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Asymptotyka trasera w polu lokalnie stacjonarnym

Passive Tracer Model

One of the simplest formulations of the problem of a turbulent diffusion and the problem of a turbulent transport is a *passive tracer model.* This model is described by the formula

$$\begin{cases} \dfrac{dX(t)}{dt} = \mathbf{V}(t, X(t)), \\ X(0) = 0, \end{cases}$$

where $\mathbf{V}(t,x)$ is some random field. It is a classical model of statistic hydrodynamics and it is described in details in monograph [7]. The physical interpretation of this model may be the observation of a particle of a pollution moving in a turbulent flow. This particle is called a tracer. The trajectory of the tracer is modeled by the process $X(t)$. The equation describes microscopic dynamics, but observations are done in a macroscopic scale. So we need to apply some rescaling in order to have the situation that the macroscopic observation shows the microscopic changes. The passive tracer model after diffusive rescaling is as follows

$$\begin{cases} \dfrac{dX^\varepsilon(t)}{dt} = \dfrac{1}{\varepsilon}\mathbf{V}\Big(\dfrac{t}{\varepsilon^2}, \dfrac{X^\varepsilon(t)}{\varepsilon}\Big) \\ X^\varepsilon(0) = 0. \end{cases} \tag{1}$$

The above equation was studied mostly in the case when the field $\mathbf{V}$ is divergent-free and stationary ([1],[4],[5],[6],[3]). Our aim is to move away from the stationarity assumption.

One of the basic questions in studying asymptotic behavior of the tracer is the question about the law of large numbers (LLN) and the central limit theorem (CLT). The law of large numbers states that there exists a deterministic vector $v_* \in \mathbb{R}^d$ (called *Stoke's drift*), such that $X(t)/t \to v_*$ a.e. If it is possible to show LLN, then we can ask about CLT, i.e. does $(\mathbf{X}(t) - v_* t)/\sqrt{t}$ converge in distribution when $t \to +\infty$ to normal vector $N(0,\kappa)$? The covariance matrix $\kappa = [\kappa_{i,j}],\ i,j = 1,\ldots d$ is called the *turbulent diffusivity* of the tracer. With the assumption that the vector field $\{\mathbf{V}(t,x), x \in \mathbb{R}^d\}$ is Markovian, CLT was proved in [1],[3]. Another important CLT was obtained in 1997 by [6] (see also [2]). The authors of this result deal with zero mean, stationary, divergent-free, Gaussian and T-dependent field $\mathbf{V}(t,x)$. T-dependence means that there exists some $T > 0$, such that covariance matrix satisfies $|\mathbf{R}(t,x)| = 0$, $|t| > T$.

Locally Stationary Fields

In our presentation we would like to move away from the stationarity assumption. A natural generalization of this assumption is a concept of *local stationarity.* We want to study the asymptotic behavior of a tracer moving in a divergence-free, locally stationary vector field. In the homogenization theory the concept of *local stationarity* occurs in [8], [9]. The passive tracer model in a locally stationary field can be described by the following equation

$$\begin{cases} \dfrac{dX^\varepsilon(t)}{dt} = \dfrac{1}{\varepsilon}\mathbf{W}\Big(\dfrac{t}{\varepsilon^2}, \dfrac{X^\varepsilon(t)}{\varepsilon}, X^\varepsilon(t)\Big), \\ X^\varepsilon(0) = 0. \end{cases} \tag{2}$$

We introduce here the third argument of the field. It is the local stationarity parameter, i.e. for fixed $y \in \mathbb{R}^d$ the field $\mathbf{W}(t, x, \varepsilon y)$ is stationary and ergodic. We are interested in a situation when $\varepsilon \ll 1$. Note that this argument changes slowly. Namely, it changes in the macroscopic scale, while the microscopic time and space variables are scaled diffusively. We will study the behavior of the tracer $X^\varepsilon(t)$, when $\varepsilon \to 0$. The above model of the field is too general for us, to say something about the asymptotic behavior of the tracer. To do so, we will study fields constructed by us.

References

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