Theorem 3.10, if E is compact and $u_m^\beta \in \mathcal{B}(P(E), U)$ satisfies*

$$v^{\beta,m}(\mu) + \varepsilon \geq \int_E c_m(x, u_m^\beta(\mu))\mu(dx) + \beta \prod\nolimits_m^{u^\beta_\cdot(\mu)}(\mu, v^{\beta,m}) \quad \text{for } \mu \in P(E), \tag{3.35}$$

then for sufficiently large m, u_m^β is a $\dfrac{4\varepsilon}{1-\beta}$ optimal control function for the cost functional J_μ^β i.e.

$$J_\mu^\beta((u_m^\beta(\pi_n)) \leq v^\beta(\mu) + \frac{4\varepsilon}{1-\beta}$$

P r o o f. If E is compact, then $P(E)$ is also compact and the convergence in (3.32) is uniform. Also, $c_m(x,a)$ converges then uniformly to $c(x,a)$. Therefore we can choose m_0 such that for $m \geq m_0$

$$\sup_{x\in E}\sup_{a\in U} |c_m(x,a) - c(x,a)| < \varepsilon$$

and

$$\sup_{\mu\in P(E)} |v^{\beta,m}(\mu) - v^\beta(\mu)| < \varepsilon$$

By Proposition 3.7, for $m \geq m_1$

$$\sup_{\mu\in P(E)}\sup_{a\in U} |\prod\nolimits_m^a(\mu, v^{\beta,m}) - \prod\nolimits^a(\mu, v^\beta)| < \varepsilon$$

An easy transformation of (3.35) gives

$$\|v^{\beta,m} - v^\beta\| + v^\beta(\mu) + \varepsilon \geq \int_E c(x, u_m^\beta(\mu))\mu(dx) - \|c - c_m\|$$
$$+\beta \prod\nolimits^{u_m^\beta(\mu)}(\mu, v^\beta) - \beta \sup_a |\prod\nolimits^a(\mu, v^\beta) - \prod\nolimits_m^a(\mu, v^{\beta,m})|$$

Therefore, for $m \geq \max\{m_0, m_1\}$, we obtain

$$\varepsilon + v^\beta(\mu) + \varepsilon \geq \int_E c(x, u_m^\beta(\mu))\mu(dx) - \varepsilon$$
$$+\beta \prod^{u_m^\beta(\mu)}(\mu, v^\beta) - \varepsilon$$

and thus

$$v^\beta(\mu) + 4\varepsilon \geq \int_E c(x, u_m^\beta(\mu))\mu(dx) + \beta \prod^{u_m^\beta(\mu)}(\mu, v^\beta)$$

from which the $4\varepsilon(1-\beta)^{-1}$ optimality of the control function u_m^β follows.

■

For actual construction of a nearly optimal control function with the use of Bellman's equation, relation (3.35) seems to present the same difficulties as (3.11), since it involves functions that are at least formally defined on the infinite dimensional space of measures $P(E)$. The difference however is that (3.35) corresponds to the approximated state and observation processes, and approximated cost function satisfying (D1)–(D3). By suitable particular choices of these approximations, the functions in (3.35) may actually turn out to depend on finite dimensional projections of measures only. In the next subsection 3.3.2.a a particular such approximation is presented.

3.3.1.b Continuous control functions

In this subsection we restrict ourselves to continuous control functions, i.e. elements of the set $\mathcal{A} = C(P(E), U)$. We shall also assume that the compact set of control parameters U is a convex subset of R^l, $l \geq 1$.

Given a fixed element $\overline{x}$ of E, let for $n = 1, 2, \ldots$, $B_n = \{x \colon \rho_E(\overline{x}, x) \leq n\}$, and define $\psi_n \in C(P(E))$ as follows

$$\psi_n(x) = \begin{cases} 1 - \rho_E(x, B_n) & \text{for } x \in B_{n+1} \\ 0 & \text{for } x \notin B_{n+1} \end{cases} \tag{3.36}$$

Let (φ_i) be a dense sequence in $C_0(E)$ the space of continuous functions vanishing at infinity. Moreover assume that (φ_i) contains a subsequence given by ψ_n, $n = 1, 2, \ldots$

We shall now approximate $\mathcal{A}$ by a family of compact classes of controls $\mathcal{A}(L,n)$ with $L>0$ and n a positive integer, defined in the following way

$$\begin{aligned}&\mathcal{A}(L,n)=\{u\in\mathcal{A},\ u(\nu)=\overline{u}(\nu(\varphi_1),\dots,\nu(\varphi_n))\\ &\text{where } \overline{u}\colon[-\|\varphi_1\|,\|\varphi_1\|]\times\ldots\times[-\|\varphi_n\|,\|\varphi_n\|]\to U\\ &\text{is Lipschitz with Lipschitz constant } L\}\end{aligned}\tag{3.37}$$

where on $[-\|\varphi_1\|,\|\varphi_1\|]\times\ldots\times[-\|\varphi_n\|,\|\varphi_n\|]$ we consider the metric $\rho_n(z,z')$ generated by the norm $\|z\|=\max\limits_i|z_i|$.

We have

Proposition 3.12 *Any function $u\in\mathcal{A}$ can be approximated uniformly on compact subsets of $P(E)$ by a sequence $u_{L,n}\in\mathcal{A}(L,n)$ with $L\to\infty$, $n\to\infty$.*

P r o o f. Notice first that it is sufficient to prove Proposition 3.12 for $U\subset R^1$. For approximation purposes we can also assume that $\{0\}\in\operatorname{int}U$. Let $r\colon[0,1]\to[0,1]$ be given by

$$r(x)=\begin{cases}0 & \text{if } x\le\frac14\\ 2(x-\frac14) & \text{if } \frac14\le x\le\frac34\\ 1 & \text{if } x\ge\frac34\end{cases}\tag{3.38}$$

Let $u\in\mathcal{A}$ be a fixed function. Since the family $\mathcal{A}(L,n)$ is increasing in L and n, it is sufficient to construct an approximating sequence $u_n\in\bigcup\limits_{L>0}\bigcup\limits_{k=1}^{\infty}\mathcal{A}(L,k)$. The construction is partitioned into several steps.

Step 1. For any compact set $H\subset P(E)$ there is n_0 such that for $\nu\in H$ and $n\ge n_0$, $r(\nu(\psi_n))=1$

Step 2. Let $p_n\colon P(E)\mapsto P(B_{n+1})$ be defined as follows

$$p_n\nu(A)=\begin{cases}\nu(\chi_A\psi_n)\cdot(\nu(\psi_n))^{-1} & \text{if } \nu(\psi_n)>0\\ \chi_A(\overline{x}) & \text{if } \nu(\psi_n)=0\end{cases}\tag{3.39}$$

Notice that for the case $\nu(\psi_n)=0$ we could have chosen any measure from $P(B_{n+1})$. Then we have that $p_n\nu\Rightarrow\nu$ as $n\to\infty$ uniformly on compact subsets of $P(E)$.

In fact, let $H \subset P(E)$ be a compact set. Then for a given $\varepsilon > 0$ there is n_0 such that for $n \geq n_0$ and $\nu \in H$, $\nu(B_n) \geq 1 - \varepsilon$.

For any $\varphi \in C(E)$ we have

$$|p_n\nu(\varphi) - \nu(\varphi)| = \{|\nu(\varphi(\psi_n - 1))| + |\nu(\varphi)|(1 - \nu(\psi_n))\}(\nu(\psi_n))^{-1}$$

$$\leq \frac{2\|\varphi\|\varepsilon}{1-\varepsilon} \quad \text{for } \nu \in H$$

Step 3. By the Stone–Weierstrass theorem (Therorem 9.28 of [28]) each $u \in \mathcal{A}$ can be uniformly approximated on $P(B_n)$ by functions $\overline{u}_n \in \bigcup_{L>0} \bigcup_{k=1}^{\infty} \mathcal{A}(L,k)$ such that

$$\sup_{\nu \in P(B_n)} |u(\nu) - \overline{u}_n(\nu)| \leq \frac{1}{n}$$

Step 4. Let

$$u_n(\nu) = \overline{u}_{n+1}(p_n\nu)r(\nu(\psi_n)) \tag{3.40}$$

We claim that u_n is the desired approximation of u. In fact, u_n is a continuous function and $u_n \in \bigcup_{L>0} \bigcup_{k=1}^{\infty} \mathcal{A}(L,k)$. Moreover, for any compact set H, by step 1, we have $r(\nu(\psi_n)) = 1$ for $\nu \in H$ and $n \geq n_0$. Therefore

$$\sup_{\nu \in H} |u_n(\nu) - u(\nu)| \leq \sup_{\nu \in H} |\overline{u}_{n+1}(p_n\nu) - u(p_n\nu)| r(\nu(\psi_n)) + \\ + \sup_{\nu \in H} |u(p_n\nu)r(\nu(\psi_n)) - u(\nu)| = I_n + II_n$$

By step 3, $I_n \leq \frac{1}{n+1}$. By step 2, $p_n\nu \Rightarrow \nu$ uniformly in $\nu \in H$; therefore by Lemma 3.6, $II_n \to 0$, and u_n in fact approximate u uniformly on compact subsets of $P(E)$.

■

As a consequence of Corollary 3.9 we obtain now the following

Theorem 3.13 *Assume (A1)–(A5) and (D1)–(D3). Let* $\mathcal{B}(P(E), U) \ni u_m \to u \in C(P(E), U)$ *uniformly on compact subsets of* $P(E)$*, as* $m \to \infty$*. Then*

$$J_\mu^{\beta,m}(u_m) \to J_\mu^\beta(u), \quad \text{as } m \to \infty \tag{3.41}$$

uniformly in μ *from compact subsets of* $P(E)$*.*

P r o o f. Let, for $u \in \mathcal{B}(P(E), U)$,

$$C_m^u(\mu) = \int_E c_m(x, u(\mu))\mu(dx)$$

and

$$C^u(\mu) = \int_E c(x, u(\mu))\mu(dx)$$

Then, using the notation of (3.29), we have

$$J_\mu^{\beta,m}(u_m) = C_m^{u_m}(\mu) + \sum_{n=1}^{\infty} \beta^n (\prod\nolimits_m^{u_m})^n(\mu, C_m^{u_m}) \tag{3.42}$$

and

$$J_\mu^\beta(u) = C^u(\mu) + \sum_{n=1}^{\infty} \beta^n (\prod\nolimits^{u})^n(\mu, C^u) \tag{3.43}$$

We show first that for any compact set $H \subset P(E)$

$$\sup_{\mu \in H} |C_m^{u_m}(\mu) - C^u(\mu)| \to 0 \quad \text{as } m \to \infty \tag{3.44}$$

In fact, for a given $\varepsilon > 0$ we can find a compact $K \subset E$ such that for each $\mu \in H$, $\mu(K) > 1 - \varepsilon$, and

$$\sup_{\mu \in H} |C_m^{u_m}(\mu) - C^u(\mu)| \le \sup_{\mu \in H} \sup_{x \in K} |c_m(x, u_m(\mu)) - c(x, u(\mu))| +$$
$$(\|c_m\| + \|c\|)\varepsilon \le \sup_{a \in U} \sup_{x \in K} |c_m(x, a) - c(x, a)| +$$
$$\sup_{\mu \in H} \sup_{x \in K} |c(x, u_m(\mu)) - c(x, u(\mu))| + (\|c_m\| + \|c\|)\varepsilon$$

Therefore by (D3) and (A5) we obtain (3.44), and we can apply now Corollary 3.9 with $F_m = C_m^{u_m}$, $F = C^u$, which gives us the convergence

$$(\prod\nolimits_m^{u_m(\mu)})^n(\mu, C_m^{u_m}) \to (\prod\nolimits^{u(\mu)})^n(\mu, C^u) \tag{3.45}$$

as $m \to \infty$, for each $n = 1, 2, \ldots$, uniformly on compact subsets of $P(E)$.

Finally, by (3.44), (3.45) and the representations (3.42), (3.43) we obtain (3.41).

■

By Proposition 3.12 and Theorem 3.13 we immediately have

Corollary 3.14 *Under (A1)–(A5), for $\mu \in P(E)$ we have*

$$\lim_{L\to\infty} \lim_{n\to\infty} \inf_{u\in\mathcal{A}(L,n)} J_\mu^\beta(u) = \inf_{u\in\mathcal{A}} J_\mu^\beta(u) \tag{3.46}$$

■

Since the controls from $\mathcal{A}(L,n)$ belong also to $\mathcal{A}$, in order to obtain a nearly optimal control function in $\mathcal{A}$, by Corollary 3.14 it is enough to find one in $\mathcal{A}(L,n)$ for sufficiently large L and n. Notice that, as for the class $\mathcal{A}$, also the controls in $\mathcal{A}(L,n)$ are defined on the infinite dimensional space of measures $P(E)$. The compactness of the class $\mathcal{A}(L,n)$ however will allow us, with the approximation introduced in the next section 3.3.2, to restrict ourselves to measures that are finite dimensional (see subsection 3.3.2b).

3.3.2 A specific approximation

This section corresponds to section 2.3.1 for the finite horizon case and leads to a specific approximation satisfying assumptions (D1)–(D3). Corresponding to this approximation, the normalized filtering process will turn out to be a process taking values in a finite dimensional space of measures. As a consequence, the measures μ in the Bellman equation corresponding to a compact state space E will be finite dimensional and this case will be discussed in 3.3.2.a below.

On the other hand, in the case of continuous control functions, we may further approximate the class of admissible controls by considering controls from the compact classes $\mathcal{A}(L,n)$ that are functions of finite dimensional measures. This will be discussed in 3.3.2.b below.

A further specific approximation, corresponding to section 2.3.2 for the finite horizon case, will be described in section 3.4 and used only in connection with a generalized version of the Bellman equation.

The specific approximation of this section is now obtained as follows. We partition the state space E and observation space R^d by choosing for each positive integer m sequences of disjoint Borel sets $B_k^m \subset E$, $D_s^m \subset R^d$, $k = 1, 2, \ldots, k_m$, $s = 1, 2, \ldots, s_m$ such that

(i) $\bigcup_{k=1}^{k_m} B_k^m = E, \quad \bigcup_{s=1}^{s_m} D_s^m = R^d$

(ii) B_k^m, D_s^m have nonempty interiors and the closures $\overline{B}_k^m$, $\overline{D}_s^m$, for $k < k_m$, $s < s_m$ are compact

(iii) $\sup\limits_{k<k_m} \operatorname{diam}(B_k^m) \to 0$

$\sup\limits_{s<s_m} \operatorname{diam}(D_s^m) \to 0$

where $\operatorname{diam}(B)$ stands for the diameter of the set B.

(iv) $B_{k_m}^m \supset B_{k_{m+1}}^{m+1}$ and $\bigcap\limits_{m=1}^{\infty} B_{k_m}^m = \emptyset$

$D_{s_m}^m \supset D_{s_{m+1}}^{m+1}$ and $\bigcap\limits_{m=1}^{\infty} D_{s_m}^m = \emptyset$

(v) for $k = 1, 2, \dots, k_m$, $s = 1, 2, \dots, s_m$, there are indices $r_1, \dots, r_{i(k)}$, $t_1, \dots, t_{j(s)}$ such that

$$B_k^m = \bigcup_{p=1}^{i(k)} B_{r_p}^{m+1}, \quad D_s^m = \bigcup_{q=1}^{j(s)} D_{t_q}^{m+1}$$

We choose next sets of selectors $\{b_k^m,\ k = 1, 2, \dots, k_m\}$, $\{d_s^m,\ s = 1, 2, \dots, s_m\}$ of (B_k^m) and (D_s^m) respectively with the following properties

$$\begin{aligned} & b_k^m \in \operatorname{int} B_k^m,\ \{b_k^m,\ k = 1, 2, \dots, k_m\} \subset \{b_k^{m+1},\ k = 1, 2, \dots, k_{m+1}\} \\ & b_{k_m}^m \to \infty \quad \text{as } m \to \infty \\ & d_s^m \in \operatorname{int} D_s^m,\ \{d_s^m,\ s = 1, 2, \dots, s_m\} \subset \{d_s^{m+1},\ s = 1, 2, \dots, s_{m+1}\} \\ & d_{s_m}^m \to \infty \quad \text{as } m \to \infty \end{aligned} \tag{3.47}$$

In what follows we shall assume that the partition (B_k^m) and selectors (b_k^m) are chosen in such way that

(B9) $P^a(b_k^m, \partial B_p^m) = 0$ for $k,\ p = 1, 2, \dots, k_m,\ a \in U$

Then we approximate the functions $r(x, y)$ and $c(x, a)$ in the following way

$$\begin{aligned} & r_m(x, y) = \Big(\int_{D_s^m} dz\Big)^{-1} \Big(\int_{D_s^m} r(b_j^m, z) dz + \frac{1}{s_m - 1} \int_{D_{s_m}^m} r(b_j^m, z) dz\Big) \\ & \qquad \text{for } x \in B_j^m \text{ and } y \in D_s^m,\ s < s_m \\ & r_m(x, y) = 0 \quad \text{for } y \in D_{s_m}^m \end{aligned} \tag{3.48}$$

and

$$c_m(x,a) = c(b_j^m, a) \quad \text{for } x \in B_j^m \tag{3.49}$$

Clearly, for a fixed x, $r_m(x,y)$ is a density function. Moreover we have

Lemma 3.15 *Under (A3) and (A4), r_m defined in (3.48) satisfy (D2). Furthermore, under (A5) c_m given by (3.49) satisfy (D3).*

The proof of the first statement follows noticing that by (A4) we have (3.26). The second part is immediate.

Let

$$P_m^a(x,\cdot) = \sum_{k=1}^{k_m} \chi_{B_k^m}(x) P^a(b_k^m, \cdot) \tag{3.50}$$

By (A1) and (A2), for $U \ni a_m \to a$, $P_m^{a_m}(x,\cdot) \Rightarrow P^a(x,\cdot)$ uniformly in x from compact subsets of E, as $m \to \infty$.

Thus P_m^a satisfies (D1), and summarizing we see that P_m^a, r_m, c_m satisfy (D1), (D2), (D3) respectively.

Now let

$$E_m = \{1, 2, \ldots, k_m\}, \quad D_m = \{d_1^m, \ldots, d_{s_m}^m\} \tag{3.51}$$

and

$$\overline{P}_m^a(k,p) = P^a(b_k^m, B_p^m) \quad \text{for } k,p \in E_m \quad a \in U \tag{3.52}$$

Notice that by (B9) and (A2) and Theorem 1.2.1(v) of [6] the mapping

$$U \ni a \mapsto \overline{P}_m^a(k,p) \quad \text{for } k,p \in E_m \tag{3.53}$$

is continuous.

Analogously to section 2.3.1, the specific approximation method defined here leads now to a controlled Markov chain $(\overline{x}_n^m)$ on E_m having transition matrix $\overline{P}_m^{a_n}(k,p)$ in the generic period n. The observations are given by the D_m-valued random variables $\overline{y}_n^m$ that satisfy the following analog of (1.1) (compare also to (1.3))

$$P\{\overline{y}_{n+1}^m = d_s^m | \overline{x}_{n+1} = k,\ \overline{Y}_n^m\} = \int_{D_s^m} r_m(b_k^m, y)dy := \overline{r}_m(k, d_s^m) \tag{3.54}$$

where $\overline{Y}_n^m := \sigma\{\overline{y}_1^m, \ldots, \overline{y}_n^m\}$, $\overline{Y}_0^m = \{\emptyset, \Omega\}$. As admissible controls we have sequences $u = (a_0, a_1, a_2, \ldots)$, where a_n is U-valued and adapted to $\overline{Y}_n^m$.

Given such an admissible control u and an initial law $\eta = (\eta_1, \ldots, \eta_{k_m}) \in P(E_m)$ for $(\overline{x}_n^m)$ consider the cost functional

$$J_\eta^{\beta,m}(u) = \sum_{n=0}^{\infty} \beta^n E_\eta^u \{c(\overline{x}_n^m, a_n)\} \tag{3.55}$$

where for simplicity we identify $c(j,a)$ with $c(b_j^m, a)$ and define

$$w^{\beta,m}(\eta) = \inf_u J_\eta^{\beta,m}(u) \tag{3.56}$$

For the given specific approximation, corresponding to (3.14), we obtain (see (1.9)) an approximating filter process $(\overline{\pi}_n^{m,u}) \in P(E_m)$ satisfying

$$\begin{aligned} \overline{\pi}_{n+1}^{m,u}(j) &= \frac{\overline{r}_m(j, \overline{y}_{n+1}^m) \sum_{k=1}^{k_m} \overline{P}_m^{a_m}(k,j) \overline{\pi}_n^{m,u}(k)}{\sum_{p=1}^{k_m} \overline{r}_m(p, \overline{y}_{n+1}^m) \sum_{k=1}^{k_m} \overline{P}_m^{a_m}(k,p) \overline{\pi}_n^{m,u}(k)} \\ &:= \overline{M}_m^{a_n}(\overline{y}_{n+1}^m, \overline{\pi}_n^{m,u})(j) \end{aligned} \tag{3.57}$$

with $j \in E_m$ and $\overline{\pi}_0^{m,u} = \eta$. For feedback controls of the form $a_n = u(\overline{\pi}_n^m)$, the filter process $(\overline{\pi}_n^{m,u})$ is again Markov with transition operator (see (3.15))

$$\overline{\prod}_m^{u(\eta)}(\eta, F) = \sum_{j=1}^{k_m} \sum_{s=1}^{s_m} F(\overline{M}_m^{u(\eta)}(d_s^m, \eta)) \overline{r}_m(j, d_s^m) \sum_{k=1}^{k_m} \overline{P}_m^{u(\eta)}(k,j) \eta_k \tag{3.58}$$

where $F \in b\mathcal{B}(P(E_m))$ and $\eta \in P(E_m)$.

With the use of the filter process $\overline{\pi}_n^{m,u}$, the cost functional $J_\eta^{\beta,m}(u)$ in (3.55) can be rewritten as

$$J_\eta^{\beta,m}(u) = \sum_{n=0}^{\infty} \beta^n E_\eta^u \Big\{ \sum_{k=1}^{k_m} c(k, u(\overline{\pi}_n^{m,u})) \overline{\pi}_n^{m,u}(k) \Big\} \tag{3.59}$$

In what follows it will be useful to introduce the mapping

$$\mathcal{L}_m \colon P(E) \ni \nu \mapsto \sum_{k=1}^{k_m} \nu(B_k^m) \delta_{b_k^m} \in P(E) \tag{3.60}$$

where $\delta_{b_k^m}$ stands for Dirac measure concentrated at b_k^m.

We have

Lemma 3.16 *$\mathcal{L}_m \nu \Rightarrow \nu$, as $m \to \infty$, uniformly on compact subsets of $P(E)$.*

P r o o f. If $H \subset P(E)$ is compact then by tightness, for any $\varepsilon > 0$, there is a compact set $K \subset E$ such that $\nu(K^c) < \varepsilon$ for $\nu \in H$. Therefore for $\varphi \in C(E)$ we have

$$\sup_{\nu \in H} |\nu(\varphi) - \mathcal{L}_m \nu(\varphi)| \leq 2\varepsilon \|\varphi\| +$$
$$+ \sup_{\nu \in H} \sum_{k=1}^{k_m} \int_{K \cap B_k^m} |\varphi(x) - \varphi(b_k^m)| \nu(dx) \to 2\varepsilon \|\varphi\|$$

since, as $m \to \infty$ diam $(B_k^m) \to 0$ for $k < k_m$ and $K \cap B_{k_m}^m \to \emptyset$.

■

Almost immediately, by Lemma 3.16 we obtain

Corollary 3.17 *If $\nu_m \Rightarrow \nu$, then $\mathcal{L}_m \nu_m \Rightarrow \nu$ as $m \to \infty$.*

■

We now apply the specific approximation outlined above to the cases of compact state space E (subsection 3.3.2.a) and of continuous control functions (subsection 3.3.2.b).

3.3.2.a Compact state space E

Similarly as in subsection 3.3.1.a we formulate first a general result in which E need not to be compact.

Theorem 3.18 *Under (A1)–(A5) and (B9) we have*

$$w^{\beta,m}(\mu(B_1^m), \ldots, \mu(B_{k_m}^m)) \to v^\beta(\mu) \quad \text{as } m \to \infty \tag{3.61}$$

uniformly in μ from compact subsets of $P(E)$.

P r o o f. Let (x_n^m) be an approximation of (x_n) with transition operator $P_m^{a_n}(x_n^m, \cdot)$ in the generic period n and density of the observation r_m, defined in (3.50), (3.48) respectively. Consider the cost functional $J_\mu^{\beta,m}(u)$ of the form (3.12) with c_m defined in (3.49). Since by Lemma 3.15 and the comment following (3.50) assumptions (D1)–(D3) hold, by Theorem 3.10 we obtain that

$$v^{\beta,m}(\mu) \to v^\beta(\mu) \quad \text{as } m \to \infty \tag{3.62}$$

uniformly on compact subsets of $P(E)$.

We shall now show that

$$v^{\beta,m}(\mu) = w^{\beta,m}(\mu(B_1^m), \ldots, \mu(B_{k_m}^m)) \tag{3.63}$$

By Remark 3.3, the functions $v^{\beta,m}$ and $w^{\beta,m}$ can be uniformly approximated by the following sequences

$$\begin{aligned} &v_0^{\beta,m}(\mu) \equiv 0 \\ &v_{n+1}^{\beta,m}(\mu) = \inf_{a \in U} \Big[\int\limits_E c_m(x,a)\mu(dx) + \beta \textstyle\prod_m^a(\mu, v_n^{\beta,m}) \Big] \end{aligned} \tag{3.64}$$

where $\prod_m^a$ is as defined in (3.15) with $u(\mu) \equiv a$, and r_m, P_m^a given by (3.48), (3.50) respectively, and

$$\begin{aligned} &w_0^{\beta,m}(\eta) \equiv 0 \\ &w_{n+1}^{\beta,m}(\eta) = \inf_{a \in U} \Big[\sum_{k=1}^{k_m} c(k,a)\eta_k + \beta \overline{\prod}_m^a(\eta, w_n^{\beta,m}) \Big] \end{aligned} \tag{3.65}$$

with $\overline{\prod}_m^a$ defined in (3.58) for $u(\eta) \equiv a$.

Therefore, to prove (3.62) it suffices to show that

$$v_n^{\beta,m}(\mu) = w_n^{\beta,m}(\mu(B_1^m), \ldots, \mu(B_{k_m}^m)) \tag{3.66}$$

We prove this by induction. For $n = 0$, (3.66) clearly holds. Assume (3.66) is satisfied for n. Then we have

$$\begin{aligned} &\textstyle\prod_m^a(\mu, v_n^{\beta,m}) = \int\limits_E \int\limits_{R^d} v_n^{\beta,m}(M_m^a(y,\mu)) r_m(x,y) dy P_m^a(\mu, dz) \\ &= \sum_{k=1}^{k_m} \sum_{s=1}^{s_m} w_n^{\beta,m}(M_m^a(d_s^m,\mu)(B_1^m), \ldots, M_m^a(d_s^m,\mu)(B_{k_m}^m)) \overline{r}_m(k, d_s^m) \\ &\sum_{j=1}^{k_m} \overline{P}_m^a(j,k)\mu(B_j^m) \end{aligned}$$

Since for $s \le s_m$, $k \le k_m$

$$M_m^a(d_s^m,\mu)(B_k^m) = \overline{M}_m^a(d_s^m, \mu(B_1^m), \ldots, \mu(B_{k_m}^m))(k)$$

we obtain that

$$\prod\nolimits_m^a(\mu, v_n^{\beta,m}) = \overline{\prod}\nolimits_m^a(\mu(B_1^m), \dots, \mu(B_{k_m}^m), w_n^{\beta,m})$$

and consequently

$$v_{n+1}^{\beta,m}(\mu) = w_{n+1}^{\beta,m}(\mu(B_1^m), \dots, \mu(B_{k_m}^m))$$

Thus by induction (3.66) holds for $n = 1, 2, \dots$, and therefore we obtain (3.63), which together with (3.62) completes the proof.

■

We may now consider control functions in the classes $\mathcal{B}(P(E_m), U)$ or $C(P(E_m), U)$ noticing that a given function $u_m \in \mathcal{B}(P(E_m), U)$ can be extended to a function $u \in \mathcal{B}(P(E), U)$ by putting

$$u(\mu) = u_m(\mu(B_1^m), \dots, \mu(B_{k_m}^m))$$

As corollary to Theorem 3.18 now we have

Corollary 3.19 *Assume (A1)–(A5), (B9) and E compact. If for a given $\varepsilon > 0$*

$$\sup_{\mu \in P(E)} |w^{\beta,m}(\mu(B_1^m), \dots, \mu(B_{k_m}^m)) - v^\beta(\mu)| < \varepsilon \tag{3.67}$$

for $m \geq m_0$, and $u_m^\beta \in \mathcal{B}(P(E_m), U)$ satisfies

$$w^{\beta,m}(\eta) + \varepsilon \geq \sum_{k=1}^{k_m} c(k, u_m^\beta(\eta))\eta_k + \beta \overline{\prod}\nolimits_m^{u_m^\beta(\eta)}(\eta, w^{\beta,m}) \tag{3.68}$$

for $\eta \in P(E_m)$, then the control with generic term $a_n = u_m^\beta(\pi_n(B_1^m), \dots, \pi_n(B_{k_m}^m))$ is $\frac{3\varepsilon}{1-\beta}$-optimal for the cost functional J_μ^β.

P r o o f . Follows directly from Corollary 3.11 and the proof of Theorem 3.18.

■

Remark 3.20 *The construction of a nearly optimal control function in the case of a compact state space E can now be reduced to finding for any $\varepsilon > 0$*

a Borel measurable function $u: P(E_m) \to U$ such that for $\eta \in P(E_m)$ the following inequality holds

$$w^{\beta,m}(\eta) + \varepsilon \geq \sum_{k=1}^{k_m} c(k, u(\eta))\eta_k + \beta \overline{\prod}_m^{u(\eta)}(\eta, w^{\beta,m}) \tag{3.69}$$

To obtain $u \in \mathcal{B}(P(E_m), U)$, satisfying (3.69) we may use the value iteration algorithm (3.65) truncating it at a sufficiently large value of n as we did in Corollary 3.2. This however does not result in a computationally convenient approach. Therefore in subsection 3.3.3.a, after a further approximation leading to control functions taking a finite number of values, we shall mention some computationally feasible approaches recalling also from the literature an algorithm for the solution of the Bellman equation associated to (3.69), namely

$$w^{\beta,m}(\eta) = \inf_{a \in U} \Big\{ \sum_{k=1}^{k_m} c(k, a)\eta_k + \beta \overline{\prod}_m^{a}(\eta, w^{\beta,m}) \Big\} \tag{3.70}$$

3.3.2.b Continuous control functions

The first step consists of defining compact classes of controls $\mathcal{A}_m(L, n)$ that correspond to the classes $\mathcal{A}(L, n)$ and consist of functions in $C(P(E_m), U)$. More precisely let

$$\begin{aligned}
&\mathcal{A}_m(L, n) = \{u \in C(P(E_m), U): u(\eta) = \overline{u}\Big(\sum_{k=1}^{k_m} \varphi_1(b_k^m)\eta_k, \ldots, \sum_{k=1}^{k_m} \varphi_n(b_k^m)\eta_k\Big),\\
&\text{where } \overline{u}: [-\|\varphi_1\|, \|\varphi_1\|] \times \ldots \times [-\|\varphi_n\|, \|\varphi_n\|] \to U\\
&\text{is Lipschitz with Lipschitz constant } L\}
\end{aligned}$$

Recalling the mapping $\mathcal{L}_m: P(E) \to P(E)$ introduced in (3.60), define also the following mappings

$$\overline{\mathcal{L}}_m: C(P(E), U) \ni u \to \overline{\mathcal{L}}_m u \quad \text{with } \overline{\mathcal{L}}_m u(\nu) = u(\mathcal{L}_m \nu)$$

and

$$\widetilde{\mathcal{L}}_m: \mathcal{A} \ni u \to \widetilde{\mathcal{L}}_m u \in C(P(E_m), U) \quad \text{with } \widetilde{\mathcal{L}}_m u(\eta) = u\Big(\sum_{k=1}^{k_m} \eta_k \delta_{b_k^m}\Big)$$

It follows immediately that

$$\overline{\mathcal{L}}_m u(\nu) = u\Big(\sum_{k=1}^{k_m} \nu(B_k^m)\delta_{b_k^m}\Big) = \widetilde{\mathcal{L}}_m u(\nu(B_1^m), \dots, \nu(B_{k_m}^m)) \tag{3.71}$$

Consequently

$$\widetilde{\mathcal{L}}_m \mathcal{A}(L,n) = \mathcal{A}_m(L,n) \tag{3.72}$$

In fact, if $u \in \mathcal{A}_m(L,n)$, then

$$(\widetilde{\mathcal{L}}_m u)(\eta) = u\Big(\sum_{k=1}^{k_m} \eta_k \delta_{b_k^m}\Big) = \overline{u}\Big(\sum_{k=1}^{k_m} \varphi_1(b_k^m)\eta_k, \dots, \sum_{k=1}^{k_m} \varphi_n(b_k^m)\eta_k\Big) \in \mathcal{A}_m(L,n)$$

On the other hand, let $\widetilde{u} \in \mathcal{A}_m(L,n)$. Then there is a Lipschitz function $\overline{u}$ with Lipschitz constant L such that

$$\widetilde{u}(\eta) = \overline{u}\Big(\sum_{k=1}^{k_m} \varphi_1(b_k^m)\eta_k, \dots, \sum_{k=1}^{k_m} \varphi_n(b_k^m)\eta_k\Big)$$

Define, for $\nu \in P(E)$,

$$\hat{u}(\nu) = \overline{u}(\nu(\varphi_1), \dots, \nu(\varphi_n)) \tag{3.73}$$

Obviously $\hat{u} \in \mathcal{A}(L,n)$ and furthermore

$$(\widetilde{\mathcal{L}}_m \hat{u})(\eta) = \overline{u}\Big(\sum_{k=1}^{k_m} \varphi_1(b_k^m)\eta_k, \dots, \sum_{k=1}^{k_m} \varphi_n(b_k^m)\eta_k\Big) = \widetilde{u}(\eta)$$

We shall need the following properties of the operator $\overline{\mathcal{L}}_m$.

Lemma 3.21 *We have*

(i) *for $u \in \mathcal{A}$, $\overline{\mathcal{L}}_m u(\nu) \to u(\nu)$ as $m \to \infty$, uniformly on compact subsets of $P(E)$*

(ii) *if $\mathcal{B}(P(E), U) \ni u_m \to u \in C(P(E), U)$ uniformly on $P(E)$, as $m \to \infty$, then $\overline{\mathcal{L}}_m u_m(\nu) \to u(\nu)$ as $m \to \infty$ uniformly on compact subsets of $P(E)$.*

P r o o f. Part (i) is an almost immediate implication of Lemma 3.16 and Lemma 3.6. For the proof of (ii) notice that, letting ρ_U be a metric compatible with the topology of U, we have

$$\rho_U(\overline{\mathcal{L}}_m u_m(\mu), u(\mu)) \le \rho_U(\overline{\mathcal{L}}_m u_m(\mu), \overline{\mathcal{L}}_m u(\mu)) + \rho_U(\overline{\mathcal{L}}_m u_m(\mu), u(\mu))$$

The first term on the right hand side converges to 0 by the assumption. The convergence of the second term follows from (i).

■

Consider now again the controlled Markov chain $(\overline{x}_n^m)$ on E_m, with transition matrix $\overline{P}_m^a(k,p)$ in the generic period n, where this time $a_n = u(\overline{\pi}_n^m)$, with $u \in \mathcal{A}_m(L,n)$ and $\overline{\pi}_n^m$ is the filtering process corresponding to the observation structure (3.54) and given by (3.57). Let the cost functional corresponding to an initial law η of $(\overline{x}_n^m)$ and control function $u \in \mathcal{A}_m(L,n)$ be given by (compare to (3.55))

$$J_\eta^{\beta,m}(u) = \sum_{n=0}^{\infty} \beta^n E_\eta^u\{c_m(\overline{x}_n^m, u(\overline{\pi}_n^m))\} \tag{3.74}$$

For given $\mu \in P(E)$ denote by $\overline{\mu}$ the vector $(\mu(B_1^m), \ldots, \mu(B_{k_m}^m))$. We have

Theorem 3.22 *Assume (A1)–(A5) and (B9). Then, for given $L > 0$, $n = 1, 2, \ldots$,*

$$\sup_{u \in \mathcal{A}(L,n)} |J_{\overline{\mu}}^{\beta,m}(\widetilde{\mathcal{L}}_m u) - J_\mu^\beta(u)| \to \infty \tag{3.75}$$

as $m \to \infty$, uniformly in μ from compact subsets of $P(E)$.

P r o o f. Assume (3.75) does not hold uniformly on a compact set $H \subset P(E)$. Then, by the compactness of the class $\mathcal{A}(L,n)$ and of the set H, there are sequences $\mathcal{A}(L,n) \ni u_m \to u$ uniformly as $m \to \infty$, and $H \ni \mu_m \Rightarrow \mu$, such that for some $\delta > 0$ and $m = 1, 2, \ldots$, we have

$$|J_{\overline{\mu}_m}^{\beta,m}(\widetilde{\mathcal{L}}_m u_m) - J_{\mu_m}^\beta(u_m)| > \delta \tag{3.76}$$

By Theorem 3.13

$$|J_{\mu_m}^\beta(u_m) - J_{\mu_m}^\beta(u)| \to 0 \quad \text{as } m \to \infty \tag{3.77}$$

and by Lemma 3.21 (ii) and Theorem 3.13 again

$$|J_{\mu_m}^{\beta,m}(\overline{\mathcal{L}}_m u_m) - J_{\mu_m}^{\beta}(u)| \to 0 \quad \text{as } m \to \infty \tag{3.78}$$

Notice that the cost functional $J_{\mu_m}^{\beta,m}(\overline{\mathcal{L}}_m u_m)$ used above corresponds to an approximating process (x_n^m), defined on E as in Section 3.3.1, with initial law μ_m, transition operator $P_m^{a_n}(x_n^m, \cdot)$ in the generic period n, where now $a_n = \overline{\mathcal{L}}_m u_m(\pi_n^{m,u})$, and $(\pi_n^{m,u})$ is given by (3.14).

Now, since the cost functions $c_m(x,a)$ and transition operators $P_m^a(x,\cdot)$ defined in (3.49), (3.50) respectively, do not change their values for $x \in B_k^m$, $1 \le k \le k_m$, we have

$$J_{\mu_m}^{\beta,m}(\overline{\mathcal{L}}_m u_m) = J_{\mathcal{L}_m \mu_m}^{\beta,m}(\overline{\mathcal{L}}_m u_m) \tag{3.79}$$

Notice also that the filtering processes (π_n^m) and $(\overline{\pi}_n^m)$, corresponding to initial laws $(\mathcal{L}_m \mu_m)$ and $\overline{\mu}_m$, and controls $a_n = \overline{\mathcal{L}}_m u_m(\pi_n^m)$ and $a_n = \widetilde{\mathcal{L}}_m u_m(\overline{\pi}_n^m)$ respectively, satisfy the following relation

$$\pi_n^m(B_k^m) = \overline{\pi}_n^m(k) \quad \text{for } 1 \le k \le k_m$$

Therefore

$$J_{\mathcal{L}_m \mu_m}^{\beta,m}(\overline{\mathcal{L}}_m u_m) = J_{\overline{\mu}_m}(\widetilde{\mathcal{L}}_m u_m) \tag{3.80}$$

and by (3.77), (3.78), (3.79) we obtain a contradiction to (3.75). This completes the proof of the theorem.

■

Corollary 3.23 *Under the assumptions of Theorem 3.22 we have*

(i) $$\inf_{u \in \mathcal{A}_m(L,n)} J_{\overline{\mu}}^{\beta,m}(u) \to \inf_{u \in \mathcal{A}(L,n)} J_{\mu}^{\beta}(u) \tag{3.80}$$

uniformly in μ from compact subsets of $P(E)$, as $m \to \infty$,

(ii) *if $\widetilde{u} \in \mathcal{A}_m(L,n)$ is ε-optimal for $J_{\overline{\mu}}^{\beta,m}$ with m sufficiently large so that*

$$\sup_{u \in \mathcal{A}(L,n)} |J_{\overline{\mu}}^{\beta,m}(\widetilde{\mathcal{L}}_m u) - J_{\mu}^{\beta}(u)| < \varepsilon \tag{3.81}$$

then any $\hat{u} \in \mathcal{A}(L,n)$ such that $\widetilde{\mathcal{L}}_m \hat{u} = \widetilde{u}$ is 3ε-optimal for J_{μ}^{β}.

P r o o f. For the proof of part (i) it is sufficient to notice that

$$\begin{aligned}&|\inf_{u\in\mathcal{A}_m(L,n)} J_{\overline{\mu}}^{\beta,m}(u) - \inf_{u\in\mathcal{A}(L,n)} J_{\mu}^{\beta}(u)| \leq \\ &\leq \sup_{u\in\mathcal{A}(L,n)} |J_{\overline{\mu}}^{\beta,m}(\widetilde{\mathcal{L}}_m u) - J_{\mu}^{\beta}(u)|\end{aligned}$$

and apply Theorem 3.22.

In part (ii), if $\tilde{u} \in \mathcal{A}_m(L,n)$ is ε-optimal for $J_{\overline{\mu}}^{\beta,m}$, then for $\hat{u} \in \mathcal{A}(L,n)$ such that $\widetilde{\mathcal{L}}_m\hat{u} = \tilde{u}$ we have

$$\begin{aligned}&J_{\mu}^{\beta}(\hat{u}) \leq J_{\overline{\mu}}^{\beta,m}(\widetilde{\mathcal{L}}_m\hat{u}) + \sup_{u\in\mathcal{A}(L,n)} |J_{\mu}^{\beta}(u) - J_{\overline{\mu}}^{\beta,m}(\widetilde{\mathcal{L}}_m u)| \\ &\leq \inf_{u\in\mathcal{A}_m(L,n)} J_{\overline{\mu}}^{\beta,m}(u) + \varepsilon + \varepsilon \leq \inf_{u\in\mathcal{A}(L,n)} J_{\mu}^{\beta}(u) + 3\varepsilon\end{aligned}$$

provided m is so large that (3.81) holds.

■

Remark 3.24 *Once $\tilde{u}$ is given, an $\hat{u} \in \mathcal{A}(L,n)$ such that $\widetilde{\mathcal{L}}_m\hat{u} = \tilde{u}$ can be constructed according to (3.73). Also in the case of continuous control functions the problem of determining a nearly optimal control function is thus reduced to the problem of determining a nearly optimal control function for $J_{\eta}^{\beta,m}$ (see 3.73) in the class $\mathcal{A}_m(L,n)$ that contains functions of finite dimensional measures of $P(E_m)$. A further approximation, allowing such controls to be determined explicitly, will be described in subsection 3.3.3.b.*

3.3.3 Further discretizations

By the results of the previous subsections 3.3.1, 3.3.2, the problem of the construction of a nearly optimal control function for the original problem is now reduced to the same problem for the partially observed Markov chain $\overline{x}_n^m$ on E_m, having transition matrix $\overline{P}_n^{a_n}(k,p)$ in the generic period n and D_m-valued observations $(\overline{y}_n^m)$ satisfying (3.54). The corresponding cost functional $J_{\eta}^{\beta,m}$ is given by (3.55). The controls (a_n) are $\overline{Y}_n^m = \sigma\{\overline{y}_1^m, \ldots, \overline{y}_n^m\}$ adapted, U-valued random variables. In the particular case, when we consider control functions that are continuous, we restrict ourselves to controls of the form $a_n = u(\overline{\pi}_n^m)$, where $u \in \mathcal{A}_m(L,n)$ and $\overline{\pi}_n^m$ is the filtering process corresponding to $(\overline{x}_n^m)$ and given by (3.57).

The purpose is to find, for the above finite state control problem, a nearly optimal control function that by the previous results can then be extended to become a nearly optimal control function for the original problem with cost function J^β_μ. We shall again distinguish between the two cases when the state space E is compact and when the control functions are restricted to be continuous.

We first summarize several simple consequences of assumptions (A1), (A2) and (B9).

Lemma 3.25 *Under (A2), and (B9)*

(i) *the mapping* $U \ni a \mapsto \overline{P}^a_m(i,j)$ *is continuous for* $i, j \in E_m$

(ii) *the mapping* $U \times P(E_m) \ni (a, \eta) \mapsto \overline{M}^a_m(y, \eta) \in P(E_m)$ *is continuous for* $y \in D_m$

(iii) *the mapping* $U \ni P(E_m) \ni (a, \eta) \mapsto \overline{\prod}^a_m(\eta, F)$ *is continuous provided* $F \in C(P(E_m))$

(iv) *for* $u \in C(P(E_m), U)$*, the transition operator* $\overline{\prod}^{u(\eta)}_m(\eta, \cdot)$ *is Feller i.e. for* $F \in C(P(E_m))$ *the mapping* $P(E_m) \ni \eta \mapsto \overline{\prod}^{u(\eta)}_m(\eta, F)$ *is continuous.*

P r o o f. The statement (i) follows from (3.53). The remaining conclusions follow from the previous ones.

■

3.3.3.a Compact state space E

The processes $(\overline{x}^m_n)$ and $(\overline{y}^m_n)$ take a finite number of admissible values. However, the set of control parameters is still infinite. Below we shall therefore also consider the possibility of discretizing the control set by using partitions $(U^H_k)_{k=1,2,\ldots,H}$, $H \to \infty$, of U, and representative elements $(\alpha^H_k)_{k=1,2,\ldots,H}$, $\alpha^H_k \in U^H_k$, such that for $H' > H$, $(U^{H'}_k)_{k=1,2,\ldots,H'}$ is a refinement of $(U^H_k)_{k=1,2,\ldots,H}$ and $(\alpha^H_k)_{k=1,2,\ldots,H}$ are contained in $(\alpha^{H'}_k)_{k=1,2,\ldots,H'}$; furthermore the diameters of U^H_k converge to 0 as $H \to \infty$.

Define the projection operator Z_H as in (2.50). From Lemma 3.25(iii) we easily obtain

Corollary 3.26 *Under (A2) and (B9) for $F \in C(P(E_m))$*

$$\overline{\prod}_m^{Z_H a}(\eta, F) \to \overline{\prod}_m^{a}(\eta, F) \quad \text{as } H \to \infty$$

uniformly in $(a, \eta) \in U \times P(E_m)$

Analogously to (3.56) denote by $w_H^{\beta,m}(\eta)$ the optimal value of the cost functional $J_\eta^{\beta,m}$ of (3.59) over the controls (a_n) that in this subsection will always take the values in $U_H = \{\alpha_1^H, \ldots, \alpha_H^H\}$.

We have

Theorem 3.27 *Under (A2), (A5) and (B9)*

$$w_H^{\beta,m}(\eta) \to w^{\beta,m}(\eta) \quad \text{as } H \to \infty \tag{3.82}$$

uniformly in $\eta \in P(E_m)$.

Moreover, if for a given $\varepsilon > 0$ and $H > H_0$

$$\sup_{\eta \in P(E_m)} |w_H^{\beta,m}(\eta) - w^{\beta,m}(\eta)| < \varepsilon \tag{3.83}$$

and $u_H \in \mathcal{B}(P(E_m), U_H)$ satisfies

$$w_H^{\beta,m}(\eta) + \varepsilon \geq \sum_{k=1}^{k_m} c(k, u_H(\eta))\eta_k + \beta \overline{\prod}_m^{u_H(\eta)}(\eta, w_H^{\beta,m}) \tag{3.84}$$

for $\eta \in P(E_m)$, then the control $a_n = u_H(\overline{\pi}_n^m)$ is, for $H > H_0$, $\frac{3\varepsilon}{1-\beta}$ optimal for the cost functional $J_\eta^{\beta,m}$ over controls a_n adapted to $\overline{Y}_n^m$ with values in U.

P r o o f. By Theorem 3.1, $w^{\beta,m}$ and $w_H^{\beta,m}$ can be uniformly approximated by sequences $w_n^{\beta,m}$ and $w_{H,n}^{\beta,m}$ respectively given by the value iterations

$$\begin{aligned} &w_0^{\beta,m}(\eta) \equiv 0 \\ &w_{n+1}^{\beta,m}(\eta) = \inf_{a \in U} \Big[\sum_{k=1}^{k_m} c(k, a)\eta_k + \beta \overline{\prod}_m^{a}(\eta, w_n^{\beta,m}) \Big] \end{aligned} \tag{3.85}$$

and

$$\begin{aligned} &w_{H,0}^{\beta,m}(\eta) \equiv 0 \\ &w_{H,n+1}^{\beta,m}(\eta) = \inf_{a \in U_H} \Big[\sum_{k=1}^{k_m} c(k, a)\eta_k + \beta \overline{\prod}_m^{a}(\eta, w_{H,n}^{\beta,m}) \Big] \end{aligned} \tag{3.86}$$

For the first statement therefore it remains to show that for each $n = 0, 1, 2, \ldots$

$$w_{H,n}^{\beta,m}(\eta) \to w_n^{\beta,m}(\eta) \quad \text{as } H \to \infty \tag{3.87}$$

uniformly in $\eta \in P(E_m)$.

Since by (A5) and (3.49) the mapping

$$U \ni a \mapsto c(k, a)$$

is continuous, in view of Corollary 3.26 we easily obtain (3.87) by induction in n. Thus (3.82) holds.

The second part of Theorem 3.27 follows from (3.83) Theorem 3.18 and (3.11) of Remark 3.3.

■

Remark 3.28 *Combining Theorem 3.27 with Corollary 3.19 we have that for H and m such that (3.83) and (3.67) are satisfied, the control $a_n = u_H(\pi_n(B_1^m), \ldots, \pi_n(B_{k_m}^m))$ is $\frac{5\varepsilon}{1-\beta}$ optimal for the cost functional J_μ^β.*

■

Although the partially observed controlled process $(\overline{x}_n^m)$, its observations $(\overline{y}_n^m)$ and controls (a_n) take now a finite number of admissible values, the corresponding filtering process $\overline{\pi}_n^m$ takes its values in the infinite space $P(E_m)$. At this stage we have two possibilities: Either we look for nearly optimal controls of a completely observable problem with states given by the filtering process $\overline{\pi}_n^m$; this approach will lead to a discretization of $P(E_m)$, making it possible to use Howard's policy iteration procedure (see [18]) for completely observed discounted problems with finite state space and finite set of control parameters. We may however also adapt the so called Sondik algorithm that concerns the construction of nearly optimal controls for partially observed Markov chains with finite state space, finite observation space, and finite set of control parameters. The two approaches are described in the following two subsections.

3.3.3.a$_1$ Discretization of $P(E_m)$

Let $(G_k^q)_{k=1,2,\ldots,k_q}, q = 1, 2, \ldots$ be a sequence partitions of $P(E_m)$, such that for $q' > q$ the partition $(G_k^{q'})_{k=1,2,\ldots,k_{q'}}$, is a subpartition of $(G_k^q)_{k=1,2,\ldots,k_q}$ and the diameters of G_k^q go to 0 as $q \to \infty$. Furthermore, assume that we are given a sequence of selectors $(e_1^q, \ldots, e_{k_q}^q)$, $e_k^q \in G_k^q$ for $k = 1, 2, \ldots, k_q$ such that $\{e_1^q, \ldots, e_{k_q}^q\} \subset \{e_1^{q'}, \ldots, e_{k_{q'}}^{q'}\}$ when $q' > q$.

Let, for $k, p = 1, 2, \ldots, k_q$

$$\overline{\prod}^a_{m,q}(k,p) = \overline{\prod}^a_m(e_k^q, G_p^q) \tag{3.88}$$

Consider now a completely observed controlled Markov process $(\widetilde{\pi}_n)$ with values in the finite set $\{1, 2, \ldots, k_q\}$ indexing the selectors for a given q and with transition matrix $\overline{\prod}^{a_n}_{m,q}(k,p)$ is the generic period n. Assume the controls $u = (a_n)$ are U_H-valued and adapted to $\sigma\{\widetilde{\pi}_1, \ldots, \widetilde{\pi}_n\}$. Given $\widetilde{\pi}_0 = p \in \{1, 2, \ldots, k_q\}$, let the corresponding cost functional $\widetilde{J}_p^{\beta,q}(u)$ be given by

$$\widetilde{J}_p^{\beta,q}(u) = \sum_{n=0}^{\infty} \beta^n E_p^u \Big\{ \sum_{k=1}^{k_m} c(k, a_n) e^q_{\widetilde{\pi}_n}(k) \Big\} \tag{3.89}$$

where $e^q_{\widetilde{\pi}_n}(k)$ is the k-th coordinate of the selector $e^q_{\widetilde{\pi}_n}$ in $P(E_m)$ (Clearly $e^q_{\widetilde{\pi}_n} = (e^q_{\widetilde{\pi}_n}(1), \ldots, e^q_{\widetilde{\pi}_n}(k_m)) \in P(E_m)$.

Denote by $\widetilde{w}_H^{\beta,m,q}(p)$ the optimal value of the cost functional $\widetilde{J}_p^{\beta,q}(u)$ over controls $u = (a_n)$ that are U_H-valued and adapted to $\sigma\{\widetilde{\pi}_1, \ldots, \widetilde{\pi}_n\}$.

Furthermore, denote by Q_q the projection operator on $\{1, 2, \ldots, k_q\}$, i.e.

$$Q_q \colon P(E_m) \ni \eta \mapsto k \quad \text{if } \eta \in G_k^q \;\; k = 1, 2, \ldots, k_q \tag{3.90}$$

Proposition 3.29 *For given m and H we have*

$$\sup_{\eta \in P(E_m)} |\widetilde{w}_H^{\beta,m,q}(Q_q\eta) - w_H^{\beta,m}(\eta)| \to 0 \tag{3.91}$$

as $q \to \infty$.

Moreover, if for a given $\varepsilon > 0$

$$\sup_{\eta \in P(E_m)} |\widetilde{w}_H^{\beta,m,q}(Q_q\eta) - w_H^{\beta,m}(\eta)| < \varepsilon \tag{3.92}$$

and $u_q \colon \{1, 2, \ldots, k_q\} \to U_H$ is such that

$$\begin{aligned} \widetilde{w}_H^{\beta,m,q}(p) + \varepsilon \;\; &\geq \sum_{k=1}^{k_m} c(k, u_q(p)) e_p^q(k) \\ &+ \beta \sum_{l=1}^{k_q} \widetilde{w}_H^{\beta,m,q}(l) \overline{\prod}^{u_q(p)}_{m,q}(p,l) \end{aligned} \tag{3.93}$$

then the control function $u_H \in \mathcal{B}(P(E_m), U_H)$ defined as $u_H(\eta) = u_q(Q_q\eta)$ is $\frac{3\varepsilon}{1-\beta}$ optimal for the cost functional $J_\eta^{\beta,m}$ over controls with values in U_H.

P r o o f. We use again the value iteration procedure from which we know that $w_H^{\beta,m}$ is approximated uniformly by a sequence of functions $w_{H,n}^{\beta,m}(\eta)$ as in (3.86) and, analogously, $\widetilde{w}_H^{\beta,m,q}$ is approximated uniformly by

$$\begin{aligned}
&\widetilde{w}_{H,0}^{\beta,m,q}(p) \equiv 0 \\
&\widetilde{w}_{H,n+1}^{p,m,q}(p) = \inf_{a \in U_H} \Big[\sum_{k=1}^{k_m} c(k,a) e_p^q(k) + \beta \sum_{l=1}^{k_q} \widetilde{w}_H^{\beta,m,q}(l) \overline{\prod}_{m,q}^a(p,l) \Big]
\end{aligned} \tag{3.94}$$

Therefore it suffices to show the convergence of

$$\widetilde{w}_{H,n}^{\beta,m,q}(Q_q\eta) \to w_{H,n}^{\beta,m}(\eta) \quad \text{as } q \to \infty \tag{3.95}$$

for each $n = 0, 1, 2, \ldots$, uniformly in $\eta \in P(E_m)$.

We use induction in n. For $n = 0$, (3.95) clearly holds. Given (3.95) for n, for $(n+1)$ we have

$$\begin{aligned}
&|\widetilde{w}_{H,n+1}^{\beta,m,q}(Q_q\eta) - w_{H,n+1}^{\beta,m}(\eta)| \le k_m \|c\| \max_k diam(G_k^q) \\
&+\beta \int_{P(E_m)} |\widetilde{w}_H^{\beta,m,q}(Q_q\zeta) - w_{H,n}^{\beta,m}(\zeta)| \overline{\prod}_m^a(e_{Q_q\eta}^q, d\zeta) \\
&+\beta \int_{P(E_m)} w_{H,n}^{\beta,m}(\zeta) \Big| \overline{\prod}_m^a(e_{Q_q\eta}^q, d\zeta) - \overline{\prod}_m^a(\eta, d\zeta) \Big| \\
&= I_q + II_q + III_q
\end{aligned} \tag{3.96}$$

As $q \to \infty$ we have by the construction of the partition G_k^q, $I_q \to 0$ and, by induction hypothesis, that $II_q \to 0$.

To estimate III_q notice first that, by Lemma 3.25 (iii), $w_{H,n}^{\beta,m} \in C(P(E_m))$. Therefore, by the same Lemma 3.25 (iii), $III_q \to 0$ as $q \to \infty$. Since the limits are uniform in $\eta \in P(E_m)$, we obtain (3.96) for $n+1$ and, by induction, for each $n = 0, 1, 2, \ldots$.

The second part of Proposition follows from Remark 3.3.

∎

Remark 3.30 *Since the Markov process $(\widetilde{\pi}_n)$ with transition matrix $\widetilde{\prod}_{m,q}^a(k,p)$ has a finite number of possible states and we are left with only a finite number of possible controls $u_q(p)$ $(u_q(p) \in U_H,\ p \in \{1, 2, \ldots, k_q\})$, we can obtain*

(3.93) by using e.g. the value iteration procedure truncated after a sufficiently large number of iterations; we may also obtain (3.93) using Howard's policy improvement procedure (see [18], or Lemma 3.37, below).

Combining the second statement of Proposition 3.29 with Remark 3.28 we have that for q such that (3.92) is satisfied the control

$$a_n = u_q(Q_q(\pi_n(B_1^m)), \ldots, \pi_n(B_{k_m}^m)))$$

is $\frac{7\varepsilon}{1-\beta}$ *optimal for the cost functional* J_μ^β*.*

■

3.3.3.a$_2$ Sondik's algorithm

This section is again concerned with the partially observed control problem of the Markov chain $(\overline{x}_n^m)$ on the finite state space E_m, having transition matrix $\overline{P}_n^{a_n}(k,p)$ and the finite D_m-valued observations $(\overline{y}_n^m)$. We shall assume that the controls (a_n) are U_H-valued i.e. finite, $(\overline{Y}_n^m)$ adapted random variables and the cost functional to be minimized is $J_\eta^{\beta,m}$ given by (3.55). For such partially observed problems where the state space E_m, the observation space D_m, and set of control parameters U_H are finite, we now describe a further method to obtain nearly optimal control functions, namely the so called Sondik algorithm.

Let us first notice that, since the optimal value $w_H^{\beta,m}(\eta)$ of the cost functional $J_\eta^{\beta,m}$ over the U_H-valued controls is a solution to the Bellman equation

$$w_H^{\beta,m}(\eta) = \min_{a \in U_H} \Big[\sum_{k=1}^{k_m} c(k,a)\eta_k + \beta \overline{\prod}_m^a(\eta, w_H^{\beta,m}) \Big] \tag{3.97}$$

we can restrict ourselves to stationary controls i.e. controls of the form $a_n = u(\overline{\pi}_n^m)$, for $u \in \mathcal{B}(P(E_m), U_H)$, where $(\overline{\pi}_n^m)$ is the filtering process corresponding to the Markov chain $(\overline{x}_n^m)$ and the observations $(\overline{y}_n^m)$.

Next we recall the notion and some basic properties of finitely transient controls. Since U_H is finite, any $u \in \mathcal{B}(P(E_m), U_H)$ is piecewise constant. Given $u \in \mathcal{B}(P(E_m), U_H)$, denote by Δ_u the set of discontinuity points of u. Moreover, let for $A \in \mathcal{B}(P(E_m))$

$$M^u(A) = \text{closure}\,\{\overline{M}_m^{u(\eta)}(y,\eta) \colon y \in D_m,\ \eta \in A\}$$

and (3.98)

$$S_0^u = P(E_m), \qquad S_{n+1}^u = M^u(S_n^u)$$

We say that a control function $u \in \mathcal{B}(P(E_m), U_H)$ is finitely transient if for some positive integer n we have $\Delta_u \cap S_n^u = \emptyset$. The smallest such number n is called index and will be denoted by n_u.

Clearly, if u is finitely transient with index n_u then, since $M^u(S_n^u) \subset S_n^u$, we have $\Delta_u \cap S_n^u = \emptyset$ for $n \geq n_u$.

Define the sequence of sets

$$\Delta_u^0 = \Delta_u,\ \Delta_u^{n+1} = \{\eta \colon \overline{M}_m^{u(\eta)} \in \Delta_u^n \text{ for some } y \in D_m\}$$

We can say equivalently that a control function $u \in \mathcal{B}(P(E_m), U_H)$ is finitely transient if and only if $\Delta_u^n = \emptyset$ for $n \geq n_u$.

A finite partition $V^u = \{V_1, V_2, \ldots, V_\alpha\}$ of $P(E_m)$ is called Markov with respect to a control function $u \in \mathcal{B}(P(E_m), U_H)$ if

a) u is constant on V_i, $1 \leq i \leq \alpha$

b) there exists a mapping γ

$$\{1, 2, \ldots, \alpha\} \times D_m \ni (i, y) \mapsto \gamma(i, y) \in \{1, 2, \ldots, \alpha\}$$

such that for any $y \in D_m$, the mapping

$$P(E_m) \ni \eta \mapsto \overline{M}_m^{u(\eta)}(y, \eta)$$

transforms sets V_i into $V_{\gamma(i,y)}$.

With a given control function $u \in \mathcal{B}(P(E_m), U_H)$ we can associate a sequence (V_n^u) $n = 0, 1, 2, \ldots$, of partitions of $P(E_m)$, $V_n^u = \{V_1^n, \ldots, V_{\alpha_n}^n\}$ such that: V_0^u is the minimal partition (i.e. the partition that consists of minimal number of sets) into sets $V_1^0, \ldots, V_{\alpha_0}^0$ on which the control function u is constant; given the partition V_n^u we construct the partition V_{n+1}^u as the subpartition of V_n^u such that for each fixed $y \in D_m$, the mapping

$$P(E_m) \ni \eta \mapsto \overline{M}_n^{u(\eta)}(y, \eta) \tag{3.99}$$

transforms sets of V_{n+1}^u into sets of V_n^u.

We have

Lemma 3.31 *If $u \in \mathcal{B}(P(E_m), U_H)$ is finitely transient, then the partition $V^u_{n_u-1}$ is Markov.*

P r o o f. It suffices to notice that the set of boundaries of the partition V^u_n is $\Delta^0_u \cup \Delta^1_u \cup \cdots \cup \Delta^n_u$, and if u is finitely transient then $\Delta^n_u = \emptyset$ for $n \geq n_u$. ∎

Denote by $v^{\beta,m}(\eta|u)$ the value of $J^{\beta,m}_\eta((a_n))$ for $a_n = u(\overline{\pi}^m_n)$. By the Markov property of $(\overline{\pi}^m_n)$ we clearly have that $v^{\beta,m}(\eta|u)$ is the unique solution to the following equation

$$\begin{aligned} & v^{\beta,m}(\eta|u) = \\ & \sum_{k=1}^{k_m}[c(k,u(\eta)) + \beta\sum_{s=1}^{s_m} v^{\beta,m}(\overline{M}^{u(\eta)}_m(d^m_s,\eta)|u)\sum_{j=1}^{k_m}\overline{r}_m(j,d^m_s)\overline{P}^{u(\eta)}_m(k,j)]\eta_k \\ & := \sum_{k=1}^{k_m} d(k,\eta|u)\eta_k \end{aligned} \tag{3.100}$$

where we implicitly defined $d(k,\eta|u)$.

Proposition 3.32 *For $u \in \mathcal{B}(P(E_m), U_H)$ the vector function $d(k,\eta|u)$, $k = 1,2,\ldots,k_m$, $\eta \in P(E_m)$ is the unique solution to the equation*

$$\begin{aligned} d(k,\eta|u) = \; & c(k,u(\eta)) + \beta\sum_{s=1}^{s_m}\sum_{j=1}^{k_m} d(j,\overline{M}^{u(\eta)}_m(d^m_s,\eta)|u) \\ & \overline{r}_m(j,d^m_s)\overline{P}^{u(\eta)}_m(k,j) \end{aligned} \tag{3.101}$$

Moreover, if u is finitely transient, then $d(k,\eta|u)$ are constant on subsets of the partition $V^u_{n_u-1} = \{V^{n_u-1}_1, V^{n_u-1}_2, \ldots, V^{n_u-1}_{\alpha_{n_u-1}}\}$ and, letting $d(k,i) = d(k,\eta|u)$ for $\eta \in V^{n_u-1}_i$, $k = 1,2,\ldots,k_m$, $i = 1,2,\ldots,\alpha_{n_u-1}$, we have that $d(k,i)$ form the unique solution to the following system of equations

$$d(k,i) = c(k,u_i) + \beta\sum_{s=1}^{s_m}\sum_{j=1}^{k_m} d(j,\gamma(i,d^m_s))\overline{r}_m(j,d^m_s)\overline{P}^{u_i}_m(k,j) \tag{3.102}$$

where by u_i we denote the value of u on $V^{n_u-1}_i$, and γ is the mapping defined by the Markov partition $V^u_{n_u-1}$.

P r o o f. By the Banach contraction principle there is a unique solution $d(k,\eta|u)$ to (3.101). Furthermore, by taking into account the definition of the operator $\overline{M}_m$ (see (3.57)), it can be easily checked that $\sum_{k=1}^{k_m} d(k,\eta|u)\eta_k$ is a solution to (3.100). Since the solution $v^{\beta,m}$ to (3.100) is unique, we therefore have

$$v^{\beta,m}(\eta|u) = \sum_{k=1}^{k_m} d(k,\eta|u)\eta_k$$

If u is finitely transient, by Lemma 3.31 the partition $V^u_{n_u-1}$ is Markov. As a consequence, the control function u is constant on the subsets of $V^u_{n_u-1}$ and the function γ is well defined. Again by the Banach contraction principle the solution $d(k,i)$ to the equation (3.102) is unique. Since the vector function that is equal to $d(k,i)$ for $\eta \in V_i^{n_u-1}$ is a solution to (3.101) and as we noticed earlier $d(k,\eta|u)$ is the unique solution to (3.101), we obtain that $d(k,\eta|u)$ is constant equal to $d(k,i)$ on $V_i^{n_u-1}$.

■

From Proposition 3.32 and equation (3.100) we see that for a finitely transient control function $u \in \mathcal{B}(P(E_m),U_H)$, the cost functional $J^{\beta,m}_\eta(u) = v^{\beta,m}(\eta|u)$ as a function of the initial law η is piecewise linear and can be calculated from the finite system of linear equations (3.102). The practical solvability of the equations (3.102) is based on the existence of the function γ, induced by the Markov partition V^u. The piecewise linearity of the cost functional for finitely transient controls gives the starting point for Sondik's algorithm. Since a generic control function $u \in \mathcal{B}(P(E_m),U_H)$ will not be finitely transient and, even if it is, it will be difficult to verify such a property, we shall approximate the cost functional $J^{\beta,m}_\eta(u)$ by piecewise linear functions. These functions are constructed from the solutions of (3.102) where the function γ is obtained from a truncation of the sequence of partitions (V^u_n) corresponding to the control u as follows: given the sequence of partitions $(V^u_n)_{n=1,2,\ldots}$, we fix a certain n and select points $\overline{\eta}_1,\ldots,\overline{\eta}_{\alpha_n}$ such that $\overline{\eta}_i \in V^n_i$, $i \le \alpha_n$, letting

$$\gamma_n(i,y) = k \quad \text{if } \overline{M}^{u(\overline{\eta}_i)}(y,\overline{\eta}_i) \in V^n_k \tag{3.103}$$

Clearly, when u is finitely transient and $n \ge n_u - 1$, then $\gamma_n \equiv \gamma$. In general however, it may happen that $\eta \in V^n_i$, $\gamma_n(i,y) = k$ and $\overline{M}^{u(\eta)}_m(y,\eta) \notin V^n_k$.

Given γ_n we thus solve the following analog of the system (3.102)

$$d_n(k,i) = c(k,u_i) + \beta \sum_{s=1}^{s_m} \sum_{j=1}^{k_m} d_n(j,\gamma_n(i,d_s^m)) \overline{r}_m(j,d_s^m) \overline{P}_m^{u_i}(k,j) \tag{3.104}$$

for $k = 1,2,\dots,k_m$, $i = 1,2,\dots,\alpha_n$, where by u_i we denote the constant value of the control function u on V_i^n.

By the Banach contraction principle, there is a unique solution to (3.104), and therefore $d_n(k,i)$ is defined in a unique way.

Letting

$$v^{\beta,m,n}(\eta|u) = \sum_{k=1}^{k_m} d_n(k,i)\eta_k \quad \text{for } \eta \in V_i^n \tag{3.105}$$

we obtain

Proposition 3.33 *For a given $u \in C(P(E_m), U_H)$ we have*

$$\sup_{\eta \in P(E_m)} |v^{\beta,m}(\eta|u) - v^{\beta,m,n}(\eta|u)| \le \frac{\beta^n}{1-\beta^n} \cdot \frac{K}{1-\beta} \tag{3.106}$$

with $K = \sup_{k=1,2,\dots,k_m} \sup_{a,a' \in U_H} |c(k,a') - c(k,a)|$.

P r o o f. It will be convenient to introduce the following operators T^u, T_n^u defined on $b\mathcal{B}(P(E_m))$ and the set of functions that map $\{1,2,\dots,k_m\} \times \{1,2,\dots,\alpha_n\}$ into R respectively,

$$T^u v(\eta) = \sum_{k=1}^{k_m} \sum_{s=1}^{s_m} v(\overline{M}_m^{u(\eta)}(d_s^m,\eta)) \sum_{j=1}^{k_m} \overline{r}_m(j,d_s^m) \overline{P}_m^{u(\eta)}(k,j)\eta_k \tag{3.107}$$

for $v \in b\mathcal{B}(P(E_m))$, $\eta \in P(E_m)$, and

$$T_n^u d(k,i) = \sum_{s=1}^{s_m} \sum_{j=1}^{k_m} d(j,\gamma_n(i,d_s^m)) \overline{r}_m(j,d_s^m) \overline{P}_m^{u_i}(k,j) \tag{3.108}$$

for a function $d\colon \{1,2,\dots,k_m\} \times \{1,2,\dots,\alpha_n\} \to R$, and $1 \le k \le k_m$, $1 \le i \le \alpha_n$.

By (3.100)

$$v^{\beta,m}(\eta|u) = C^u(\eta) + \sum_{p=1}^{n-1} \beta^p (T^u)^p C^u(\eta) + \beta^n (T^u)^n v^{\beta,m}(\eta|u) \tag{3.109}$$

with $C^u(\eta) := \sum_{k=1}^{k_m} c(k, u(\eta))\eta_k$.

Furthermore by (3.104), (3.105), given $\eta \in V_i^n$, we have

$$\begin{aligned} v^{\beta,m,n}(\eta|u) = \quad & \sum_{k=1}^{k_m} \overline{c}(k,i)\eta_k + \sum_{p=1}^{n-1} \sum_{k=1}^{k_m} \beta^p (T_n^u)^p \overline{c}(k,i)\eta_k \\ & + \beta^n \sum_{k=1}^{k_m} (T_n^u)^n d_n(k,i)\eta_k \end{aligned} \tag{3.110}$$

with $\overline{c}(k,i) := c(k, u_i)$, where by $(T^u)^p$, $(T_n^u)^p$ we denote the p-th iterations of the operators T^u, T_n^u respectively.

Notice now that, by the definition of the operator $\overline{M}_m^u$ and the construction of the function γ_n (see (3.103)), it can be checked that

$$\sum_{p=1}^{n-1} \beta^p (T^u)^p C^u(\eta) = \sum_{p=1}^{n-1} \beta^p \sum_{k=1}^{k_m} (T_n^u)^p \overline{c}(k,i)\eta_k$$

for $\eta \in V_i^n$.

Therefore

$$v^{\beta,m}(\eta|u) - v^{\beta,m,n}(\eta|u) = \beta^n [(T^u)^n v^{\beta,m}(\eta|u) - \sum_{k=1}^{k_m} (T_n^u)^n d_n(k,i)\eta_k]$$

for $\eta \in V_i^n$.

Consequently,

$$\begin{aligned} & \sup_{\eta \in P(E_m)} |v^{\beta,m}(\eta|u) - v^{\beta,m,n}(\eta|u)| \leq \\ & \leq \sup_{\eta \in P(E_m)} \beta^n |(T^u)^n v^{\beta,m}(\eta|u) - (T^u)^n v^{\beta,m,n}(\eta|u)| \\ & + \sup_{i=1,2,\ldots,\alpha_n} \sup_{\eta \in V_i^n} \beta^n |(T^u)^n v^{\beta,m,n}(\eta|u) - \sum_{k=1}^{k_m} (T_n^u)^n d_n(k,i)\eta_k| \end{aligned} \tag{3.111}$$

Since for $\eta \in V_i^n$

$$\begin{aligned} & |(T^u)^n v^{\beta,m,n}(\eta|u) - \sum_{k=1}^{k_m} (T_n^u)^n d_n(k,i)\eta_k| \leq \\ & \leq \sup_{\eta \in P(E_m)} \sup_{i,j=1,2,\ldots,\alpha_n} \Big| \sum_{k=1}^{k_m} (d_n(k,i) - d_n(k,j))\eta_k \Big| \end{aligned}$$

and by (3.102)

$$\begin{aligned}&\sup_{\eta\in P(E_m)}\sup_{i,j=1,2,\ldots,\alpha_n}\Big|\sum_{k=1}^{k_m}(d_n(k,i)-d_n(k,j))\eta_k\Big|\\ &\le(1-\beta)^{-1}\sup_{k=1,\ldots,k_m}\sup_{a,a'\in U_H}|c(k,a)-c(k,a')|\end{aligned}$$

from (3.111) we obtain (3.106).

■

It will be important to have an interpretation of the function $v^{\beta,m,n}$ defined in (3.105) as value of a certain cost functional corresponding to a controlled Markov process.

To this effect consider the pair $(\overline{\pi}_n^m,\overline{y}_n^m)$, consisting of the filtering process $\overline{\pi}_n^m$, corresponding to the Markov chain $\overline{x}_n^m$ with transition matrix in the generic period n equal to $\overline{P}_n^{a_n}(k,p)$, and the observation process $\overline{y}_n^m$, where for $\overline{y}_0^m$ we take a fixed element of D_m.

We have almost immediately that (see (3.57) and (3.58))

Lemma 3.34 *If the control a_n in the generic period n is of the form $a_n = u(\overline{\pi}_n^m,\overline{y}_n^m)$ with $u\in\mathcal{B}(P(E_m)\times D_m,U_H)$, the pair $(\overline{\pi}_n^m,\overline{y}_n^m)$ forms a Markov process with respect to the σ-field $\overline{Y}_n^m$ with transition operator*

$$\begin{aligned}&\overline{T}^{u(\eta,y)}(\eta,y,v)=\\ &\sum_{k=1}^{k_m}\sum_{s=1}^{s_m}v(\overline{M}_m^{u(\eta,y)}(d_s^m,\eta),d_s^m)\sum_{j=1}^{k_m}\overline{r}_m(j,d_s^m)\overline{P}_m^{u(\eta,y)}(k,j)\eta_k\end{aligned}\tag{3.112}$$

for $v\in b\mathcal{B}(P(E_m)\times D_m)$.

■

Recalling now that the cost functional $J_\eta^{\beta,m}((a_n))$, corresponding to a control (a_n), where a_n is adapted to $\overline{Y}_n^m$, can be written as follows

$$J_\eta^{\beta,m}((a_n))=\sum_{n=0}^{\infty}\beta^n E_\eta^u\Big\{\sum_{k=1}^{k_m}c(k,a_n)\overline{\pi}_n^m(k)\Big\}\tag{3.113}$$

we see that the partially observed control problem of the process $\overline{x}_n^m$ with observations $\overline{y}_n^m$ and adapted controls a_n can be replaced by the completely observed control problem of the Markov process $(\overline{\pi}_n^m,\overline{y}_n^m)$ with transition operator $\overline{T}^{a_n}(\overline{\pi}_n^m,\overline{y}_n^m,\cdot)$ in the generic period n.

In particular

Corollary 3.35 *If for $u \in \mathcal{B}(P(E_m), U_H)$, the partition $V_n^u = \{V_1^n, \ldots, V_{\alpha_n}^n\}$ and the function γ_n (see (3.103)) have been constructed, if*

$$\begin{aligned} &a_0 = u_i \quad \text{for an initial law } \eta \in V_i^n \\ &a_1 = u_{\gamma_n(i,y_1)} \\ &\ldots \\ &a_p = u_{\gamma_n^p(i,y_1,\ldots,y_p)} \\ &\ldots \end{aligned} \tag{3.114}$$

where for simplicity we use y_j to denote $\overline{y}_j^m$, for $j = 1, 2, \ldots, p$, u_i stands for the value of u on V_i^n and for $p = 1, 2, \ldots,$

$$\gamma_n^p(i, y_1, \ldots, y_p) = \gamma_n(\gamma_n(\ldots \gamma_n(\gamma_n(i, y_1), y_2), \ldots), y_p)$$

then we have

$$J_\eta^{\beta,m}((a_p)) = v^{\beta,m,n}(\eta|u) \tag{3.115}$$

We still need an additional definition, namely for a given control function $u \in \mathcal{B}(P(E_m), U_H)$ let

$$\overline{v}^{\beta,m,n}(\eta|u) := \min_{i=1,2,\ldots,\alpha_n} \sum_{k=1}^{k_m} d_n(k,i)\eta_k \tag{3.116}$$

Clearly this function $\overline{v}^{\beta,m,n}$ satisfies

$$\overline{v}^{\beta,m,n}(\eta|u) \leq v^{\beta,m,n}(\eta|u) \quad \text{for } \eta \in P(E_m)$$

and represents the concave hull of the piecewise linear function $v^{\beta,m,n}(\eta|u)$.

We show below that $\overline{v}^{\beta,m,n}$ too is the value of the cost functional $J_\eta^{\beta,m}$ corresponding to a certain control for the process given by the completely observable pair $(\overline{\pi}_p^m, \overline{y}_p^m)_{p=1,2\ldots}$.

Lemma 3.36 *There is an adapted control $(\overline{a}_p)$ for the pair $(\overline{\pi}_p^m, \overline{y}_p^m)$ for which we have*

$$J_\eta^{\beta,m}((\overline{a}_p)) = \overline{v}^{\beta,m,n}(\eta|u) \tag{3.117}$$

P r o o f. For $j = 1, 2, \ldots, \alpha_n$, define the sets

$$W_j = \{\eta \in P(E_m): \overline{v}^{\beta,m,n}(\eta|u) = \sum_{k=1}^{k_m} d_n(k,j)\eta_k\}$$

and choose representative points $\overline{\eta}^j \in W_j$.

We construct now $(\overline{a}_p)$ as follows:

– if the initial law η is such that $v^{\beta,m,n}(\eta|u) = \overline{v}^{\beta,m,n}(\eta|u)$ we put $\overline{a}_p = a_p$, $p = 0, 1, 2, \ldots$, with (a_p) defined by (3.114)

– if, for the initial law η, $v^{\beta,m,n}(\eta|u) > \overline{v}^{\beta,m,n}(\eta|u)$ and $\eta \in W_j$ construct $(\overline{a}_p)$ as (a_p) with the initial law η replaced by $\overline{\eta}^j$.

By a direct calculation one can check that for the strategy $(\overline{a}_p)$ as defined above, (3.117) holds.

■

For the purpose of being now able to describe Sondik's algorithm we recall the Howard-Blackwell policy improvement procedure (see e.g. [18]).

Lemma 3.37 *Assume we are given a completely observed controlled Markov process (z_n) on a state space Z, with transition operator $P^{a_n}(z_n, \cdot)$ in the generic period n, control (a_n), and corresponding cost functional*

$$J^{\beta}_{z_0}((a_n)) = \sum_{n=0}^{\infty} \beta^n E_{z_0}\{c(z_n, a_n)\} \tag{3.118}$$

where c is a bounded cost function, and the set of admissible control parameters U is finite. Let (a_n) be a control strategy that is defined for each initial state $z = z_0 \in Z$ and takes values $(a_n(z))$. Define the corresponding value function as

$$w(z) = J^{\beta}_{z}((a_n(z))) \tag{3.119}$$

and denote by u a Borel measurable function $u: Z \to U$, for which

$$\min_{a \in U}[c(z,a) + P^a(z,w)] \tag{3.120}$$

is attained for $a = u(z)$ (policy improvement).

Then we have

$$J_z^\beta((u(z_n))) \le w(z) \tag{3.121}$$

and if $w(z) \not\equiv \inf_{(a_n)} J_z^\beta(a_n)$, *there is* $z \in Z$ *for which the strong inequality in (3.121) holds.*

■

Lemma 3.37 is now used to obtain our main result, on which Sondik's algorithm is based, and this result can be formulated as follows

Theorem 3.38 *Assume that for given* $u \in \mathcal{B}(P(E_m), U_H)$ *and positive integer* n, *the partition* $V_n^u = \{V_1^n, \ldots, V_{\alpha_n}^n\}$ *and the functions* γ_n, $v^{\beta,m,n}$ *and* $\overline{v}^{\beta,m,n}$ *have been constructed.*

Let $\hat{u} \in \mathcal{B}(P(E_m), U_H)$ *be such that*

$$\min_{a \in U_H} \Big[\sum_{k=1}^{k_m} c(k,a)\eta_k + \beta \overline{\prod}_m^a(\eta, \overline{v}^{\beta,m,n}) \Big] \tag{3.122}$$

is attained for $a = \hat{u}(\eta)$.

Then

$$\begin{aligned} J_\eta^{\beta,m}((\hat{u}(\overline{\pi}_p^m))) &\le \overline{v}^{\beta,m,n}(\eta|u) \le v^{\beta,m,n}(\eta|u) \le \\ &\le J_\eta^{\beta,m}((u(\overline{\pi}_p^m))) + \frac{\beta^n}{1-\beta^n} \cdot \frac{K}{1-\beta} \end{aligned} \tag{3.123}$$

with $K = \sup_{k=1,2,\ldots,k_m} \sup_{a,a' \in U_H} |c(k,a) - c(k,a')|$.

Moreover,

$$\begin{aligned} &\sup_{\eta \in P(E_m)} |\overline{v}^{\beta,m,n}(\eta|u) - w_H^{\beta,m}(\eta)| \le \\ &\le (1-\beta)^{-1} \sup_{\eta \in P(E_m)} |\overline{v}^{\beta,m,n}(\eta|u) - \min_{a \in U_H} \Big[\sum_{k=1}^{k_m} c(k,a)\eta_k \\ &+\beta \overline{\prod}_m^a(\eta, \overline{v}^{\beta,m,n}) \Big]| \end{aligned} \tag{3.124}$$

where $w_H^{\beta,m}$ *is defined in section 3.3.3a.*

P r o o f. By Lemma 3.36, $\overline{v}^{\beta,m,n}$ is the value function corresponding to the control $(\overline{a}_p)$. According to Lemma 3.37 applied to the pair $(\overline{\pi}_p^m, \overline{y}_p^m)$ with transition operator $\overline{T}^a(\eta, y, \cdot)$ and cost functional $J_\eta^{\beta,m}$, construct now the control function $\overline{u}\colon P(E_m) \times D_m \mapsto U_H$, for which

$$\min_{a \in U_H} \Big[\sum_{k=1}^{k_m} c(k,a)\eta_k + \beta \overline{T}^a(y, \eta, \overline{v}^{\beta,m,n}) \Big]$$

is achieved.

Since (see 3.116) $\overline{v}^{\beta,m,n}$ does not depend on y

$$\overline{T}^a(y, \eta, \overline{v}^{\beta,m,n}) = \overline{\prod}_m^a(\eta, \overline{v}^{\beta,m,n})$$

for $y \in D_m$ and, consequently, we can take $\overline{u} = \hat{u}$. By (3.121), $J_\eta^{\beta,m}((\hat{u}(\overline{\pi}_p^m))) \leq \overline{v}^{\beta,m,n}(\eta|u)$.

The second part of the inequality (3.123) follows from Proposition 3.33.

It remains to show (3.124). By (3.97) we have

$$\begin{aligned}
&\sup_{\eta \in P(E_m)} |w_H^{\beta,m}(\eta) - \overline{v}^{\beta,m,n}(\eta|u)| \leq \sup_{\eta \in P(E_m)} \Big\{ \Big| \min_{a \in U_H} \Big[\sum_{k=1}^{k_m} c(k,a)\eta_k \\
&+\beta \overline{\prod}_m^a(\eta, w_H^{\beta,m}) \Big] - \min_{a \in U_H} \Big[\sum_{k=1}^{k_m} c(k,a)\eta_k + \beta \overline{\prod}_m^a(\eta, \overline{v}^{\beta,m,n}) \Big] \Big| \\
&+\Big| \min_{a \in U_H} \Big[\sum_{k=1}^{k_m} c(k,a)\eta_k + \beta \overline{\prod}_m^a(\eta, \overline{v}^{\beta,m,n}) \Big] - \overline{v}^{\beta,m,n}(\eta|u) \Big| \Big\} \\
&\leq \sup_{\eta \in P(E_m)} |w_H^{\beta,m}(\eta) - \overline{v}^{\beta,m,n}(\eta|u)| + \\
&+ \sup_{\eta \in P(E_m)} | \min_{a \in U_H} \Big[\sum_{k=1}^{k_m} c(k,a)\eta_k + \beta \overline{\prod}_m^a(\eta, \overline{v}^{\beta,m,n}) \Big] - \overline{v}^{\beta,m,n}(\eta|u)|
\end{aligned}$$

from which we obtain (3.124).

■

The algorithm

Sondik's algorithm, whose convergence is guaranteed by the results of this subsection, can now be synthesized in the following steps:

Step 1: Choose two levels of accuracy, a level $\varepsilon > 0$, and a much smaller level $\varepsilon^* > 0$ that will be the actually desired one, as well as an initial control function $u \in \mathcal{B}(P(E_m), U_H)$

Step 2: Choose an integer $n \geq 1$

Step 3: Corresponding to the chosen u and n, determine the function $\overline{v}^{\beta,m,n}(\eta|u)$

Step 4: Perform the policy improvement in (3.122) to obtain a new control function $\hat{u}$ and determine $\hat{\varepsilon}$ as the value of the right hand side of (3.124)

Step 5: If $\hat{\varepsilon} > \varepsilon$, increase the originally chosen value of n to n' and return to step 3; otherwise continue to

Step 6: If $\hat{\varepsilon} > \varepsilon^*$, replace u by $\hat{u}$ and ε by $\hat{\varepsilon}$ and return to step 2; otherwise stop.

Once the algorithm is stopped, the first inequality in (3.123) together with the inequality (3.124) imply that the control function $\hat{u}$, determined in step 4, when applied to the filter $(\overline{\pi}_n^m)$, is $\hat{\varepsilon}$-optimal for the control problem described at the beginning of this subsection which, we recall, has $J_\mu^{\beta,m}$ as its cost functional with minimal value $w_H^{\beta,m}$ over the controls in U_H.

Furthermore, recalling the Bellman equation (3.97), by (3.122) and (3.124) we may now write

$$\begin{aligned} & w_H^{\beta,m}(\eta) + \beta\hat{\varepsilon} \geq \\ & \geq \sum_{k=1}^{k_m} c(k, \hat{u}(\eta))\eta_k + \beta \overline{\prod}_m^{\hat{u}(\eta)}(\eta, \overline{v}^{\beta,m,n}) \geq \\ & \geq \sum_{k=1}^{k_m} c(k, \hat{u}(\eta))\eta_k + \beta \overline{\prod}_m^{\hat{u}(\eta)}(\eta, w_H^{\beta,m}) - \beta\hat{\varepsilon} \end{aligned}$$

which shows that the control function $\hat{u}$ obtained from Sondik's algorithm satisfies relation (3.84) of Theorem 3.27 for $\varepsilon = 2\beta\hat{\varepsilon}$. Combining this finally with Corollary 3.19 we have that, for H and m large enough that (3.83) and (3.67) hold, the control with generic term

$$a_n = \hat{u}(\pi_n(B_1^m), \ldots, \pi_n(B_{k_m}^m))$$

is, with $\varepsilon = 2\beta\hat{\varepsilon}$, $\frac{5\varepsilon}{1-\beta}$-optimal for the original cost function J_μ^β.

3.3.3.b Continuous control functions

We now restrict ourselves to the controls of the form $a_n = u(\overline{\pi}_n^m)$, where $u \in \mathcal{A}_m(L,n)$ and $\overline{\pi}_n^m$ is the filtering process corresponding to $(\overline{x}_n^m)$, defined in (3.57).

As already pointed out in subsection 3.3.3.a although $\overline{x}_n^m$ and $\overline{y}_n^m$ take a finite number of values, $(\overline{\pi}_n^m)$ still takes its values in the infinite space $P(E_m)$. Of the two possibilities described in 3.3.3.a for the actual construction of a nearly optimal control function, here we consider only the analog of the first one which is based on the discretization of the space $P(E_m)$.

For a given partition $(G_k^q)_{k=1,2,\ldots,k_q}$ of $P(E_m)$ with representative elements $\{e_1^q, \ldots, e_{k_q}^q\}$ we thus define a projection operator $\hat{Q}_q$ as

$$\hat{Q}_q \colon P(E_m) \ni \eta \mapsto e^q_{Q_q \eta} \tag{3.125}$$

where Q_q is as in (3.90). Define furthermore a transition operator $\hat{\prod}$ on $\{e_1^q, \ldots, e_{k_q}^q\}$ as

$$\hat{\prod}_m^{u(e_k^q)}(e_k^q, e_p^q) = \overline{\prod}_m^{u(e_k^q)}(e_k^q, G_p^q) \tag{3.126}$$

with $u \in \mathcal{A}_m(L,n)$.

Denote by $(\hat{\pi}_n)$ a Markov process on $\{e_1^q, \ldots, e_{k_q}^q\}$ with transition matrix $\hat{\prod}_m^{u(e_k^q)}(e_k^q, e_p^q)$, and let the corresponding cost functional $\hat{J}^{\beta,q}_{e_p^q}(u)$ be given by

$$\hat{J}^{\beta,q}_{e_p^q}(u) = \sum_{n=0}^{\infty} \beta^n E^u_{e_p^q} \Big\{ \sum_{k=1}^{k_m} c(k, u(\hat{\pi}_n)) \hat{\pi}_n(k) \Big\} \tag{3.127}$$

for $u \in \mathcal{A}_m(L,n)$, where by $\hat{\pi}_n(k)$ we denote the k-th coordinate of $\hat{\pi}_n$ in $P(E_m)$.

We have

Theorem 3.39 *Under (A2), (A5) and (B9) we have for given m*

$$\sup_{u \in \mathcal{A}_m(L,n)} \sup_{\eta \in P(E_m)} |J^{\beta,m}_\eta(u) - \hat{J}^{\beta,q}_{\hat{Q}_q \eta}(u)| \to 0 \tag{3.128}$$

as $q \to \infty$.

P r o o f. For a given $u \in \mathcal{A}_m(L,n)$, consider a process $(\breve{\pi}_n)$ on $P(E_m)$ with transition operator

$$\check{\prod}_m^{u(\eta)}(\eta,\cdot) = \overline{\prod}_m^{u(\hat{Q}_q\eta)}(\hat{Q}_q\eta,\cdot)$$

and corresponding cost functional $\check{J}_\eta^{\beta,q}(u)$ defined as follows

$$\check{J}_\eta^{\beta,q}(u) = \sum_{n=0}^{\infty} \beta^n E_\eta^u \Big\{ \sum_{k=1}^{k_m} c(k, u(\hat{Q}_q\breve{\pi}_n^u))\hat{Q}_q\breve{\pi}_n^u(k) \Big\}$$

Letting

$$C_q^u(\eta) = \sum_{k=1}^{k_m} c(k, u(\hat{Q}_q\eta))\hat{Q}_q\eta$$

we have

$$\check{J}_\eta^{\beta,q}(u) = C_q^u(\eta) + \sum_{n=1}^{\infty} \beta^n (\check{\prod}_m^u)^n C_q^u(\eta) \tag{3.129}$$

and clearly

$$\hat{J}_{\hat{Q}_q\eta}^{\beta,q}(u) = \check{J}_\eta^{\beta,q}(u) \tag{3.130}$$

Assume now, for some sequences $u_q \in \mathcal{A}_m(L,n)$, $\eta_q \in P(E_m)$ $q = 1,2,\ldots$, we have

$$|J_{\eta_q}^{\beta,m}(u_q) - \check{J}_{\eta_q}^{\beta,q}(u_q)| > \delta \tag{3.131}$$

for $q = 1,2,\ldots$.

By the compactness of $\mathcal{A}_m(L,n)$ and $P(E_m)$ we may assume that $u_q \to u$ and $\eta_q \to \eta$ as $q \to \infty$.

However by Lemma 3.25 (iii) if $b\mathcal{B}(P(E_m)) \ni F_q \to F \in C(P(E_m))$, uniformly as $q \to \infty$, we have

$$\begin{aligned} \overline{\prod}_m^{u_q(\eta)}(\eta, F_q) &\to \overline{\prod}_m^{u(\eta)}(\eta, F) \\ \check{\prod}_m^{u_q(\eta)}(\eta, F_q) &\to \overline{\prod}_m^{u(\eta)}(\eta, F) \end{aligned} \tag{3.132}$$

as $q \to \infty$, uniformly in $\eta \in P(E_m)$.

Since by the continuity of $c(k,a)$ with respect to $a \in U$, $C_q^u(\eta) \to C^u(\eta) = \sum_{k=1}^{k_m} c(k, u(\eta))\eta_k$, uniformly in $\eta \in P(E_m)$ and $u \in \mathcal{A}_m(L,n)$, and

$$J_\eta^{\beta,m}(u) = C^u(\eta) + \sum_{n=1}^{\infty} \beta^n (\overline{\prod}_m^u)^n C^u(\eta)$$

by (3.129), (3.132) we obtain a contradiction to (3.131). Consequently

$$\sup_{u \in \mathcal{A}_m(L,n)} \sup_{\eta \in P(E_m)} |J_\eta^{\beta,m}(u) - \check{J}_\eta^{\beta,q}(u)| \to 0 \quad \text{as } q \to \infty$$

and in view of (3.130) we have (3.128).

■

By Theorem 3.39 we immediately have

Corollary 3.40 *Under the assumptions of Theorem 3.39 if* $u \in \mathcal{A}_m(L,n)$ *is an* ε*-optimal control function for the cost functional* $\hat{J}^{\beta,q}_{\hat{Q}_q\eta}$, $\eta \in P(E_m)$, *and* q *is so large that*

$$\sup_{u \in \mathcal{A}_m(L,n)} \sup_{\eta \in P(E_m)} |J_\eta^{\beta,m}(u) - \hat{J}^{\beta,q}_{\hat{Q}_q\eta}(u)| < \varepsilon \tag{3.133}$$

then u *is a* 3ε*-optimal control function for the cost functional* $J_\eta^{\beta,m}$.

We have now reduced the problem of the construction of a nearly optimal control function for $J_\eta^{\beta,m}$ in the class $\mathcal{A}_m(L,n)$ to that for $\hat{J}^{\beta,q}_{\hat{Q}_q\eta}$ with q sufficiently large.

Notice now that the controls for $\hat{J}^{\beta,q}_{\hat{Q}_q\eta}$ are obtained by applying a control function $u \in \mathcal{A}_m(L,n)$ to the completely observed Markov process $(\hat{\pi}_n)$, restricted to take values in the finite set $\{e_1^q, \ldots, e_{k_q}^q\}$. A crucial consequence of this is that, for a given $u \in \mathcal{A}_m(L,n)$, we need only a finite number of its values, i.e. $\overline{u}(\zeta_i)$, $i = 1, \ldots, k_q$, where

$$\zeta_i = \Big(\sum_{k=1}^{k_m} \varphi_1(b_k^m) e_i^q(k), \ldots, \sum_{k=1}^{k_m} \varphi_n(b_k^m) e_i^q(k) \Big) \tag{3.134}$$

with $\overline{u}\colon [-\|\varphi_1\|, \|\varphi_1\|] \times \ldots \times [-\|\varphi_n\|, \|\varphi_n\|] \to U$ the Lipschitz function corresponding to u in the definition of the class $\mathcal{A}_m(L,n)$, and b_k^m the selectors in (3.47).

It therefore suffices to consider the values $a^1, \ldots, a^{k_q} \in U$ corresponding to the various control functions in $\mathcal{A}_m(L,n)$, when the process $(\hat{\pi}_n)$ is in the states $e_1^q, \ldots, e_{k_q}^q$ respectively, with the restriction that $a^1, \ldots, a^{k_q}$ satisfy the following Lipschitz condition

$$\rho_U(a^j, a^l) \le L \max_{i=1,\ldots,n} \Big| \sum_{k=1}^{k_m} \varphi_i(b_k^m)(e_j^q(k) - e_l^q(k)) \Big| \tag{3.135}$$

for $1 \le j,\, l \le k_q$. Condition (3.135) implies in fact that, given the values of a Lipschitz function $\overline{u}$ with constant L at the points $\zeta_1, \ldots, \zeta_{k_q}$. i.e. given $a^i = \overline{u}(\zeta_i)$, $1 \le i \le k_q$, by linear interpolation we obtain again a Lipschitz function $\overline{\overline{u}}$ with the same constant L, that is defined on $[-\|\varphi_1\|, \|\varphi_1\|] \times \ldots \times [-\|\varphi_n\|, \|\varphi_n\|]$ and that takes the same values at the points ζ_i. This procedure gives us the possibility to construct control functions in $\mathcal{A}_m(L, n)$ from their values at the points e^q_j. In what follows we shall denote by $U^q(L)$ the set of vectors $a = (a^1, \ldots, a^{k_q}) \in U^{k_q}$ that satisfy condition (3.135). As a consequence of the foregoing, in what follows we shall also use $\hat{J}^{\beta,q}_{e^q_p}(a)$, $a \in U^q(L)$, to denote the value of the cost functional (3.127) that corresponds to a control function in $\mathcal{A}_m(L, n)$ which at e^q_j has value a^j for $j = 1, \ldots, k_q$.

We immediately have

Lemma 3.41

$$\inf_{u \in \mathcal{A}_m(L,n)} \hat{J}^{\beta,q}_{e^q_p}(u) = \inf_{a \in U^q(L)} \hat{J}^{\beta,q}_{e^q_p}(a)$$

The admissible control values $a^1, \ldots, a^{k_q}$ now belong to U, which is still infinite. For actual computation we therefore again introduce a partition $(U^H_k)_{k=1,\ldots,H}$ of U and choose representative elements forming a set $U_H = \{\alpha^H_1, \ldots, \alpha^H_H\}$. In the same way as described at the beginning of section 3.3.3.a assume that the partition $(G^q_k)_{k=1,\ldots,k_q}$ of $P(E_m)$ satisfies

(B10) the mapping $U \ni a \to \overline{\prod}^a_m(e^q_k, G^q_p)$ is continuous for $1 \le k,\, p \le k_q$.

Let $U^q_H(L)$ denote the subset of $U^q(L)$ consisting of vectors $a \in U^{k_q}_H$.

For easier reference we state as lemma the following fact

Lemma 3.42 *Given k_q, L and H, there exists an operator $\hat{Z}_H \colon U^{k_q} \to U^{k_q}_H$ such that*

(i) $\hat{Z}_H(a) \in U^q_H(L)$ *for* $a \in U^q(L)$

Moreover

(ii) $\hat{Z}_H(a) \to a$ *uniformly in* $a \in U^{k_q}$ *as* $H \to \infty$.

From the Lemma we obtain

Corollary 3.43 *Under (B10) there exists H_0 such that, for $H > H_0$,*

$$\sup_{a \in U^q(L)} |\hat{J}^{\beta,q}_{e^q_p}(a) - \hat{J}^{\beta,q}_{e^q_p}(\hat{Z}_H a)| < \varepsilon \tag{3.136}$$

To find a nearly optimal control function in $\mathcal{A}_m(L,n)$, by Corollary 3.43, Lemma 3.41 and the discussion preceding it, we are now left with the problem of finding an ε-optimal vector in $U^q_H(L)$, namely $a^* \in U^q_H(L)$ such that

$$\hat{J}^{\beta,q}_{e^q_p}(a^*) \leq \inf_{a \in U^q_H(L)} \hat{J}^{\beta,q}_{e^q_p}(a) + \varepsilon \tag{3.137}$$

In this way the problem has been reduced to a finite search problem that can be approached by adapting any of the existing methods of global optimization with constraints (see e.g. [33], [41]).

Given an ε-optimal $a^* \in U^q_H(L)$, let $u^* \in \mathcal{A}_m(L,n)$ be the function obtained by linear interpolation from a^* as mentioned below (3.135).

We have

Theorem 3.44 *If H is sufficiently large that (3.136) holds, then*

$$\hat{J}^{\beta,q}_{e^q_p}(u^*) \leq \inf_{u \in \mathcal{A}_m(L,n)} \hat{J}^{\beta,q}_{e^q_p}(u) + 2\varepsilon \tag{3.138}$$

P r o o f. By (3.136) and (3.137) we have

$$\hat{J}^{\beta,q}_{e^q_p}(a^*) \leq \inf_{a \in U^q(L)} \hat{J}^{\beta,q}_{e^q_p}(a) + 2\varepsilon \tag{3.139}$$

By the construction of u^*, Lemma 3.41 and the discussion preceding it, (3.139) is equivalent to the statement of the Theorem.

■

Remark 3.45 *Combining Theorem 3.44 with Corollary 3.40 we have that, if q is so large that (3.133) holds, the control $u^* \in \mathcal{A}_m(L,n)$ is 4ε-optimal for $J^{\beta,m}_\eta$. Combining this in turn with Corollary 3.23, if m is such that (3.81) holds, the extension of the control according to (3.73) is 5ε-optimal for J^β_μ over $\mathcal{A}(L,n)$. Finally, by Corollary 3.14 this extension will be 6ε-optimal for J^β_μ over $\mathcal{A}$ if also L and n are taken sufficiently large.*

3.4 Approximating operators separated in the variables

In this section we present an alternative general approximation approach, that can be used instead of the specific approximations studied in 3.3.2 and 3.3.3 and that is applicable only for the case when the state space E is compact.

Namely, following section 2.3.2 we assume that the approximating transition operators $P_m^a(x, dz)$ are of the form (2.28) i.e.

$$P_m^a(x, dz) = \sum_{i=1}^{m} \varphi_i^m(x)\gamma_i^m(a, dz) \tag{3.140}$$

with $\varphi_i^m \in b\mathcal{B}(E)$, $\varphi_i^m \geq 0$, and $\gamma_i^m(a, dz)$ being for $a \in U$ finite measures on E, such that for $B \in \mathcal{B}(E)$, the mappings $U \ni a \to \gamma_i^m(a, B)$ are Borel measurable and

$$\sum_{i=1}^{m} \varphi_i^m(x)\gamma_i^m(a, E) = 1 \quad \text{for} \quad x \in E,\ a \in U$$

As mentioned in 2.3.2 this form includes in particular the case (3.50) studied in 3.3.2.

For a Markov process (x_n^m) with transition operator $P_m^{a_n}(x_n^m, dz)$ as in (3.140), observations (y_n^m), $y_n^m \in R^d$, satisfying

$$P\{y_{n+1}^m \in A | x_0^m, x_1^m, \ldots, x_{n+1}^m, Y_m^n\} = \int_A r(x_{n+1}^m, y)\, dy \tag{3.141}$$

for $n = 0, 1, 2, \ldots$, $A \in \mathcal{B}(R^d)$, $Y_m^n = \sigma\{y_1^m, \ldots, y_n^m\}$, $Y_m^0 = \{\emptyset, \Omega\}$, taking controls $u = (a_n)$, with a_n adapted to Y_m^n for $n = 0, 1, 2, \ldots$, we obtain analogously to (3.14) the following representation formula for the associated filtering process $(\pi_n^{m,u})$:

$$\pi_0^{m,u}(A) \quad = \mu(A), \text{ where } \mu \text{ is the initial law of } (x_n^m)$$

$$\pi_{n+1}^{m,u}(A) \quad = \frac{\sum_{i=1}^{m} \int_A r(z, y_{n+1}^m)\gamma_i^m(a_n, dz)\pi_n^{m,u}(\varphi_i^m)}{\sum_{i=1}^{m} \int_E r(z, y_{n+1}^m)\gamma_i^m(a_n, dz)\pi_n^{m,u}(\varphi_i^m)} \tag{3.142}$$

for $A \in \mathcal{B}(E)$, $n = 0, 1, 2, \ldots$.

Then the cost functional $J_\mu^{\beta,m}$ of (3.12), where (see section 2.3.2) we may take $c_m = c$, can be rewritten in terms of the filtering process $(\pi_n^{m,u})$ as follows

$$\begin{aligned}
J_\mu^{\beta,m}(u) \quad &= \sum_{n=0}^{\infty} \beta^n E_\mu^u \Big\{ \int_E c(x, a_n)\pi_n^{m,u}(dx) \Big\} = \\
&= \int_E c(x, a_0)\mu(dx) + \sum_{n=1}^{\infty} \beta^n E_\mu^u \Big\{ \sum_{i=1}^{m} \int_E c(z, a_n) r(z, y_n^m) \\
&\quad \gamma_i^m(a_{n-1}, dz)\pi_{n-1}^{m,u}(\varphi_i^m)\Big(\sum_{j=1}^{m} \int_E r(z, y_n^m)\gamma_j^m(a_{n-1}, dz)\pi_{n-1}^{m,u}(\varphi_j^m) \Big)^{-1} \Big\} \\
&:= \int_E c(x, a_0)\mu(dx) + \sum_{n=1}^{\infty} \beta^n E_\mu^u \{ C_m(a_n, y_n^m, a_{n-1}, \pi_{n-1}^{m,u}(\varphi_1^m), \ldots, \\
&\quad \pi_{n-1}^{m,u}(\varphi_m^m)) \}
\end{aligned} \tag{3.143}$$

where we implicitly defined the function C_m.

Notice that in the above cost functional the filtering process $(\pi_n^{m,u})$ appears only through the values $(\pi_n^{m,u}(\varphi_i^m))$ with $i = 1, 2, \ldots, m$; furthermore by (3.142) the statistics $\pi_n^{m,u}(\varphi_i^m)$ can be calculated in a recursive way, namely

$$\begin{aligned}\pi_{n+1}^{m,u}(\varphi_i^m) \quad &= \sum_{j=1}^{m} \int\limits_E \varphi_i^m(z) r(z, y_{n+1}^m) \gamma_j^m(a_n, dz) \pi_n^{m,u}(\varphi_j^m) \\ &\quad \Big(\sum_{k=1}^{m} \int\limits_E r(z, y_{n+1}^m) \gamma_k^m(a_n, dz) \pi_n^{m,u}(\varphi_k^m) \Big)^{-1} \\ &:= M_m^{a_n}(y_{n+1}^m, \pi_n^{m,u}(\varphi_1^m), \ldots, \pi_n^{m,u}(\varphi_m^m))(i)\end{aligned} \tag{3.144}$$

Notice moreover that the function $M_m^{a_n}$ is the same as in (3.14) except that we take r_m identically equal to r.

As in the previous sections of this chapter 3, the purpose is to determine a nearly optimal control function which, when applied to the true observations and filter values, yields nearly optimal controls.

To this effect, in the present case of approximating operators separated in the variables, we cannot apply directly the convergence results of section 3.3.1.a.

Therefore in subsection 3.4.1 we introduce a generalized Bellman equation for the original problem, which is used in subsection 3.4.2 to show that ε-optimal control functions for the approximating problem are nearly optimal for the original problem.

3.4.1 Generalized Bellman equation

Notice first that for the original cost functional $J_\mu^\beta(u)$ we have

$$\begin{aligned}&J_\mu^\beta(u) = \int\limits_E c(x, a_0)\mu(dx) + \sum_{n=1}^{\infty} \beta^n E_\mu^u \Big\{ \int\limits_E c(x, a_n) \\ &M^{a_{n-1}}(y_n, \pi_{n-1}^u)(dx) \Big\} = \int\limits_E c(x, a_0)\mu(dx) + \\ &+ \sum_{n=1}^{\infty} \beta^n E_\mu^u \{ C(a_n, y_n, a_{n-1}, \pi_{n-1}^u) \}\end{aligned} \tag{3.145}$$

with $M^a(y, \pi)(\cdot)$ as in (1.8) and where the function C is defined implicitly.

Theorem 3.46 *Under (A1)–(A5) there exists a unique continuous bounded function w^β, $w^\beta\colon R^d \times P(E) \times U \mapsto R$, that satisfies the following equation*

$$w^\beta(y,\mu,a) = \inf_{a_1 \in U}[C(a_1,y,a,\mu) + \beta \int\limits_{R^d} w^\beta(\zeta, M^a(y,\mu), a_1) \\ \int\limits_E \int\limits_E r(z,\zeta) P^{a_1}(z_1,dz) M^a(y,\mu)(dz_1) d\zeta] \tag{3.146}$$

Moreover, w^β has the following interpretation

$$w^\beta(y,\mu,a) = \inf_{(a_n), a_0=a} E_\mu \Big\{ \sum_{n=1}^{\infty} \beta^{n-1} C(a_n, y_n, a_{n-1}, \pi_{n-1}) | y_1 = y \Big\} \quad P_\mu \text{a.e.} \tag{3.147}$$

where we set $a_0 = a$ and the infimum is taken over all sequences (a_n) that are adapted to $\sigma\{y_1, \ldots, y_n\}$.

Furthermore, there exists a Borel measurable function $u^\beta\colon R^d \times P(E) \times U \mapsto U$ such that

$$w^\beta(y,\mu,a) = C(u^\beta(y,\mu,a), y, a, \mu) \\ +\beta \int\limits_{R^d} w^\beta(\zeta, M^a(y,\mu), u^\beta(y,\mu,a)) \\ \int\limits_E \int\limits_E r(z,\zeta) P^{u^\beta(y,\mu,a)}(z_1,dz) M^a(y,\mu)(dz_1) d\zeta \tag{3.148}$$

and in addition

$$w^\beta(y,\mu,a) = E_\mu \Big\{ \sum_{n=1}^{\infty} \beta^{n-1} C(\hat{a}_n, y_n, \hat{a}_{n-1}, \pi_{n-1}) | y_1 = y \Big\} \tag{3.149}$$

with $\hat{a}_n = u^\beta(y_n, \pi_{n-1}, \hat{a}_{n-1})$, $\hat{a}_0 = a$.

P r o o f. By Proposition 1.4 the mapping

$$T\colon C(R^d \times P(E) \times U) \ni w \mapsto Tw(y,\mu,a) = \\ = \inf_{a_1 \in U}[C(a_1,y,a,\mu) + \beta \int\limits_{R^d} w(\zeta, M^a(y,\mu), a_1) \int\limits_E \int\limits_E r(z,\zeta) \\ P^{a_1}(z_1,dz) M^a(y,\mu)(dz_1) d\zeta] \tag{3.150}$$

is a contraction in $C(R^d \times P(E) \times U)$, and therefore by the Banach contraction principle there exists a unique function $w^\beta \in C(R^d \times P(E) \times U)$ for which (3.146) is satisfied.

Since the right hand side of (3.146) under the infimum sign is a continuous function with respect to $a_1 \in U$, by using any of the existing measurable selection theorems there exists a Borel measurable function u^β for which the infimum is attained i.e. (3.148) holds.

It remains to show the representation formulae (3.147) and (3.149).

Let (y_n) be the original observation process in (1.1) and $u = (a_n)$ a control, a_n adapted to Y^n.

From (3.146) we have for all a_{n+1}

$$\begin{aligned}
&w^\beta(y_{n+1}, \pi_n^u, a_n) \leq C(a_{n+1}, y_{n+1}, a_n, \pi_n^u) + \\
&+\beta \int_{R^d} w^\beta(\zeta, M^{a_n}(y_{n+1}, \pi_n^u), a_{n+1}) \int_E \int_E r(z,\zeta) P^{a_{n+1}}(z_1, dz) \\
&M^{a_n}(y_{n+1}, \pi_n^u)(dz_1) d\zeta
\end{aligned} \tag{3.151}$$

Multiplying both sides of (3.151) by β^n, summing the above inequalities for $n = 0, 1, \ldots, k-1$ and taking conditional expectation given $y_1 = y$ we obtain

$$\begin{aligned}
&E_\mu^u \Big\{ \sum_{n=0}^{k-1} \beta^n C(a_{n+1}, y_{n+1}, a_n, \pi_n^u) | y_1 = y \Big\} \geq \\
&\geq E_\mu^u \Big\{ \sum_{n=0}^{k-1} \beta^{n+1} \Big[-\int_{R^d} w^\beta(\zeta, M^{a_n}(y_{n+1}, \pi_n^u), a_{n+1}) \\
&\int_E \int_E r(z,\zeta) P^{a_{n+1}}(z_1, dz) M^{a_n}(y_{n+1}, \pi_n^u)(dz_1) d\zeta \\
&+ w^\beta(y_{n+2}, \pi_{n+1}^u, a_{n+1}) \Big] | y_1 = y \Big\} \\
&- E_\mu^u \{ \beta^k w^\beta(y_{k+1}, \pi_k^u, a_k) | y_1 = y \} + w^\beta(y, \mu, a_0) \\
&= -\beta^k E_\mu^u \{ w^\beta(y_{k+1}, \pi_k^u, a_k) | y_1 = y \} + w^\beta(y, \mu, a_0)
\end{aligned} \tag{3.152}$$

with equality for $a_n = \hat{a}_n$.

Letting $k \to \infty$ in (3.152) we have

$$w^\beta(y,\mu,a_0) \le \sum_{n=0}^{k-1} \beta^n E^u_\mu \{C(a_{n+1}, y_{n+1}, a_n, \pi^u_n) | y_1 = y\}$$

with equality for $a_n = \hat{a}_n$, from which (3.147) and (3.149) follow.

■

Corollary 3.47 *For $v^\beta(\mu)$ as defined in (3.1) we have, under the assumptions of Theorem 3.46,*

$$v^\beta(\mu) = \inf_a \Big[\int_E c(x,a)\mu(dx) + \beta \int_{R^d} w^\beta(y,\mu,a) \int_E r(z,y) P^a(\mu, dz) dy \Big] \quad (3.153)$$

Moreover, assuming that E is compact, for given $\varepsilon > 0$ one can find a compact set $L \subset R^d$ and functions $u_0 \in \mathcal{B}(P(E), U)$, $u \in \mathcal{B}(R^d \times P(E) \times U, U)$ satisfying the following inequalities

$$\sup_{x \in E} R(x, L^c) \le \frac{\varepsilon}{\|c\|} \quad (3.154)$$

(with R defined in (A4) and, see (A5), $\|c\| = \sup_{x\in E} \sup_{a \in U} c(x,a)$),

$$\begin{aligned} &\int_E c(x, u_0(\mu))\mu(dx) \\ &+\beta \int_{R^d} w^\beta(y,\mu,u_0(\mu)) \int_E r(z,y) P^{u_0(\mu)}(\mu, dz) dy \\ &\le v^\beta(\mu) + \varepsilon \quad \text{for } \mu \in P(E) \end{aligned} \quad (3.155)$$

$$\begin{aligned} &C(u(y,\mu,a), y, a, \mu) + \beta \int_{R^d} w^\beta(\zeta, M^a(y,\mu), u(y,\mu,a)) \\ &\int_E \int_E r(z,\zeta) P^{u(y,\mu,a)}(z_1, dz) M^a(y,\mu)(dz_1) d\zeta \le \\ &\le w^\beta(y,\mu,a) + \varepsilon \quad \text{for } y \in L,\ \mu \in P(E),\ a \in U \end{aligned} \quad (3.156)$$

The control (a_n^*)*, defined as*

$$\begin{aligned} & a_0^* = u_0(\mu) \\ & \dots \\ & a_{n+1}^* = u(y_{n+1}, \pi_n^u, a_n^*) \\ & \dots \end{aligned} \tag{3.157}$$

is $4\varepsilon(1-\beta)^{-1}$ *optimal for the cost functional* $J_\mu^\beta(u)$.

In addition, if for $u_0^\beta \in \mathcal{B}(P(E), U)$ *the infinimum in (3.153) is attained and for* $u^\beta \in \mathcal{B}(R^d \times P(E) \times U, U)$*, (3.148) holds, then the control*

$$\begin{aligned} & \hat{a}_0^* = u_0^\beta(\mu) \\ & \dots \\ & \hat{a}_{n+1}^* = u^\beta(y_{n+1}, \pi_n, \hat{a}_n) \\ & \dots \end{aligned} \tag{3.158}$$

is optimal.

P r o o f. Notice first that the existence of L, u_0 and u satisfying (3.154)-(3.156) is immediate. Using the representation formulae (3.145) and (3.147) we have for any $u = (a_n)$

$$\begin{aligned} J_\mu^\beta(u) &= \int\limits_E c(x, a_0)\mu(dx) + E_\mu^u\Big\{E_\mu^u\Big\{\sum_{n=1}^{\infty} \beta^n C(a_n, y_n, a_{n-1}, \pi_{n-1})|y_1\Big\}\Big\} \geq \\ &\geq \int\limits_E c(x, a_0)\mu(dx) + \beta E_\mu^u\{w^\beta(y_1, \mu, a_0)\} = \\ &= \int\limits_E c(x, a_0)\mu(dx) + \beta \int\limits_{R^d} w^\beta(y, \mu, a_0) \int\limits_E r(z, y) P^{a_0}(\mu, dz) dy \end{aligned}$$

with (see (3.149)) equality for $u = (\hat{a}_n)$, where $\hat{a}_n$ is given by (3.158), and the existence of u_0^β follows from the continuity in a of the right hand side of (3.153). Therefore we have (3.153) and the optimality of the control $(\hat{a}_n)$.

It remains to show the near optimality of the control (a_n^*).

Notice first that by the inequality (3.154), we have for $n = 1, 2, \ldots$,

$$P_\mu\{y_n \notin L\} = E_\mu\{P_\mu\{y_n \notin L | x_n, Y^{n-1}\}\}$$
$$= E_\mu\Big\{\int_{L^c} r(x_n, y)dy\Big\} \le \frac{\varepsilon}{\|c\|}$$

Therefore by the inequalities (3.155) and (3.156) we obtain

$$J_\mu^\beta((a_n^*)) \le \int_E c(x, u_0(\mu))\mu(dx) + E_\mu^u\Big\{E_\mu^u\Big\{\sum_{n=1}^\infty \beta^n$$
$$C(a_n^*, y_n, a_{n-1}^*, \pi_{n-1})\chi_L(y_n)|y_1\Big\}\Big\} + \varepsilon \cdot \frac{\beta}{1-\beta}$$
$$\le \int_E c(x, u_0(\mu))\mu(dx) + E_\mu^u\Big\{E_\mu^u\Big\{\sum_{n=1}^\infty \beta^n (w^\beta(y_n, \pi_{n-1}, a_{n-1}^*)$$
$$-\beta \int_{R^d} w^\beta(\zeta, M^{a_{n-1}^*}(y_n, \pi_{n-1}), u(y_n, \pi_{n-1}, a_{n-1}^*))$$
$$\int_E\int_E r(z, \zeta) P^{u(y_n, \pi_{n-1}, a_{n-1}^*)}(z_1, dz) M^{a_{n-1}^*}(y_n, \pi_{n-1})(dz_1)d\zeta$$
$$\chi_L(y_n)|y_1\Big\}\Big\} + 2\varepsilon\frac{\beta}{1-\beta} =$$
$$= \int_E c(x, u_0(\mu))\mu(dx) + E_\mu^u\Big\{E_\mu^u\Big\{\sum_{n=2}^\infty \beta^n w^\beta(y_n, \pi_{n-1}, a_{n-1}^*)$$
$$(\chi_L(y_n) - \chi_L(y_{n-1}))|y_1\Big\}\Big\} + \beta E_\mu^u\{\chi_L(y_1) w^\beta(y_1, \pi_0, a_0^*)\}$$
$$+2\varepsilon\frac{\beta}{1-\beta} \le \int_E c(x, u_0(\mu))\mu(dx) + \|c\| E_\mu^u\Big\{\sum_{n=2}^\infty \beta^n \chi_L(y_n)$$
$$\chi_{L^c}(y_{n-1})\Big\} + \beta E_\mu^u\{w^\beta(y_1, \mu, a_0^*)\} + 2\varepsilon\frac{\beta}{1-\beta}$$
$$\le v^\beta(\mu) + \varepsilon + \varepsilon\frac{\beta^2}{1-\beta} + 2\varepsilon\frac{\beta}{1-\beta} \le v^\beta(\mu) + \frac{4\varepsilon}{1-\beta}$$

from which the near optimality of (a_n^*) follows.

■

3.4.2 Convergence of approximations

In what follows we shall assume that the state space E is compact and consider the setting as described in the beginning of this section 3.4. The purpose is to determine a nearly optimal control function for J_μ^β.

We start with the following auxiliary result

Proposition 3.48 *Assume E is compact, (A1)–(A4) hold, the sequence of transition kernels P_m^a satisfies (D1) and a sequence $w_m \in b\mathcal{B}(R^d \times P(E) \times U)$ is uniformly bounded and converges uniformly on compact subsets of $R^d \times P(E) \times U$ to $w \in C(R^d \times P(E) \times U)$.*

Then for any compact set $L \subset R^d$

$$\begin{aligned}&\int_{R^d} w_m(\zeta, M_m^a(y,\mu), a_1) \int_E \int_E r(z,\zeta) P_m^{a_1}(z_1, dz) M_m^a(y,\mu)(dz_1) d\zeta \to \\ &\int_{R^d} w(\zeta, M^a(y,\mu), a_1) \int_E \int_E r(z,\zeta) P^{a_1}(z_1, dz) M^a(y,\mu)(dz_1) d\zeta\end{aligned} \tag{3.159}$$

as $m \to \infty$, uniformly in $(y, \mu, a, a_1) \in L \times P(E) \times U \times U$.

P r o o f. Since w_m are uniformly bounded, under (A4) and using the compactness of E it suffices to show (3.159) replacing integration over R^d by integration over any compact set $L_1 \subset R^d$, namely

$$\begin{aligned}&\int_{L_1} w_m(\zeta, M_m^a(y,\mu), a_1) \int_E \int_E r(z,\zeta) P_m^{a_1}(z_1, dz) M_m^a(y,\mu)(dz_1) d\zeta \to \\ &\int_{L_1} w(\zeta, M^a(y,\mu), a_1) \int_E \int_E r(z,\zeta) P^{a_1}(z_1, dz) M^a(y,\mu)(dz_1) d\zeta\end{aligned} \tag{3.160}$$

as $m \to \infty$, uniformly in $(y, \mu, a, a_1) \in L \times P(E) \times U \times U$.

To show (3.160) it is sufficient to prove that

$$w_m(\zeta, M_m^a(y,\mu), a_1) \to w(\zeta, M^a(y,\mu), a_1) \tag{3.161}$$

and

$$\int_E \int_E r(z,\zeta) P_m^{a_1}(z_1, dz) M_m^a(y,\mu)(dz_1) \to \tag{3.162}$$
$$\int_E \int_E r(z,\zeta) P^{a_1}(z_1, dz) M^a(y,\mu)(dz_1)$$

as $m \to \infty$, uniformly in $(\zeta, y, \mu, a, a_1) \in L_1 \times L \times P(E) \times U \times U$.

We show (3.161) and (3.162) by contradiction. Suppose (3.161) does not hold. Then for $\delta > 0$ there exist $\zeta_m \to \zeta$, $y_m \to y$, $\mu_m \Rightarrow \mu$, $a_m \to a$, $a_1^m \to a_1$ such that for $m = 1, 2, \ldots$,

$$|w_m(\zeta_m, M_m^{a_m}(y_m, \mu_m), a_1^m) - w(\zeta_m, M^{a_m}(y_m, \mu_m), a_1)| > \delta \tag{3.163}$$

By Proposition 1.4 and the continuity of w

$$w(\zeta_m, M^{a_m}(y_m, \mu_m), a_1^m) \to w(\zeta, M^a(y, \mu), a_1) \tag{3.164}$$

as $m \to \infty$.

Since by Proposition 3.5

$$M_m^{a_m}(y_m, \mu_m) \Rightarrow M^a(y, \mu) \tag{3.165}$$

and $w_m \to w$, as $m \to \infty$ uniformly on compact sets, we also have

$$w_m(\zeta_m, M_m^{a_m}(y_m, \mu_m), a_1^m) \to w(\zeta, M^a(y, \mu), a_1)$$

as $m \to \infty$, which together with (3.164) contradicts (3.163). Therefore (3.161) holds.

Suppose now (3.162) does not hold.

In that case there is $\delta > 0$ and sequences $\zeta_m \to \zeta$, $y_m \to y$, $\mu_m \Rightarrow \mu$, $a_m \to a$, $a_1^m \to a_1$ such that

$$\Big| \int_E \int_E r(z,\zeta_m) P_m^{a_1^m}(z_1, dz) M_m^{a_m}(y_m,\mu_m)(dz_1) \tag{3.166}$$
$$- \int_E \int_E r(z,\zeta_m) P^{a_1^m}(z_1, dz) M^{a_m}(y_m,\mu_m)(dz_1) \Big| > \delta$$

for $m = 1, 2, \ldots$

By (1.18)

$$M^{a_m}(y_m,\mu_m) \Rightarrow M^a(y,\mu)$$

as $m \to \infty$, and therefore by (1.21)

$$\begin{aligned}&\int_E\int_E r(z,\zeta_m)P^{a_1^m}(z_1,dz)M^{a_m}(y_m,\mu_m)(dz_1)\\ &\to \int_E\int_E r(z,\zeta)P^{a_1}(z_1,dz)M^a(y,\mu)(dz_1)\end{aligned} \tag{3.167}$$

as $m \to \infty$.

On the other hand by (D1) and (3.165)

$$\begin{aligned}&\int_E\int_E r(z,\zeta_m)P_m^{a_1^m}(z_1,dz)M_m^{a_m}(y_m,\mu_m)(dz_1)\\ &\to \int_E\int_E r(z,\zeta)P^{a_1}(z_1,dz)M^a(y,\mu)(dz_1)\end{aligned}$$

as $m \to \infty$ and we obtain a contradiction to (3.166).

The proof of Proposition 3.48 is complete.

■

Assume now, that the transition kernels P_m^a are of the particular form (2.28) and that

(C11) for $\varphi \in C(E)$, $i = 1, 2, \ldots, m$, the mappings $U \ni a \mapsto \int_E \varphi(z)\gamma_i^m(a,dz)$ are continuous.

By analogy to Theorem 3.46 and Corollary 3.47 we have

Proposition 3.49 *Under (A3), (A5), (C11) there exists a unique function* $R^d \times [0,\|\varphi_1^m\|] \times \ldots \times [0,\|\varphi_m^m\|] \times U \ni (y,\eta_1,\ldots,\eta_m,a) \mapsto w_m^\beta(y,\eta_1,\ldots,\eta_m,a)$ *that is continuous bounded and satisfies the following equation*

$$\begin{aligned}&w_m^\beta(y,\eta_1,\ldots,\eta_m,a) = \inf_{a_1\in U}[C_m(a_1,y,a,\eta_1,\ldots,\eta_m)+\\ &+\beta\int_{R^d} w_m^\beta(\zeta, M_m^a(y,\eta_1,\ldots,\eta_m)(1),\ldots,M_m^a(y,\eta_1,\ldots,\eta_m)(m),a_1)\\ &\sum_{i=1}^m\int_E r(z,\zeta)\gamma_i^m(a_1,dz)M_m^a(y,\eta_1,\ldots,\eta_m)(i)d\zeta]\end{aligned} \tag{3.168}$$

where C_m is the function defined in (3.143).

Moreover, with (y_n^m) and (π_n^m) the observations and filtering processes from (3.141) and (3.142) respectively,

$$w_m^\beta(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a) = \inf_{(a_n)\ a_0=a} E_\mu^u\Big\{\sum_{n=1}^{\infty}\beta^{n-1}C_m(a_n, y_n^m, a_{n-1}, \pi_{n-1}^{m,u}(\varphi_1^m), \ldots, \pi_{n-1}^{m,u}(\varphi_m^m)|y_1^m = y\Big\} \tag{3.169}$$

where $y \in R^d$, $a \in U$, we set $a_0 = a$ and the infimum is taken over all sequences (a_n) that are adpted to $\sigma\{y_1^m, \ldots, y_n^m\}$.

Furthermore, for $v^{\beta,m}$ as defined in (3.13) we have

$$v^{\beta,m}(\mu) = \inf_a \Big[\int_E c(x,a)\mu(dx) + \beta\int_{R^d} w_m^\beta(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a) \sum_{i=1}^m \int_E r(z,y)\gamma_i^m(a, dz)\mu(\varphi_i^m)dy\Big] \tag{3.170}$$

Finally, for a given $\varepsilon > 0$ we can choose a compact set $L \subset R^d$ and functions $u_0^m \in \mathcal{B}(P(E), U)$ and $u^m \in \mathcal{B}(R^d \times [0, \|\varphi_1^m\|] \times \ldots \times [0, \|\varphi_m^m\|] \times U, U)$ such that the following inequalities hold

$$\sup_{x\in E} R(x, L^c) \le \frac{\varepsilon}{\|c\|} \tag{3.171}$$

$$\int_E c(x, u_0^m(\mu))\mu(dx) + \beta\int_{R^d} w_m^\beta(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), u_0^m(\mu)) \sum_{i=1}^m \int_E r(z,y)\gamma_i^m(u_0^m(\mu), dz)\mu(\varphi_i^m)dy \le v^{\beta,m}(\mu) + \varepsilon \tag{3.172}$$

for $\mu \in P(E)$

and

$$\begin{aligned}&C_m(u^m(y,\eta_1,\ldots,\eta_m,a),y,a,\eta_1,\ldots,\eta_m)+\\&+\beta\int\limits_{R^d} w_m^\beta(\zeta,M_m^a(y,\eta_1,\ldots,\eta_m)(1),\ldots,M_m^a(y,\eta_1,\ldots,\eta_m)(m),\\&u^m(y,\eta_1,\ldots,\eta_m,a))\sum_{i=1}^m\int\limits_E r(z,\zeta)\gamma_i^m(u^m(y,\eta_1,\ldots,\eta_m,a),dz)\\&M_m^a(y,\eta_1,\ldots,\eta_m)(i)d\zeta\le w_m^\beta(y,\eta_1,\ldots,\eta_m,a)+\varepsilon\end{aligned}\tag{3.173}$$

for $y\in L$, $\eta_i\in[0,\|\varphi_i^m\|]$, $i=1,2,\ldots,m$, $a\in U$.

The control $(a_n^{m^*})$ *of the form*

$$\begin{aligned}&a_0^{m^*}=u_0^m(\mu)\\&\ldots\\&a_{n+1}^{m^*}=u^m(y_{n+1}^m,\pi_n^{m,u}(\varphi_1^m),\ldots,\pi_n^{m,u}(\varphi_m^m),a_n^*)\\&\ldots\end{aligned}\tag{3.174}$$

for $n=0,1,2,\ldots$, *is* $4\varepsilon(1-\beta)^{-1}$ *optimal for the cost functional* $J_\mu^{\beta,m}$ *in (3.143).*

Remark 3.50 *If for* $u_0^\beta\in\mathcal{B}(P(E),U)$ *and* $u^\beta\in\mathcal{B}(R^d\times P(E)\times U,U)$ *the infima in (3.170) and (3.168) are attained respectively, then the control* $\hat{a}_0=u_0^\beta(\mu),\ldots,\hat{a}_{n+1}=u^\beta(y_{n+1}^m,\pi_n^{m,u}(\varphi_1^m),\ldots,\pi_n^{m,u}(\varphi_m^m),\hat{a}_n)$ *is optimal for* $J_\mu^{\beta,m}$.

P r o o f. Notice that under (A3), (A5), (C11) the operator

$$\begin{aligned}&T_m w(y,\eta_1,\ldots,\eta_m,a):=\inf_{a_1\in U}[C_m(a_1,y,a,\eta_1,\ldots,\eta_m)+\\&+\beta\int\limits_{R_d} w(\zeta,M_m^a(y,\eta_1,\ldots,\eta_m)(1),\ldots,M_m^a(y,\eta_1,\ldots,\eta_m)(m),a_1)\\&\sum_{i=1}^m\int\limits_E r(z,\zeta)\gamma_i^m(a_1,dz)M^a(y,\eta_1,\ldots,\eta_m)(i)d\zeta]\end{aligned}\tag{3.175}$$

in a contraction on $C(R^d \times [0, \|\varphi_1^m\|] \times \ldots \times [0, \|\varphi_m^m\|] \times U)$ and follow the considerations of subsection 3.4.1.

$\blacksquare$

Our main approximation result is now

Theorem 3.51 *Assume E is compact and (A1)–(A5) as well as (D1), (C11) hold.*

Then for any compact set $L \subset R^d$

$$w_m^\beta(\eta, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a) \to w^\beta(\eta, \mu, a) \tag{3.176}$$

as $m \to \infty$, uniformly in $(y, \mu, a) \in L \times P(E) \times U$ and

$$v^{\beta,m}(\mu) \to v^\beta(\mu) \tag{3.177}$$

as $m \to \infty$ uniformly on $P(E)$.

Furthermore, given $\varepsilon > 0$ there exist a compact set $L \subset R^d$ and functions $u_0^m \in \mathcal{B}(P(E), U)$ and $u^m \in \mathcal{B}(R^d \times [0, \|\varphi_1^m\|] \times \ldots \times [0, \|\varphi_m^m\|] \times U, U)$ satisfying (3.171), (3.172) and (3.173) respectively and the control a_n^* defined as

$$\begin{aligned} & a_0^* = u_0^m(\mu) \\ & \ldots \\ & a_{n+1}^* = u^m(y_{n+1}, \pi_n^u(\varphi_1^m), \ldots, \pi_n^u(\varphi_m^m), a_n^*) \\ & \ldots \end{aligned} \tag{3.178}$$

for $n = 0, 1, 2, \ldots$ is nearly optimal for the cost functional J_μ^β with y_n and π_n^u being now the original observations and true filter process from (1.1) and (1.7) respectively.

Remark 3.52 *If, given $\varepsilon > 0$, $L \subset R^d$ satisfies (3.171) and m_0 is such that for $m \geq m_0$ also the inequalities (3.181)–(3.185) below hold, then from the proof of Theorem 3.51, to be given next, it follows that the control (a_n^*) in (3.178) is more precisely $\dfrac{24\varepsilon}{(1-\beta)^2}$ optimal for J_μ^β.*

P r o o f. Let ϕ denote the function that is identically equal to zero. Notice first that, by the Banach contraction principle, the functions w^β and w^β_m, solutions of (3.146) and (3.168) respectively, can be uniformly approximated by the iterations $T^k\phi$ and $T^k_m\phi$ of the contraction (with constant β) operators T and T_m defined in (3.150) and (3.175) respectively. To prove (3.176) it therefore suffices to show that for each $k = 0, 1, 2, \dots$

$$(T^k_m\phi)(y, \mu(\varphi^m_1), \dots, \mu(\varphi^m_m), a) \to (T^k\phi)(y, \mu, a) \tag{3.179}$$

uniformly on compact subsets of $R^d \times P(E) \times U$.

The proof of (3.179) is by induction in $k = 0, 1, 2, \dots$. Since $T^0_m\phi = 0 = T^0\phi$, (3.179) holds for $k = 0$.

Assume (3.179) holds for k. Then by Proposition 3.48 we have

$$\begin{aligned}&\int\limits_{R^d}(T^k_m\phi)(\zeta, M^a_m(y, \mu(\varphi^m_1), \dots, \mu(\varphi^m_m))(1), \dots,\\ &M^a_m(y, \mu(\varphi^m_1), \dots, \mu(\varphi^m_m))(m), a_1)\\ &\sum_{i=1}^m \int\limits_E r(z, \zeta)\gamma^m_i(a_1, dz) M^a_m(y, \mu(\varphi^m_1), \dots, \mu(\varphi^m_m))(i) d\zeta\\ &\to \int\limits_{R^d}(T^k\phi)(\zeta, M^a(y, \mu), a_1) \int\limits_E\int\limits_E r(z, \zeta)P^{a_1}(z_1, dz)M^a(y, \mu)(dz_1)d\zeta\end{aligned}$$

as $m \to \infty$, uniformly in $(y, \mu, a, a_1) \in L \times P(E) \times U \times U$, where $L \subset R^d$ is any compact set.

Since by the defining relations (3.143) and (3.145) as well as by (3.18)

$$\begin{aligned}&C_m(a_1, y, a, \mu(\varphi^m_1), \dots, \mu(\varphi^m_m)) = \sum_{i=1}^m \int\limits_E c(z, a_1)r(z, y)\gamma^m_i(a, dz)\\ &\mu(\varphi^m_i)\Big(\sum_{j=1}^m \int\limits_E r(z, y)\gamma^m_j(a, dz)\mu(\varphi^m_j)\Big)^{-1} =\\ &= \int\limits_E c(z, a_1)M^a_m(y, \mu)(dz) \to \int\limits_E c(z, a_1)M^a(y, \mu)(dz) = C(a_1, y, a, \mu)\end{aligned} \tag{3.180}$$

as $m \to \infty$ uniformly in $(y, \mu, a, a_1) \in L \times P(E) \times U \times U$, we obtain that

$$(T^{k+1}_m\phi)(y, \mu(\varphi^m_1), \dots, \mu(\varphi^m_m), a) \to (T^{k+1}\phi)(y, \mu, a)$$

as $m \to \infty$ uniformly in $(y, \mu, a, a_1) \in L \times P(E) \times U \times U$.

Therefore (3.179) holds for $k+1$ and by induction, it holds for any positive integer. The convergence (3.176) is thus proved. By (3.153) and (3.170) as well as by (3.176) and (D1) we have for $L \subset R^d$ satisfying (3.171)

$$\sup_{\mu \in P(E)} |v^{\beta,m}(\mu) - v^{\beta}(\mu)| \leq \sup_{\mu \in P(E)} \sup_{a \in U} \Big| \int_{R^d} w_m^{\beta}(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a)$$
$$\int_E r(z,y) P_m^a(\mu, dz) dy - \int_{R^d} w^{\beta}(y, \mu, a) \int_E r(z,y) P^a(\mu, dz) dy \Big|$$
$$\leq \sup_{\mu \in P(E)} \sup_{a \in U} \Big[\sup_{y \in L} |w_m^{\beta}(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a) - w^{\beta}(y, \mu, a)|$$
$$+ \frac{2\varepsilon}{1-\beta} + \Big| \int_{R^d} w^{\beta}(y, \mu, a) \int_E r(z,y)(P_m^a(\mu, dz) - P^a(\mu, dz)) dy \Big| \Big]$$
$$\to \frac{2\varepsilon}{1-\beta} \quad \text{as } m \to \infty$$

thus implying (3.177).

It remains to show the near optimality of the control (a_n^*) in (3.178) for the cost functional J_μ^β. For this purpose we shall more precisely show that there exists m_0 such that for $m \geq m_0$ the control (a_n^*) is $\frac{24\varepsilon}{(1-\beta)^2}$ optimal for J_μ^β. Given $\varepsilon > 0$ let then $L \subset R^d$ satisfy (3.171) and notice that, by Proposition 3.49 we can choose functions u_0^m and u^m so that they satisfy (3.172) and (3.173) respectively.

By (3.176), Proposition 3.48, (3.177), (3.180) there exists furthermore m_0 such that for $m \geq m_0$

$$\sup_{y \in L} \sup_{\mu \in P(E)} \sup_{a \in U} \Big| w_m^{\beta}(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a) \int_E r(z,y) P_m^a(\mu, dz)$$
$$- w^{\beta}(y, \mu, a) \int_E r(z,y) P^a(\mu, dz) \Big| < \varepsilon \tag{3.181}$$

$$\sup_{\mu \in P(E)} |v^{\beta,m}(\mu) - v^{\beta}(\mu)| < \varepsilon \tag{3.182}$$

$$\sup_{a_1 \in U} \sup_{a \in U} \sup_{y \in L} \sup_{\mu \in P(E)} |C_m(a_1, y, a, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m))$$
$$- C(a_1, y, a, \mu)| < \varepsilon \tag{3.183}$$

$$\sup_{y\in L}\sup_{y'\in L}\sup_{\mu\in P(E)}\sup_{a_1\in U}\sup_{a\in U}\Big|w_m^\beta(y', M_m^a(y,\mu(\varphi_1^m),\ldots,\mu(\varphi_m^m))(1),\ldots,$$

$$M_m^a(y,\mu(\varphi_1^m),\ldots,\mu(\varphi_m^m))(m),a_1)\sum_{i=1}^m\int_E r(z,y')\gamma_i^m(a_1,dz)$$

$$M_m^a(y,\mu(\varphi_1^m),\ldots,\mu(\varphi_m^m))(i)-w^\beta(y',M^a(y,\mu),a_1)$$

$$\int_E\int_E r(z,y')P^{a_1}(x,dz)M^a(y,\mu)(dx)\Big|<\varepsilon \tag{3.184}$$

$$\sup_{\mu\in P(E)}\sup_{y\in L}\sup_{a\in U}|w_m^\beta(y,\mu(\varphi_1^m),\ldots,\mu(\varphi_m^m),a)-w^\beta(y,\mu,a)|<\varepsilon \tag{3.185}$$

By the choice of L, by (3.181)–(3.182) and by the fact that u_0^m satisfies (3.172) we now have for $m\geq m_0$ and $\mu\in P(E)$

$$\int_E c(x,u_0^m(\mu))\mu(dx)+\beta\int_{R^d}w^\beta(y,\mu,u_0^m(\mu))\int_E r(x,y)P^{u_0^m(\mu)}(\mu,dz)dy$$

$$\leq\int_E c(x,u_0^m(\mu))\mu(dx)+\beta\varepsilon(\|w^\beta\|+\|w_m^\beta\|)+$$

$$+\beta\int_{R^d}w_m^\beta(y,\mu(\varphi_1^m),\ldots,\mu(\varphi_m^m),u_0^m(\mu)) \tag{3.186}$$

$$\sum_{i=1}^m\int_E r(z,y)\gamma_i^m(u_0^m(\mu),dz)\mu(\varphi_i^m)dy+\varepsilon\leq\frac{2\varepsilon}{1-\beta}+v^{\beta,m}(\mu)+2\varepsilon\leq$$

$$\leq\frac{2\varepsilon}{1-\beta}+3\varepsilon+v^\beta(\mu)\quad\text{for }\mu\in P(E)$$

Similarly, by the choice of L, by (3.183)–(3.185) and by the fact that u^m

satisfies (3.173) we have for $m \geq m_0$ and $y \in L$, $\mu \in P(E)$, $a \in U$

$$
\begin{aligned}
&C(u^m(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a), y, a, \mu) + \beta \int_{R^d} w^\beta(\zeta, M^a(y, \mu), \\
&u^m(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a)) \int_E \int_E r(z, \zeta) P^{u^m(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a)}(z_1, dz) \\
&M^a(y, \mu)(dz_1) d\zeta \leq C_m(u^m(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a), y, a, \\
&\mu(\varphi_1^m), \mu(\varphi_2^m), \ldots, \mu(\varphi_m^m)) + \varepsilon + \beta\varepsilon(\|w^\beta\| + \|w_m^\beta\|)) + \\
&\beta \int_{R^d} w_m^\beta(\zeta, M_m^a(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m))(1), \ldots, M_m^a(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m))(m), \\
&u^m(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a) \sum_{i=1}^m \int_E r(z, \zeta) \gamma_i^m(u^m(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a), dz) \\
&M_m^a(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m))(i) d\zeta + \varepsilon \leq 2\varepsilon + \frac{2\varepsilon}{1-\beta} \\
&+ w_m^\beta(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), a) + \varepsilon \leq 4\varepsilon + \frac{2\varepsilon}{1-\beta} + w^\beta(y, \mu, a)
\end{aligned}
\tag{3.187}
$$

At this point notice that (3.186) and (3.187) correspond to (3.155) and (3.156) respectively if ε is put equal to $\dfrac{6\varepsilon}{1-\beta}$. By Corollary 3.47 we therefore have that the control (a_n^*) in (3.178) is $\dfrac{24\varepsilon}{(1-\beta)^2}$ optimal for J_μ^β.

■

By Theorem 3.51, the construction of a nearly optimal control function for J_μ^β is thus reduced to the problem of determining a nearly optimal control function for $J_\mu^{\beta,m}$ that satisfies (3.172) and (3.173) and that involves the filter only through the values of the statistic $\pi_n^{m,u}(\varphi_i^m)$. To practically construct functions satisfying (3.172) and (3.173), we have to consider further discretizations both of the observation and the control spaces. Recalling the definition of the projection operators W_H, Z_H in (2.48), (2.50), and that

$U^H = \{Z_H a, a \in U\}$, consider then the following relations

$$
\begin{aligned}
& w^\beta_{m,H}(y,\eta_1,\ldots,\eta_m,a) = \inf_{a_1\in U}[C_m(Z_H a_1, W_H y, Z_H a, \eta_1,\ldots,\eta_m) \\
& +\beta \int_{R^d} w^\beta_{m,H}(\zeta, M_m^{Z_H a}(W_H y,\eta_1,\ldots,\eta_m)(1),\ldots, \\
& M_m^{Z_H a}(W_H y,\eta_1,\ldots,\eta_m)(m), Z_H a)\sum_{i=1}^m \int_E r(z,\zeta) \\
& \gamma_i^m(Z_H a_1, dz) M_m^{Z_H a}(W_H y,\eta_1,\ldots,\eta_m)(i) d\zeta]
\end{aligned}
\tag{3.188}
$$

$$
\begin{aligned}
& v_H^{\beta,m}(\mu) = \inf_{a\in U}\Big[\int_E c(x,Z_H a)\mu(dx) + \beta\int_{R^d} w^\beta_{m,H}(W_H y, \\
& \mu(\varphi_1^m),\ldots,\mu(\varphi_m^m), Z_H a)\sum_{i=1}^m \int_E r(z,y)\gamma_i^m(Z_H a, dz)\mu(\varphi_i^m) dy\Big]
\end{aligned}
\tag{3.189}
$$

that correspond to (3.168) and (3.170) respectively.

From the definition in (3.188) notice that the functions $w^\beta_{m,H}$ depend on y and a only through the finite number of values of $W_H y$ and $Z_H a$; furthermore, the infimum over a_1 is actually a minimum over the finite set of values of $Z_H a_1 \in U^H$.

Proposition 3.53 *Under (A3)–(A5), and (C11), for E compact there exists a unique function $w^\beta_{m,H} \in b\mathcal{B}(R^d \times [0,\|\varphi_1^m\|] \times \ldots \times [0,\|\varphi_m^m\|] \times U)$ that is a solution to (3.188). One also has*

$$
w^\beta_{m,H}(y,\eta_1,\ldots,\eta_m,a) \to w^\beta_m(y,\eta_1,\ldots,\eta_m,a) \tag{3.190}
$$

as $H \to \infty$, uniformly in $\eta_i \in [0,\|\varphi_i^m\|]$, $i = 1,2,\ldots,m$, $a \in U$, and y belonging to a compact set $L \subset R^d$. Moreover, for fixed $y \in R^d$, $a \in U$ the mapping

$$
[0,\|\varphi_1^m\|] \times \ldots \times [0,\|\varphi_m^m\|] \ni (\eta_1,\ldots,\eta_m) \to w^\beta_{m,H}(y,\eta_1,\ldots,\eta_m,a)
$$

is continuous. Furthermore, for $v_H^{\beta,m}$ defined in (3.189) one has $v_H^{\beta,m} \in C(P(E))$ and

$$
v_H^{\beta,m}(\mu) \to v^{\beta,m}(\mu) \tag{3.191}
$$

as $H \to \infty$, uniformly in $\mu \in P(E)$.

Finally, given $\varepsilon > 0$, if H_0 is such that for $H > H_0$, $u_0^H \in \mathcal{B}(P(E), U^H)$ and $u^H \in \mathcal{B}(W_H R^d \times [0, \|\varphi_1^m\|] \times \ldots \times [0, \|\varphi_m^m\|] \times U^H, U^H)$ satisfy the following inequalities

$$\begin{aligned}
&\int_E c(x, u_0^H(\mu))\mu(dx) + \beta \int_{R^d} w_{m,H}^\beta(y, \mu(\varphi_1^m), \ldots, \mu(\varphi_m^m), u_0^H(\mu)) \\
&\sum_{i=1}^m \int_E r(z,y)\gamma_i^m(u_0^H(\mu), dz)\mu(\varphi_i)dy \le v_H^{\beta,m}(\mu) + \varepsilon \quad \text{for } \mu \in P(E)
\end{aligned} \tag{3.192}$$

$$\begin{aligned}
&C_m(u^H(W_H y, \eta_1, \ldots, \eta_m, a), W_H y, a, \eta_1, \ldots, \eta_m) + \\
&+\beta \int_{R^d} w_{m,H}^\beta(\zeta, M_m^a(W_H y, \eta_1, \ldots, \eta_m)(1), \ldots, M_m^a(W_H y, \eta_1, \ldots, \eta_m)(m), \\
&u^H(W_H y, \eta_1, \ldots, \eta_m, a)) \sum_{i=1}^m \int_E r(z,\zeta)\gamma_i^m(u^H(W_H y, \eta_1, \ldots, \eta_m, a), dz) \\
&M_m^a(W_H y, \eta_1, \ldots, \eta_m)(i)d\zeta \le w_{m,H}^\beta(W_H y, \eta_1, \ldots, \eta_m, a) + \varepsilon
\end{aligned} \tag{3.193}$$

for $y \in R^d$, $\eta_i \in [0, \|\varphi_i^m\|]$, $i = 1, 2, \ldots, m$, $a \in U^H$ respectively, then the control $a_n^{H^*}$ defined by analogy to (3.178) as

$$\begin{aligned}
&a_0^{H^*} = u_0^H(\mu) \\
&\ldots \\
&a_{n+1}^{H^*} = u^H(W_H y_{n+1}^m, \pi_n^{m,u}(\varphi_1^m), \ldots, \pi_n^{m,u}(\varphi_m^m), a_n^{H^*}) \\
&\ldots
\end{aligned} \tag{3.194}$$

is, for $n = 0, 1, 2, \ldots$, nearly optimal for the cost functional $J_\mu^{\beta,m}$.

Remark 3.54 *If, given $\varepsilon > 0$, $L \subset R^d$ satisfies (3.171) and H_0 is such that for $H > H_0$ also the inequalities (3.195)–(3.199) below hold, then from the* *proof given below it follows (see also Remark 3.52) that the control $(a_n^{H^*})$ in (3.194) is more precisely $\frac{24\varepsilon}{(1-\beta)^2}$ optimal for $J_\mu^{\beta,m}$.*

$$\sup_{y\in L}\sup_i \sup_{\eta_i\in[0,\|\varphi_i^m\|]} \sup_{a\in U}\Big|w_{m,H}^\beta(y,\eta_1,\dots,\eta_m,a)$$
$$\sum_{i=1}^m r(z,y)\gamma_i^m(a,dz)\eta_i - w_m^\beta(y,\eta_1,\dots,\eta_m,a)\sum_{i=1}^m\int_E r(z,y)$$
$$\gamma_i^m(a,dz)\eta_i\Big| < \varepsilon \tag{3.195}$$
$$\sup_{\mu\in P(E)} |v_H^{\beta,m}(\mu) - v^{\beta,m}(\mu)| < \varepsilon \tag{3.196}$$
$$\sup_{a_1\in U}\sup_{a\in U}\sup_{y\in L}\sup_i \sup_{\eta_i\in[0,\|\varphi_i^m\|]}$$
$$|C_m(a_1,W_Hy,a,\eta_1,\dots,\eta_m) - C_m(a_1,y,a,\eta_1,\dots,\eta_m)| < \varepsilon \tag{3.197}$$
$$\sup_{y\in L}\sup_{y'\in L}\sup_i \sup_{\eta_i\in[0,\|\varphi_i^m\|]}\sup_{a_1\in U}\sup_{a\in U}\Big|w_{m,H}^\beta(y',$$
$$M_m^a(W_Hy,\eta_1,\dots,\eta_m)(1),\dots,M_m^a(W_Hy,\eta_1,\dots,\eta_m)(m),a_1)$$
$$\sum_{i=1}^m\int_E r(z,y')\gamma_i^m(a_1,dz)M_m^a(W_Hy,\eta_1,\dots,\eta_m)(i)$$
$$-w_m^\beta(y',M_m^a(y,\eta_1,\dots,\eta_m)(1),\dots,M_m^a(y,\eta_1,\dots,\eta_m)(m),a_1)$$
$$\sum_{i=1}^m\int_E\int_E r(z,y')\gamma_i^m(a_1,dz)M_m^a(y,\eta_1,\dots,\eta_m)(i)\Big| < \varepsilon \tag{3.198}$$
$$\sup_{y\in L}\sup_i \sup_{\eta_i\in[0,\|\varphi_i^m\|]} \sup_{a\in U}|w_{m,H}^\beta(W_Hy,\eta_1,\dots,\eta_m,a)$$
$$-w_m^\beta(y,\eta_1,\dots,\eta_m,a)| < \varepsilon \tag{3.199}$$

P r o o f. The existence and uniqueness of $w_{m,H}^\beta$ follows from the Banach contraction principle. To prove (3.190), by analogy to the proof of Theorem 3.51 it is sufficient to show the convergence of the iterations of the contraction operators that are used to prove the existence of solutions to (3.188) and (3.168) respectively.

The contiunuity of $w_{m,H}^\beta(y,\eta_1,\dots,\eta_m,a)$ with respect to $\eta_1,\dots,\eta_m$ follows from the continuity in $\eta_1,\dots,\eta_m$ of the iterations of the contraction operators that approximate uniformly $w_{m,H}^\beta$, recalling that $\inf_{a_1}$ is actually a minimum over the finite set of values of $Z_H a_1$. Consequently, by (3.189) we have $v_H^{\beta,m} \in C(P(E))$ as well as the convergence (3.191). To obtain the near optimality of the control $(a_n^{H^*})$ in (3.194), one may proceed analogously as in the proof of Theorem 3.51 using Proposition 3.49, instead of Corollary 3.47. ∎

Although the functions $w_{m,H}^{\beta}(y, \eta_1, \ldots, \eta_m, a)$ depend on y, a only through the finite number of values of $W_H y$, $Z_H a$, they still depend on an infinite number of values of $\eta_i \in [0, \|\varphi_i^m\|]$, $i = 1, 2, \ldots, m$. To make the construction of the nearly optimal control $(a_n^{H^*})$ feasible, one has therefore to perform also a discretization of η_i which does not create additional problems because of the continuity of $w_{m,H}^{\beta}$ with respect to $(\eta_1, \ldots, \eta_m)$. Since this further discretization would repeat arguments already discussed previously in subsection 2.3.2, it is left to the reader.

3.4.3 The approximation procedure

Summarizing, the procedure to obtain a nearly optimal control function for J_μ^β in the context of this section 3.4 is as follows:

a) For sufficiently large m and H, use an iterative procedure of the value iteration type to determine a uniformly approximating solution to (3.188). Due to the contraction property of the operator on the right hand side of (3.188) any degree of approximation can thereby be reached and the computations can actually be carried out since, after an additional discretization of η_i ($i = 1, 2, \ldots, m$), all quantities involved are finite-valued.

   The minimizing values of a_1 (actually of $Z_H a_1 \in U^H$), obtained at the last iteration corresponding to the various possible values of $W_H y$, $\eta_1, \ldots, \eta_m, Z_H a$, lead to a function $u^H \in \mathcal{B}(W_H(R^d) \times [0, \|\varphi_1^m\|] \times \ldots \times [0, \|\varphi_m^m\|] \times U^H, U^H)$.

b) With the (uniform) approximation to $w_{m,H}^{\beta}$ from (3.188), determine according to (3.189) an approximation to $v_H^{\beta,m}(\mu)$ for the given initial measure μ.

Provided the integrals in (3.189) with respect to x, y and z can be carried out, this computation is again feasible due to the fact that the "inf" is actually a "min" over the finite number of values of $Z_H a$. The minimizing a is the value, at the given initial measure, of a function $u_0^H \in \mathcal{B}(P(E), U^H)$.

Concerning the functions u_0^H and u^H thus constructed, the results of subsection 3.4.2 allow to conclude the following:

i) For a sufficiently large number of iterations to solve (3.188), if H is chosen sufficiently large that conditions (3.192) and (3.193) as well as (3.195)–(3.199) are satisfied for a given $\varepsilon > 0$, by Proposition 3.53 as well as by Remark 3.54, the control $(a_n^{H^*})$ is $\dfrac{24\varepsilon}{(1-\beta)^2}$ optimal for $J_\mu^{\beta,m}$.

ii) Based on (3.192) and (3.193), and taking also (3.196), (3.197), (3.199) into account, we have that the control functions $u_0^m(\mu) := u_0^H(\mu)$ and

$$u^m(y, \eta_1, \ldots, \eta_m, a) := u^H(W_H y, \eta_1, \ldots, \eta_m, Z_H a)$$

satisfy (3.172) and (3,173) for suitable values of ε.

If therefore also m is chosen sufficiently large, by Theorem 3.51 we have that the control $(a_n^{H^*})$ given accordingly to (3.178) by

$$\begin{aligned} a_0^{H^*} &= u_0^m(\mu) \\ &\ldots \\ a_{n+1}^{H^*} &= u^m(y_{n+1}, \pi_n^u(\varphi_1^m), \ldots, \pi_n^u(\varphi_m^m), a_n^{H^*}) \\ &\ldots \end{aligned}$$

is nearly optimal for J_μ^β.

3.5 Filter approximation and near optimal control values

The previous sections of this Chapter 3 were devoted to the construction of nearly optimal control functions which, when applied to the true filter process (π_n) with values in $P(E)$ and defined in (1.7), yield nearly optimal controls.

In the context of section 3.3 the nearly optimal control functions belong either to the space $\mathcal{B}(P(E), U)$ or to $C(P(E), U)$ and can thus be applied directly to (π_n) to yield controls of the form $a_n = u(\pi_n)$. In the context of section 3.4 the control functions belong to $\mathcal{B}(R^d \times R^m \times U, U)$, they depend on (π_n) only through the values of the statistic $(\pi_n(\varphi_1^m), \ldots, \pi_n(\varphi_m^m))$, but depend also on the values of the current observation and previous control; they thus yield controls of the form $a_n = u(y_n, \pi_{n-1}(\varphi_1^m), \ldots, \pi_{n-1}(\varphi_m^m), a_{n-1})$.

Since the true filter process (π_n) takes its values in the infinite dimensional space of measures $P(E)$, it cannot be computed in practice. On the other hand, to able to determine nearly optimal control functions, already in sections 3.3 and 3.4 we considered approximating finite dimensional filter processes. More precisely, in subsection 3.3.2 we considered the process $(\bar{\pi}_n^{m,u})$ with values in the simplex $P(E_m)$ that can be computed recursively by (3.57). Analogously, in section 3.4 we considered the process $(\pi_n^{m,u})$, whose finite-dimensional statistics $\pi_n^{m,u}(\varphi_i^m)$ $(i = 1, 2, \ldots, m)$ can be computed recursively by (3.144).

Although these finite-dimensional processes are computable, they are based on the approximating fictitions observations $(\overline{y}_n^m)$ defined in (3.54) and (3.141) respectively, and are thus fictitions processes themselves.

The purpose of this section is now to define a real approximating finite-dimensional filtering process, that can be computed, and to show that the nearly optimal control functions determined in sections 3.3 and 3.4, provided they are continuous, still yield nearly optimal control values when applied to the approximating filtering process.

We shall do this in two subsections, the first subsection 3.5.1 corresponding to the context of section 3.3, the second 3.5.2 corresponding to section 3.4. A common feature of the two subsections is that, since the approximating filter process itself is not Markov, we shall consider pairs of processes, each pair consisting of an approximating filter to be used as argument of the control function and a "true" filtering process.

3.5.1 Filter approximation and near optimal control values in the context of section 3.3

Given the initial measure $\mu \in P(E)$ for the process (x_n), let $\overline{\mu} \in P(E_m)$ be the vector $(\mu(B_1^m), \ldots, \mu(B_{k_m}^m))$ as specified below (3.74). Starting from $\overline{\mu}$ as initial measure we shall construct a computable real process $\overline{\pi}_n^{m(\overline{\mu})} \in P(E_m)$ (see (3.204) below) that will serve as approximating filtering process; it will then be shown that any of the nearly optimal control functions derived in section 3.3, provided they are continuous, yield nearly optimal controls when applied to $(\overline{\pi}_n^{m(\overline{\mu})})$. There are essentially two classes of nearly optimal control functions in section 3.3. The first class, corresponding to subsections 3.3.2a and 3.3.3a, belongs to the space $\mathcal{B}(P(E_m), U)$. Each control function from this class can be applied directly to $\overline{\pi}_n^{m(\overline{\mu})}$ to yield controls $a_n = u(\overline{\pi}_n^{m(\overline{\mu})})$, but the results below hold only if first we verify that u is also continuous. The second class, corresponding to subsections 3.3.2b and 3.3.3b, belongs to the space $C(P(E), U)$. Although each control function u from this second class is already continuous, the process $(\overline{\pi}_n^{m(\overline{\mu})})$ with values in $P(E_m)$ has first to be lifted to the space $P(E)$ via the procedure connected with the operator $\widetilde{\mathcal{L}}_m$ introduced in subsection 3.3.2.b.

Both classes of controls can however be treated in one single approach by assuming that the nearly optimal control function u belongs to $\mathcal{A} = C(P(E), U)$ and considering controls obtained as $a_n = \widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\overline{\mu})})$.

As already mentioned, we shall now consider pairs of processes, each consisting of an approximating and a "true" filtering process. For reasons that will become apparent below, we shall actually consider three such pairs, each corresponding to one of the three processes (π_n), (π_n^m), $(\overline{\pi}_n^m)$ defined in (1.7), (3.14) and (3.57) respectively.

To define the three pairs, it will be convenient to consider the operators

$$Q(y,\nu)(A) = \frac{\int\limits_A r(z,y)\nu(dz)}{\int\limits_E r(z,y)\nu(dz)} \tag{3.200}$$

for $y \in R^d$, $\nu \in P(E)$, $A \in \mathcal{B}(E)$, and

$$\zeta(\mu, a)(A) = \int\limits_E P^a(x, A)\mu(dx) \tag{3.201}$$

for $a \in U$, $\mu \in P(E)$, $A \in \mathcal{B}(E)$.

Given a fixed control function $u \in \mathcal{A}$, for $\nu, \mu \in P(E)$, $\eta \in P(E_m)$ and $M^a(y,\pi)$ as in (1.8), $M_m^a(y,\pi)$ as in (3.14), $\overline{M}_m^a(y,\overline{\pi})$ as in (3.57), define

$$\begin{cases} \pi_0^{(\nu)}(\cdot) = \nu(\cdot) \quad & \pi_{n+1}^{(\nu)}(\cdot) = M^{a_n}(y_{n+1}, \pi_n^{(\nu)})(\cdot) \text{ with } a_n = u(\pi_n^{(\nu)}) \\ \pi_0^{(\mu,\nu)}(\cdot) = \mu(\cdot) \quad & \pi_{n+1}^{(\mu,\nu)}(\cdot) = Q(y_{n+1}, \zeta(\pi_n^{(\mu,\nu)}, u(\pi_n^{(\nu)})))(\cdot) \end{cases} \tag{3.202}$$

$$\begin{cases} \pi_0^{m(\nu)}(\cdot) = \nu(\cdot) & \pi_{n+1}^{m(\nu)}(\cdot) = M_m^{a_n}(y_{n+1}, \pi_n^{m(\nu)})(\cdot) \text{ with } a_n = \overline{\mathcal{L}}_m u(\pi_n^{m(\nu)}) \\ \pi_0^{m(\mu,\nu)}(\cdot) = \mu(\cdot) & \pi_{n+1}^{m(\mu,\nu)}(\cdot) = Q(y_{n+1}, \zeta(\pi_n^{m(\mu,\nu)}, \overline{\mathcal{L}}_m u(\pi_n^{m(\nu)}))(\cdot) \end{cases} \tag{3.203}$$

$$\begin{cases} \overline{\pi}_0^{m(\eta)}(\cdot) = \eta(\cdot) & \overline{\pi}_{n+1}^{m(\eta)}(\cdot) = \overline{M}_m^{a_n}(W_m y_{n+1}, \overline{\pi}_n^{m(\eta)})(\cdot) \text{ with} \\ & \qquad a_n = \widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\eta)}) \\ \widetilde{\pi}_0^{m(\mu,\eta)}(\cdot) = \mu(\cdot) & \widetilde{\pi}_{n+1}^{m(\mu,\eta)}(\cdot) = Q(y_{n+1}, \zeta(\widetilde{\pi}_n^{m(\mu,\eta)}, \widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\eta)}))(\cdot) \end{cases} \tag{3.204}$$

where W_m is the projection of R^d to D_m defined as in (2.48) and where y_n denote the "real" observations, namely the observations, according to (1.1), of the given state process (x_n) that starts with initial law μ and is controlled in the generic period n by a control taking the values $a_n = u(\pi_n^{(\nu)})$, $a_n = \overline{\mathcal{L}}_m u(\pi_n^{m(\nu)})$ and $a_n = \widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\eta)})$ respectively. The "real" filtering processes (recall that the state process (x_n) has initial measure μ) are therefore given by $\pi_n^{(\mu,\nu)}$, $\pi_n^{m(\mu,\nu)}$ and $\widetilde{\pi}_n^{m(\mu,\nu)}$ respectively.

The processes $\pi_n^{(\nu)}$, $\pi_n^{m(\nu)}$ and $\overline{\pi}_n^{m(\eta)}$ will thus play the role of arguments of the control function.

Notice also that, for $\eta = \overline{\nu} = (\nu(B_1^m), \dots, \nu(B_{k_m}^m))$, we have $\pi_n^{m(\mu,\nu)}(\cdot) = \widetilde{\pi}_n^{m(\mu,\overline{\nu})}(\cdot)$.

Lemma 3.55 *The pairs $(\pi_n^{(\nu)}, \pi_n^{(\mu,\nu)})$, $(\pi_n^{m(\nu)}, \pi_n^{m(\mu,\nu)})$ and $(\overline{\pi}_n^{m(\eta)}, \widetilde{\pi}_n^{m(\mu,\eta)})$ form Markov processes with transition operators T, T_m and $\overline{T}_m$ defined in the following way: for $F \in b\mathcal{B}(P(E) \times P(E))$, $f \in b\mathcal{B}(P(E_m) \times P(E))$, $\mu, \nu \in P(E)$, $\eta \in P(E_m)$ we have*

$$\begin{aligned} TF(\nu,\mu) &:= E_{\mu,\nu}\{F(\pi_1^{(\nu)}, \pi_1^{(\mu,\nu)})\} = \\ &= \int_E \int_{R^d} F(M^{u(\nu)}(y,\nu), Q(y, \zeta(\mu, u(\nu))))r(z,y)dyP^{u(\nu)}(\mu, dz) \end{aligned} \tag{3.205}$$

$$T_mF(\nu,\mu) := E_{\mu,\nu}\{F(\pi_1^{m(\nu)},\pi_1^{m(\mu,\nu)})\} =$$
$$= \int_E \int_{R^d} F(M_m^{\overline{\mathcal{L}}_m u(\nu)}(y,\nu), Q(y,\zeta(\mu,\overline{\mathcal{L}}_m u(\nu))))r(z,y)dyP^{\overline{\mathcal{L}}_m u(\nu)}(\mu,dz) \tag{3.206}$$

$$\overline{T}_m f(\eta,\mu) := E_{\mu,\eta}\{f(\overline{\pi}_1^{m(\eta)},\widetilde{\pi}_1^{m(\mu,\eta)})\} =$$
$$= \int_E \int_{R^d} f(\overline{M}_m^{\widetilde{\mathcal{L}}_m u(\eta)}(W_m y,\eta), Q(y,\zeta(\mu,\widetilde{\mathcal{L}}_m u(\eta))))r(z,y)dyP^{\widetilde{\mathcal{L}}_m u(\eta)}(\mu,dz) \tag{3.207}$$

respectively.

Moreover, under (A1)–(A4), T is Feller. Assuming additionally (B9), we have the Feller property of $\overline{T}_m$ as well.

P r o o f . The proof of the Markov property and of the form of the operators T, T_m, $\overline{T}_m$ respectively, is analogous to that of Lemma 1.3. The Feller property of T and $\overline{T}_m$ can be shown by arguments similar to those of Proposition 1.4 and Corollary 1.5.

■

Proposition 3.56 *Under (A1)–(A4) and (D1),(D2) if $\mathcal{B}(P(E)\times P(E)) \ni F_m \mapsto F \in C(P(E)\times P(E))$, as $m\to\infty$ uniformly on compact subsets of $P(E)\times P(E)$ and the family $\{F_m, m=1,2,\ldots\}$ is uniformly bounded, we have that for $k=1,2,\ldots,$*

$$(T_m)^k F_m(\nu,\mu) \to (T)^k F(\nu,\mu) \quad \text{as } m\to\infty \tag{3.208}$$

uniformly in (ν,μ) from compact subsets of $P(E)\times P(E)$, where $(T_m)^k$ and $(T)^k$ denote the k-th iterates of T_m and T respectively.

P r o o f . We show (3.208) by induction. Therefore we prove first (3.208) for $k=1$.

By Proposition 3.5, for $\varphi \in C(E)$

$$|M_m^{\overline{\mathcal{L}}_m u(\nu)}(y,\nu)(\varphi) - M^{\overline{\mathcal{L}}_m u(\nu)}(y,\nu)(\varphi)| \to 0 \tag{3.209}$$

as $m \to \infty$, uniformly in (y,ν) from compact subsets of $R^d \times P(E)$.

Now,

$$Q(y,\zeta(\mu,\overline{\mathcal{L}}_m u(\nu))) \Rightarrow Q(y,\zeta(\mu,u(\nu))) \tag{3.210}$$

as $m \to \infty$, uniformly in (y,μ,ν) from compact subsets of $R^d \times P(E) \times P(E)$.

In fact, if (3.210) does not hold, then there are sequences $y_m \to y$, $\mu_m \Rightarrow \mu$, $\nu_m \Rightarrow \nu$ such that for $\varphi \in C(E)$ and some $\delta > 0$ we have

$$|Q(y_m,\zeta(\mu_m,\overline{\mathcal{L}}_m u(\nu_m)))(\varphi) - Q(y_m,\zeta(\mu_m,u(\nu_m)))(\varphi)| > \delta \tag{3.211}$$

Since

$$Q(y_m,\zeta(\mu_m,\overline{\mathcal{L}}_m u(\nu_m)))(\varphi) = M^{a_m}(y_m,\mu_m)(\varphi)$$

with $a_m = \overline{\mathcal{L}}_m u(\nu_m)$

and

$$Q(y_m,\zeta(\mu_m,u(\nu_m)))(\varphi) = M^{\overline{a}_m}(y_m,\mu_m)(\varphi)$$

with $\overline{a}_m = u(\nu_m)$ and by Lemma 3.21(i), $a_m \to u(\nu)$, according to Proposition 1.4 we obtain

$$|M^{a_m}(y_m,\mu_m)(\varphi) - M^{u(\nu)}(y_m,\mu)(\varphi)| \to 0$$

and

$$|M^{\overline{a}_m}(y_m,\mu_m)(\varphi) - M^{u(\nu)}(y_m,\mu)(\varphi)| \to 0$$

a contradiction to (3.211).

Thus (3.210) holds.

Since

$$M^{\overline{\mathcal{L}}_m u(\nu)}(y,\nu) = Q(y,\zeta(\nu,\overline{\mathcal{L}}_m u(\nu)))$$

by (3.209) and (3.210) it is immediate that

$$M_m^{\overline{\mathcal{L}}_m u(\nu)}(y,\nu) \Rightarrow Q(y,\zeta(\nu,u(\nu)) = M^{u(\nu)}(y,\nu)$$

as $m \to \infty$, uniformly in (y,ν) from compact subsets of $R^d \times P(E)$.

Now, for $\mathcal{B}(P(E) \times P(E)) \ni F_m \to F \in C(P(E) \times P(E))$ as $m \to \infty$, uniformly on compact subsets of $P(E) \times P(E)$, and $\|F_m\|$ uniformly bounded we have

$$|T_m F_m(\mu,\nu) - TF(\mu,\nu)| \leq \int_E \int_{R^d} |F_m(M_m^{\overline{\mathcal{L}}_m u(\nu)}(y,\nu),$$
$$Q(y,\zeta(\mu,\overline{\mathcal{L}}_m u(\nu)))) - F(M^{u(\nu)}(y,\nu), Q(y,\zeta(\mu,u(\nu)))|r(z,y)dy$$
$$P^{\overline{\mathcal{L}}_m u(\nu)}(\mu,dz) + \Big|\int_E \int_{R^d} F(M^{u(\nu)}(y,\nu), Q(y,\zeta(\mu,u(\nu))))r(z,y)dy$$
$$(P^{\overline{\mathcal{L}}_m u(\nu)}(\mu,dz) - P^{u(\nu)}(\mu,dz)))\Big| = I_m + II_m$$

Let $H \subset P(E)$ be a compact set. By (A2) and (A4) for any $\varepsilon > 0$ we can find compact sets $K \subset E$ and $L \subset R^d$ such that for $\mu \in H$

$$\mu(K^c) < \varepsilon \quad P^{\overline{\mathcal{L}}_m u(\nu)}(\mu,K^c) < \varepsilon \quad \text{for } m = 1,2,\ldots;\ \nu \in P(E)$$

and

$$\sup_{z\in K} \int_{L^c} r(z,y)dy \leq \varepsilon$$

Then for an additional compact set $H_1 \subset P(E)$ we have

$$I_m \leq 2(\|F\| + \|F_m\|)\varepsilon + \sup_{\mu\in H} \sup_{\nu\in H_1} \sup_{y\in L} |F_m(M_m^{\overline{\mathcal{L}}_m u(\nu)}(y,\nu),$$
$$Q(y,\zeta(\mu,\overline{\mathcal{L}}_m u(\nu))) - F(M^{u(\nu)}(y,\nu), Q(y,\zeta(\mu.u(\nu))))|$$

and by (3.209), (3.210), Lemma 3.8 and the compactness of the set $\{(M^{u(\nu)}(y,\nu), Q(y,\zeta(\mu,u(\nu)))), y \in L, \mu \in H, \nu \in H_1\}$ we obtain $I_m \to 0$, as $m \to \infty$ uniformly in $(\mu,\nu) \in H \times H_1$.

Moreover, we have

$$II_m \leq 2\|F\|\varepsilon + \sup_{\mu\in H} \sup_{\nu\in H_1} \sup_{y\in L} \Big|\int_E F(M^{u(\nu)}(y,\nu),$$
$$Q(y,\zeta(\mu,u(\nu))))r(z,y)(P^{\overline{\mathcal{L}}_m u(\nu)}(\mu,dz) - P^{u(\nu)}(\mu,dz))\Big|$$

By Lemma 3.6 with $M_1 = R^d \times P(E) \times P(E) \times P(E)$, $M_2 = R$, $K = L \times H \times H_1 \times \widetilde{H}$, function F there defined as (compare with the proof of

(3.21))

$$\begin{aligned}&F\colon R^d \times P(E) \times P(E) \times P(E) \ni (y, \nu_1, \nu_2, \nu_3)\\&\mapsto \int_E \varphi(y, \nu_1, \nu_2, z)\nu_3(dz)\end{aligned}$$

where

$$\begin{aligned}&\widetilde{H} = \{P^{u(\nu)}(\mu, \cdot),\ \nu \in H_1,\ \mu \in H\}\\&\varphi(y, \nu_1, \nu_2, z) = F(M^{u(\nu_2)}(y, \nu_2), Q(y, \zeta(\nu_1, u(\nu_2))))r(z, y)\end{aligned}$$

we obtain $II_m \to 0$ as $m \to \infty$, uniformly in $(\mu, \nu) \in H \times H_1$.

Therefore

$$\sup_{\nu \in H_1} \sup_{\mu \in H} |T_m F_m(\nu, \mu) - TF(\nu, \mu)| \to 0$$

as $m \to \infty$, and we proved (3.208) for $k = 1$.

Assume now, that (3.208) is satisfied for k. Then

$$\overline{F}_m^k(\nu, \mu) \colon= (T_m)^k F_m(\nu, \mu) \to (T)^k F(\nu, \mu) \colon= \overline{F}^k(\nu, \mu)$$

as $m \to \infty$, uniformly in (ν, μ) from compact subsets of $P(E) \times P(E)$. By step $k = 1$

$$(T_m)^{k+1} F_m(\nu, \mu) = T_m \overline{F}_m^k(\nu, \mu) \to T\overline{F}^k(\nu, \mu) = (T)^{k+1} F(\nu, \mu)$$

as $m \to \infty$, uniformly in (ν, μ) from compact subsets of $P(E) \times P(E)$, and therefore (3.208) holds for $k + 1$.

Finally, by induction (3.208) is satisfied for any positive integer k.

■

The near optimality of the control $a_n = \widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\overline{\mu})})$ follows now from

Corollary 3.57 *Under the assumptions of Proposition 3.56, assuming additionaly also (A5), for any $u \in \mathcal{A}$*

$$J_\mu^\beta((\widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\overline{\mu})}))) \to J_\mu^\beta(u(\pi_n)) \tag{3.212}$$

as $m \to \infty$, uniformly in μ from compact subsets of $P(E)$.

P r o o f. We have to show that for each $i = 0, 1, 2, \ldots$,

$$
\begin{aligned}
&E_{\mu,\overline{\mu}}\Big\{ \int\limits_E c(x, \widetilde{\mathcal{L}}_m u(\overline{\pi}_i^{m(\overline{\mu})}))\widetilde{\pi}_i^{m(\mu,\overline{\mu})}(dx)\Big\} \\
&\to E_{\mu}\Big\{ \int\limits_E c(x, u(\pi_i))\pi_i(dx)\Big\} \quad \text{as } m \to \infty,
\end{aligned}
\tag{3.213}
$$

uniformly in μ from compact subsets of $P(E)$.

Notice first that

$$
\begin{aligned}
&E_{\mu,\overline{\mu}}\Big\{ \int\limits_E c(x, \widetilde{\mathcal{L}}_m u(\overline{\pi}_i^{m(\overline{\mu})}))\widetilde{\pi}_i^{m(\mu,\overline{\mu})}(dx)\Big\} = \\
&= E_{\mu,\mu}\Big\{ \int\limits_E c(x, \overline{\mathcal{L}}_m u(\pi_i^{m(\mu)})\pi_i^{m(\mu,\mu)}(dx)\Big\}
\end{aligned}
\tag{3.214}
$$

Thus, by Lemma 3.16, (3.213) is satisfied for $i = 0$.

For $i > 0$ we have

$$
\begin{aligned}
&E_{\mu,\mu}\Big\{ \int\limits_E c(x, \overline{\mathcal{L}}_m u(\pi_i^{m(\mu)}))\pi_i^{m(\mu,\mu)}(dx)\Big\} = \\
&= E_{\mu,\mu}\{C_m(\pi_i^{m(\mu)}, \pi_i^{m(\mu,\mu)}\} = (T_m)^i C_m(\mu, \mu)
\end{aligned}
\tag{3.215}
$$

with

$$C_m(\nu, \mu) := \int\limits_E c(x, \overline{\mathcal{L}}_m u(\nu))\mu(dx)$$

By Lemma 3.21(i), $C_m(\nu, \mu) \to C(\nu, \mu) := \int_E c(x, u(\nu))\mu(dx)$ as $m \to \infty$, uniformly in (ν, μ) from compact subsets of $P(E) \times P(E)$. Noting again from (3.202) that $\pi_n^{(\mu)} = \pi_n^{(\mu,\mu)} = \pi_n$, from Proposition 3.56 we obtain

$$
\begin{aligned}
&(T_m)^i C_m(\mu, \mu) \to (T)^i C(\mu, \mu) = \\
&= E_{\mu,\mu}\Big\{ \int\limits_E c(x, u(\pi_i^{(\mu)}))\pi_i^{(\mu,\mu)}(dx)\Big\} = \\
&= E_{\mu}\Big\{ \int_E c(x, u(\pi_i))\pi_i(dx)\Big\}
\end{aligned}
\tag{3.216}
$$

as $m \to \infty$, uniformly in μ from compact subsets of $P(E)$. Summarizing (3.214)–(3.216), we obtain (3.213) and consequently (3.212).

■

Corollary 3.57 concludes our approach for the construction of nearly optimal controls in infinite horizon problems with discounting when no measure transformation is used and the context is that of Section 3.3:

First determine a nearly optimal control function u by using any of the methods described in section 3.3 and make sure that this function is continuous (automatically true for some of the methods). Applying the extension procedure (described following (3.66) or below (3.71)) this function can be considered as an element of the class $\mathcal{A} = C(P(E), U)$. For an initial measure $\mu \in P(E)$ and corresponding $\eta = \overline{\mu} = (\mu(B_1^m), \ldots, \mu(B_{k_m}^m))$, compute then the approximating filter process $(\overline{\pi}_n^{m(\overline{\mu})})$ according to (3.204) for a sufficiently large value of m. Since $\overline{\pi}_n^{m(\overline{\mu})} \in P(E_m)$, this process can actually be determined.

A nearly optimal control is then obtained by choosing in the generic period n a control value a_n given by $a_n = \widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\overline{\mu})})$.

3.5.2 Filter approximation and near optimal control values in the context of section 3.4

This subsection parallels the previous one and considers the context of section 3.4. We shall in fact construct an approximating real filtering process $\pi_n^{m(\mu)} \in P(E)$ (see (3.220) below), whose finite dimensional statistics $(\pi_n^{m(\mu)}(\varphi_1), \ldots, \pi_n^{m(\mu)}(\varphi_m^m))$ can be computed recursively according to (3.144) and where we use the real observations (y_n) instead of the fictitious ones (y_n^m). We shall then show that the nearly optimal control functions derived in section 3.4, provided they are continuous, yield nearly optimal controls when applied to $(\pi_n^{m(\mu)})$.

At this stage let us point out the double usage of the index m: In section 3.4 it was used to index the approximation induced by considering the approximating transition operators $P_m^a(x, dz)$ in (3.140); it determines the number of elements in the statistic $(\pi_n^u(\varphi_1^m), \ldots, \pi_n^u(\varphi_m^m))$ to be used as arguments of a nearly optimal control function u^m as in (3.178). In this section it will index a certain approximating process (see (3.220) below) and also determine the number of elements to be considered in the statistic

$(\pi_n^{m(\mu)}(\varphi_1^m), \ldots, \pi_n^{m(\mu)}(\varphi_m^m))$ when computing it according to the recursions specified below (3.220) that correspond to those in (3.144). Theorem 3.51 in section 3.4 states that, for m sufficiently large, the control function u^m is nearly optimal. Similarly, in Proposition 3.59 below we shall show that, for m sufficiently large, the approximating process in (3.220) that will provide the arguments to be used in the nearly optimal control functions comes close in a certain sense to a limiting process related to the true filter (π_n).

Since it will be convenient to let the two indices vary independently from one another, below we shall denote by $\overline{m}$ the index in the first usage, while we shall leave it as m in the second usage. Of the statistic $(\pi_n^{m(\nu)}(\varphi_1^m), \ldots, \pi_n^{m(\nu)}(\varphi_m^m))$ we shall then use only the first $\overline{m}$ components as arguments of the nearly optimal control function thereby requiring $m > \overline{m}$. To ensure consistency, we shall then also have to require that, for $m' > m$

$$\{\varphi_1^{m'}, \ldots, \varphi_{m'}^{m'}\} \supset \{\varphi_1^m, \ldots, \varphi_m^m\} \tag{3.217}$$

Furthermore, we shall require

(C12) $\varphi_i^m \in C(E) \quad$ for $i = 1, 2, \ldots, m$

Fix now a positive integer $\overline{m}$ and recall from section 3.4 that the nearly optimal control functions are pairs of Borel functions of the form

$$\begin{aligned} &u_0: P(E) \mapsto U \\ &u: R^d \times [0, \|\varphi_1^{\overline{m}}\|] \times \ldots \times [0, \|\varphi_{\overline{m}}^{\overline{m}}\|] \times U \mapsto U \end{aligned} \tag{3.218}$$

that we shall now require to be continuous.

Again, since the approximating process $(\pi_n^{m(\mu)})$ itself will not be Markov, we shall consider pairs of processes, each pair consisting of an approximating and a "real" filter. Actually, since the control functions in the present context depend also on the values of the current observation and the previous control, we shall more precisely consider quadruples of processes. Notice however that, since here the approximating filter $(\pi_n^{m(\mu)})$ takes values in $P(E)$ (wee need to compute only its statistic $(\pi_n^{m(\mu)}(\varphi_1^{\overline{m}}), \ldots, \pi_n^{m(\mu)}(\varphi_{\overline{m}}^{\overline{m}}))$, contrary to the previous subsection here we shall consider only two such quadruples, each corresponding to one of the two processes (π_n) and (π_n^m) in (1.7) and (3.14) respectively.

Given a fixed pair (u_0, u) of continuous functions of the form (3.218), for $\nu, \mu \in P(E)$ let then

$$q_{n+1} := (y_{n+1}, \pi_n^{(\nu)}, \pi_n^{(\mu,\nu)}, a_n), \tag{3.219}$$

where

$$\begin{aligned}
&\pi_0^{(\nu)}(\cdot) = \nu(\cdot), \quad \pi_0^{(\mu,\nu)}(\cdot) = \mu(\cdot), \quad a_0 = u_0(\nu) \\
&a_{n+1} = u(y_{n+1}, \pi_n^{(\nu)}(\varphi_1^{\overline{m}}), \dots, \pi_n^{(\nu)}(\varphi_{\overline{m}}^{\overline{m}}), a_n) \\
&\pi_{n+1}^{(\nu)}(\cdot) = M^{a_n}(y_{n+1}, \pi_n^{(\nu)})(\cdot), \quad \pi_{n+1}^{(\mu,\nu)}(\cdot) = M^{a_n}(y_{n+1}, \pi_n^{(\mu,\nu)})(\cdot)
\end{aligned}$$

and

$$q_{n+1}^m := (y_{n+1}, \pi_n^{m(\nu)}, \pi_n^{m(\mu,\nu)}, a_n), \tag{3.220}$$

where

$$\begin{aligned}
&\pi_0^{m(\nu)}(\cdot) = \nu(\cdot), \quad \pi_0^{m(\mu,\nu)}(\cdot) = \mu(\cdot), \quad a_0 = u_0(\nu) \\
&a_{n+1} = u(y_{n+1}, \pi_n^{m(\nu)}(\varphi_1^{\overline{m}}), \dots, \pi_n^{m(\nu)}(\varphi_{\overline{m}}^{\overline{m}}), a_n) \\
&\pi_{n+1}^{m(\nu)}(\cdot) = M_m^{a_n}(y_{n+1}, \pi_n^{m(\nu)})(\cdot), \quad \pi_{n+1}^{m(\mu,\nu)}(\cdot) = M^{a_n}(y_{n+1}, \pi_n^{m(\mu,\nu)})(\cdot)
\end{aligned}$$

where, again, y_n denote the real observations of (x_n) according to (1.1) when the controls are given by (a_0, a_n) as specified below (3.219) and (3.220) respectively.

Lemma 3.58 *The processes q_{n+1}, q_{n+1}^m are Markov with respect to the σ-field Y^{n+1} and have the transition operators T and T_m defined below where $F \in b\mathcal{B}(R^d \times P(E) \times P(E) \times U)$*

$$\begin{aligned}
TF(y, \nu_1, \nu_2, a) = &\int_E \int_{R^d} F(\overline{y}, M^a(y, \nu_1), M^a(y, \nu_2), \\
&u(y, \nu_1(\varphi_1^{\overline{m}}), \dots, \nu_1(\varphi_{\overline{m}}^{\overline{m}}), a)) r(z, \overline{y}) d\overline{y} M^a(y, \nu_2)(dz)
\end{aligned} \tag{3.221}$$

$$\begin{aligned}
T_m F(y, \nu_1, \nu_2, a) = &\int_E \int_{R^d} F(\overline{y}, M_m^a(y, \nu_1), M^a(y, \nu_2), \\
&u(y, \nu_1(\varphi_1^{\overline{m}}), \dots, \nu_1(\varphi_{\overline{m}}^{\overline{m}}), a)) r(z, \overline{y}) d\overline{y} M^a(y, \nu_2)(dz)
\end{aligned} \tag{3.222}$$

Moreover, under (A1)–(A4), (C12), the operator T is Feller, and, assuming additionally (C11), we have the Feller property also of T_m.

P r o o f. The proof of Markov property follows the considerations of Lemma 1.3 (see also Lemma 3.55). The Feller property is a simple implication of Proposition 1.4 and the assumption (C12).

■

Proposition 3.59 *Assume (A1)–(A4), (C12) and (D1),(D2). Then if* $\mathcal{B}(R^d \times P(E) \times P(E) \times U) \ni F_m \to F \in C(R^d \times P(E) \times P(E) \times U)$ *as* $m \to \infty$ *uniformly on compact subsets of* $R^d \times P(E) \times P(E) \times U$ *and the functions* F_m *are uniformly bounded, we have for* $k = 1, 2, \ldots$

$$(T_m)^k F_m(y, \nu_1, \nu_2, a) \to (T)^k F(y, \nu_1, \nu_2, a) \tag{3.223}$$

as $m \to \infty$ *uniformly in* (y, ν_1, ν_2, a) *from compact subsets of* $R^d \times P(E) \times P(E) \times U$.

P r o o f. By the proof of Proposition 3.56 it suffices to show (3.223) for $k = 1$. Let L and Γ, Γ_1 be compact subsets of R^d, and $P(E)$ respectively. For a given $\varepsilon > 0$ one can find compact sets $K \subset E$, $L_1 \subset R^d$ such that

$$\sup_{\nu_2 \in \Gamma} \sup_{y \in L} \sup_{a \in U} M^a(y, \nu_2)(K^c) \leq \varepsilon$$

and

$$\sup_{z \in K} \int_{L_1^c} r(z, y) dy \leq \varepsilon$$

Then

$$\begin{aligned}
&\sup_{y \in L} \sup_{\nu_1 \in \Gamma} \sup_{\nu_2 \in \Gamma} \sup_{a \in U} |T_m F_m(y, \nu_1, \nu_2, a) - TF(y, \nu_1, \nu_2, a)| \\
&\leq 2\varepsilon(\|F\| + \|F_m\|) + \sup_{\overline{y} \in L_1} \sup_{y \in L} \sup_{\nu_1 \in \Gamma} \sup_{\nu_2 \in \Gamma} \sup_{a \in U} \\
&|F_m(\overline{y}, M^a_m(y, \nu_1), M^a(y, \nu_2), u(y, \nu_1(\varphi_1^{\overline{m}}), \ldots, \nu_1(\varphi_{\overline{m}}^{\overline{m}}), a)) \\
&-F(\overline{y}, M^a(y, \nu_1), M^a(y, \nu_2), u(y, \nu_1(\varphi_1^{\overline{m}}), \ldots, \nu_1(\varphi_{\overline{m}}^{\overline{m}}), a))|
\end{aligned}$$

By Proposition 3.5, Lemma 3.8 and the compactness of the set $\{(\overline{y}, M^a(y, \nu_1), M^a(y, \nu_2), u(y, \nu_1(\varphi_1^{\overline{m}}), \ldots, \nu_1(\varphi_{\overline{m}}^{\overline{m}}), a)$ with $\overline{y} \in L_1, y \in L, \nu_1 \in \Gamma, \nu_2 \in \Gamma, a \in U\}$ we obtain (3.223) for $k = 1$.

■

The near optimality of the control $a_0^m = u_0(\mu)$, $a_n^m = u(y_n, \pi_{n-1}^{m(\mu)}(\varphi_1^{\overline{m}}), \ldots, \pi_{n-1}^{m(\mu)}(\varphi_{\overline{m}}^{\overline{m}}), a_{n-1}^m)$ follows now from

Corollary 3.60 *Under the assumptions of Proposition 3.59 for* $u_0 \in C(P(E),U)$, $u \in C(R^d \times [\|0, \varphi_1^{\overline{m}}\|] \times \ldots \times [\|0, \varphi_{\overline{m}}^{\overline{m}}\|] \times U, U)$,

$$\begin{aligned} &a_0^m = u_0(\mu) \\ &\ldots \\ &a_{n+1}^m = u(y_{n+1}, \pi_n^{m(\mu)}(\varphi_1^{\overline{m}}), \ldots, \pi_n^{m(\mu)}(\varphi_{\overline{m}}^{\overline{m}}), a_n^m) \\ &\ldots \end{aligned} \tag{3.224}$$

and

$$\begin{aligned} &a_0 = u_0(\mu) \\ &\ldots \\ &a_{n+1} = u(y_{n+1}, \pi_n(\varphi_1^{\overline{m}}), \ldots, \pi_n(\varphi_{\overline{m}}^{\overline{m}}), a_n) \\ &\ldots \end{aligned} \tag{3.225}$$

We have

$$J_\mu^\beta(a_n^m) \to J_\mu^\beta(a_n) \tag{3.226}$$

as $m \to \infty$ *uniformly in* μ *from compact subsets of* $P(E)$.

P r o o f. By (3.145)

$$\begin{aligned} &J_\mu^\beta(a_n^m) = \int_E c(x, u_0(\mu))\mu(dx) + \\ &+ \sum_{n=1}^{\infty} \beta^n E_\mu^u \{C(u(y_n, \pi_{n-1}^{m(\mu)}(\varphi_1^{\overline{m}}), \ldots, \pi_{n-1}^{m(\mu)}(\varphi_{\overline{m}}^{\overline{m}}), a_{n-1}^m), y_n, \\ &a_{n-1}^m, \pi_{n-1}^{(\mu,\mu)}\} = \int_E c(x, u_0(\mu))\mu(dx) + \\ &+ \sum_{n=1}^{\infty} \beta^n E_\mu^u \{F(y_n, \pi_{n-1}^{m(\mu)}, \pi_{n-1}^{(\mu,\mu)}, a_{n-1}^m)\} = \\ &= \int_E c(x, u_0(\mu))\mu(dx) + \sum_{n=1}^{\infty} \beta^n E_\mu^u \{(T_m)^{n-1} F(y_1, \mu, \mu, u_0(\mu))\} \end{aligned} \tag{3.227}$$

defining thus implicitly the function F that is easily seen to be continuous.

Similarly, noticing that $\pi_n^{(\mu,\mu)} = \pi_n$

$$\begin{aligned}
J_\mu^\beta(a_n) &= \int_E c(x,u_0(\mu))\mu(dx) + \\
&+ \sum_{n=1}^{\infty} \beta^n E_\mu^u \{C(u(y_n, \pi_{n-1}(\varphi_1^{\overline{m}}), \ldots, \pi_{n-1}(\varphi_{\overline{m}}^{\overline{m}}), a_{n-1}), y_n, a_{n-1}, \pi_{n-1})\} \\
&= \int_E c(x,u_0(\mu))\mu(dx) + \sum_{n=1}^{\infty} \beta^n E_\mu^u \{F(y_n, \pi_{n-1}, \pi_{n-1}, a_{n-1})\} = \\
&= \int_E c(x,u_0(\mu))\mu(dx) + \sum_{n=1}^{\infty} \beta^n E_\mu^u \{(T)^{n-1} F(y_1, \mu, \mu, u_0(\mu))\}
\end{aligned} \tag{3.228}$$

Therefore by Proposition 3.59 we obtain (3.226).

■

Analogously to Corollary 3.57, the previous Corollary 3.60 concludes our approach for the construction of nearly optimal controls in infinite horizon problems with discounting and in the context of section 3.4:

First determine for a sufficiently large $\overline{m}$ a nearly optimal control function pair (u_0, u) according to the method of section 3.4 and make sure that u_0 and u are continuous.

For the initial measure $\mu \in P(E)$ and for sufficiently large m, with $m > \overline{m}$, compute, according to (3.144) and with the use of the real observations y_n, the statistics $(\pi_n^{m(\mu)}(\varphi_1^m), \ldots, \pi_n^{m(\mu)}(\varphi_m^m))$. A nearly optimal control is then given by (3.224).