# Mathematical and physical aspect of the solution to the Lamb-Daniłowska problem for nonlocal thermoviscoelastic body

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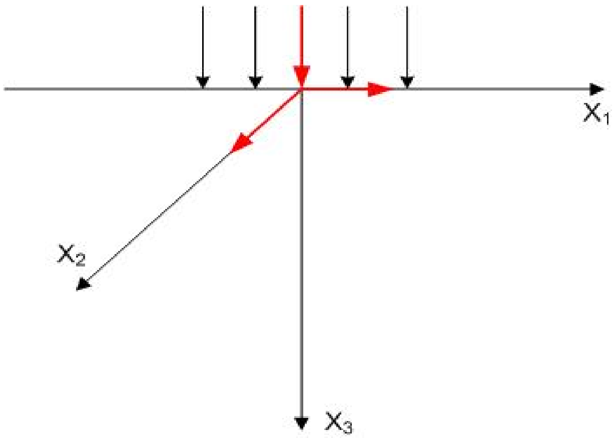
We consider the initial- boundary value problem for half-space of thermoviscoelastic body. Biot model describes the viscoelastic properties of the body and Gurtin-Pipkin describes the thermal properties of medium. We assume that on the surface of half-space acting the mechanical and thermal forces per unit area. Physical fields in that medium i.e. displacements, strain, temperature are described by system of partial differential and integral equations. For this system of equation we formulated the Lamb- Daniłowska initial-boundary value problem . We obtain the solution of that problem using Laplace-Fourier transformation and Cagniard- de Hoop method. The full geometry of propagation in this medium termo-mechanical and mechano-thermal waves are described.




## Basic notations:

θ – the temperature , u - the displacemet vector of the body, q- heat flux , e - internal energy

a(t), b(t), – the heat relaxation function

β(t) – the coefficient of thermo-mechanical coupling

λ(t) , μ(t) – Lame relaxation functions

υ - the velocity

c - specific heat

$\theta_0$ - the reference temperature

## System of equations describing nonlocal model of the thermoviscoelastic body

$$\partial_t(\rho\upsilon) - \nabla^T\sigma = 0$$
$$\partial_t(\rho e) + \nabla^T\bar{q} = 0$$
$$\sigma_{ik} = (\lambda *_t \varepsilon_{ll})\delta_{ik} + 2(\mu *_t \varepsilon_{ik}) - (\beta *_t \theta)\delta_{ik}\ ;\ \ \bar{q}_k = -a *_t \partial_k\theta;\ \ e = e_0 + (c *_t \theta) + \theta_0(\beta *_t \varepsilon_{ll});$$
$$\upsilon = \partial_t u\ ;\ \ \varepsilon_{ik} = \frac{1}{2}(\partial_k u_i + \partial_i u_k)$$

## Lamb –Daniłowska problem for the nonlocal model of the thermoviscoelastic body

$$\partial_t(\rho\upsilon_i) = \lambda * (\partial_k Tr(\varepsilon_{mn})\delta_{ik}) + 2\mu * (\partial_k\varepsilon_{ik}) - \beta * (\partial_i\theta)$$
$$\partial_t((\rho c * \theta) + \theta_0\beta * Tr(\varepsilon_{mn})) = a * \partial_{nn}^2\theta$$
$$\partial_t\varepsilon_{ik} = d_{ik} = \tfrac{1}{2}(\partial_i\upsilon_k + \partial_k\upsilon_i)$$
$$\upsilon_i = 0,\ \ \varepsilon_{ik} = 0,\ \ \theta = 0 \quad for\ \ t \le 0$$
$$\sigma_{i3} = -p_i, \qquad q_3 + \kappa(\theta - \theta_0) = \varphi \quad for \quad x_3 = 0$$

Using Laplace transformation with respect to *t* and the Fourier transformation with respect to $x_1$, $x_2$ we get

$$\begin{cases} \rho\hat{\hat{\upsilon}}_i = \bar{\lambda} l_k \hat{\hat{\varepsilon}}\delta_{ik} + 2\bar{\mu} l_k \hat{\hat{\varepsilon}}_{ik} - \bar{\beta} l_k \hat{\hat{\theta}}\delta_{ik} \\ \left(s(\rho\bar{c}) - \bar{a} l^2\right)\hat{\hat{\theta}} + s\rho\theta_0\bar{\beta}\hat{\hat{\varepsilon}} = 0 \\ s\hat{\hat{\varepsilon}}_{ik} = \tfrac{1}{2}\left(l_k\hat{\hat{\upsilon}}_i + l_i\hat{\hat{\upsilon}}_k\right)\ or\ \ s\hat{\hat{\varepsilon}}_{ii} = l_i\hat{\hat{\upsilon}}_i,\ \ \hat{\hat{\varepsilon}} = \hat{\hat{\varepsilon}}_{ik}\delta_{ik} \end{cases}$$

$$l_1 = -i\xi_1,\ \ l_2 = -i\xi_2,\ \ l_3 = -\partial_3$$
$$l^2 = l_1^2 + l_2^2 + l_3^2$$

After some calculations we obtain

$$\begin{cases} \left(\frac{s}{\bar{c}_1^2} - l^2\right)\hat{\hat{\varepsilon}} + \frac{\bar{\beta} l^2}{\bar{\lambda}+2\bar{\mu}}\hat{\hat{\theta}} = 0 \\ \frac{\rho s \theta_0 \bar{\beta}}{\bar{a}}\hat{\hat{\varepsilon}} + \left(\frac{s}{\bar{c}_T^2} - l^2\right)\hat{\hat{\theta}} = 0 \end{cases}, \qquad \text{where} \qquad \bar{c}_1^2 = \frac{\bar{\lambda}+2\bar{\mu}}{\rho} \quad \bar{c}_T^2 = \frac{\bar{a}}{\rho\bar{c}}$$

$$l_i l_j \hat{\hat{\varepsilon}}_{(ij)} = \frac{l_i^2 l_j^2}{\bar{l}_1^2}\hat{\hat{\varepsilon}}_1 + \frac{l_i^2 l_j^2}{\bar{l}_3^2}\hat{\hat{\varepsilon}}_3 + + \frac{1}{2}\left(l_i^2 \hat{\hat{\varepsilon}}^0_{(jj)} + l_j^2 \hat{\hat{\varepsilon}}^0_{(ii)}\right), \quad l_i \hat{\hat{\upsilon}}_i = s\left(\frac{l_i^2}{\bar{l}_1^2}\hat{\hat{\varepsilon}}_1 + \frac{l_i^2}{\bar{l}_3^2}\hat{\hat{\varepsilon}}_3 + \hat{\hat{\varepsilon}}^0_{(ii)}\right)$$

Where: $\hat{\hat{\varepsilon}}_1 = A_1 e^{-\gamma_1 x_3}$, $\hat{\hat{\varepsilon}}_3 = A_3 e^{-\gamma_3 x_3}$, $\hat{\hat{\varepsilon}}^0_{11} = B_1 e^{-\gamma_2 x_3}$, $\hat{\hat{\varepsilon}}^0_{22} = B_2 e^{-\gamma_2 x_3}$, $\hat{\hat{\varepsilon}}^0_{33} = B_3 e^{-\gamma_2 x_3}$

$\bar{l}_2^2 = \dfrac{s}{\bar{c}_2^2}$ and $\bar{l}_1^2, \bar{l}_3^2$ – the roots of characteristic polynomial $\dfrac{s}{\bar{c}_T^2}\bar{\delta}^2 \bar{l}^2 - \left(\dfrac{s}{\bar{c}_T^2} - \bar{l}^2\right)\left(\dfrac{s}{\bar{c}_1^2} - \bar{l}^2\right) = 0$, where $\bar{\delta}^2 = \dfrac{\bar{\beta}^2 \theta_0}{\rho \bar{c} \bar{c}_1^2}$.

The important quantities in our investigation are:

$\gamma_2 = \sqrt{\bar{l}_2^2 + \omega^2}$, $\gamma_1 = \sqrt{\bar{l}_1^2 + \omega^2}$, $\gamma_3 = \sqrt{\bar{l}_3^2 + \omega^2}$, $\omega^2 = \xi_1^2 + \xi_2^2$,

$$\bar{l}_1^2 = \frac{s}{2}\left(\left(\frac{1}{\bar{c}_1^2} + \frac{1}{\bar{c}_T^2} + \frac{\bar{\delta}^2}{\bar{c}_T^2}\right) + \sqrt{\left(\frac{1}{\bar{c}_1^2} + \frac{1}{\bar{c}_T^2} + \frac{\bar{\delta}^2}{\bar{c}_T^2}\right)^2 - \frac{4}{\bar{c}_1^2 \bar{c}_T^2}}\right), \quad \bar{l}_3^2 = \frac{s}{2}\left(\left(\frac{1}{\bar{c}_1^2} + \frac{1}{\bar{c}_T^2} + \frac{\bar{\delta}^2}{\bar{c}_T^2}\right) - \sqrt{\left(\frac{1}{\bar{c}_1^2} + \frac{1}{\bar{c}_T^2} + \frac{\bar{\delta}^2}{\bar{c}_T^2}\right)^2 - \frac{4}{\bar{c}_1^2 \bar{c}_T^2}}\right)$$

**We present now in explicite form the quantities which will be applied in boundary conditions**

$$\bar{\beta}\hat{\bar{\theta}} = -\left(\frac{s}{\bar{c}_1^2} - \bar{l}_1^2\right)\frac{\rho\bar{c}_1^2}{\bar{l}_1^2}\hat{\varepsilon}_1 - \left(\frac{s}{\bar{c}_1^2} - \bar{l}_3^2\right)\frac{\rho\bar{c}_1^2}{\bar{l}_3^2}\hat{\varepsilon}_3$$

$$\hat{\varepsilon}_{33} = \frac{\gamma_1^2}{\bar{l}_1^2}\hat{\varepsilon}_1 + \frac{\gamma_3^2}{\bar{l}_3^2}\hat{\varepsilon}_3 + \hat{\varepsilon}_{33}^0 \; ; \; \hat{\varepsilon}_{23} = -\frac{l_2\gamma_1}{\bar{l}_1^2}\hat{\varepsilon}_1 - \frac{l_2\gamma_3}{\bar{l}_3^2}\hat{\varepsilon}_3 - \frac{l_2^2\hat{\varepsilon}_{33}^0 + \gamma_2^2\hat{\varepsilon}_{22}^0}{2l_2\gamma_2} \; ; \; \hat{\varepsilon}_{13} = -\frac{l_1\gamma_1}{\bar{l}_1^2}\hat{\varepsilon}_1 - \frac{l_1\gamma_3}{\bar{l}_3^2}\hat{\varepsilon}_3 - \frac{l_1^2\hat{\varepsilon}_{33}^0 + \gamma_2^2\hat{\varepsilon}_{11}^0}{2l_1\gamma_2}$$

**The boundary condition have in this problem the following form**

$$\bar{\lambda}\hat{\varepsilon} + 2\bar{\mu}\hat{\varepsilon}_{33} - \beta\hat{\bar{\theta}} = -\hat{p}_3 \; ; \; 2\bar{\mu}\hat{\varepsilon}_{23} = -\hat{p}_2 ; \;\; 2\bar{\mu}\hat{\varepsilon}_{13} = -\hat{p}_1 ; \;\; \bar{a}\partial_3\hat{\bar{\theta}} = \hat{p}_0$$

**Introducing the above function into boundary conditions we obtain the following system of equations**

$$\begin{cases} \left(2\omega^2 - \dfrac{s}{\bar{c}_2^2}\right)C_1 + \left(2\omega^2 - \dfrac{s}{\bar{c}_2^2}\right)C_3 + 2B_3 = \dfrac{\bar{p}_3}{\bar{\mu}} \\ 2\omega^2\gamma_1\gamma_2 C_1 + 2\omega^2\gamma_2\gamma_3 C_3 + \left(2\omega^2 - \dfrac{s}{\bar{c}_2^2}\right)B_3 = -\dfrac{\gamma_2}{\bar{\mu}}\left(\hat{p}_1 l_1 + \hat{p}_2 l_2\right) \\ \gamma_1\left(\bar{l}_1^2 - \dfrac{s}{\bar{c}_1^2}\right)C_1 + \gamma_3\left(\bar{l}_3^2 - \dfrac{s}{\bar{c}_1^2}\right)C_3 = -\dfrac{\hat{p}_0}{\bar{\mu}}\dfrac{\bar{\beta}\chi^2}{\bar{a}} \end{cases}$$

**The main determinant of the system of equation has form**

$$W = \begin{vmatrix} \left(2\omega^2 - \dfrac{s}{\bar{c}_2^2}\right) & \left(2\omega^2 - \dfrac{s}{\bar{c}_2^2}\right) & 2 \\ 2\omega^2\gamma_1\gamma_2 & 2\omega^2\gamma_2\gamma_3 & \left(2\omega^2 - \dfrac{s}{\bar{c}_2^2}\right) \\ \gamma_1\left(\bar{l}_1^2 - \dfrac{s}{\bar{c}_1^2}\right) & \gamma_3\left(\bar{l}_3^2 - \dfrac{s}{\bar{c}_1^2}\right) & 0 \end{vmatrix}$$

**Remark**: for $\bar{\beta} = 0$ we get well known Rayleigh's determinant

$$W_R = \begin{vmatrix} \left(2\omega^2 - \dfrac{s}{\bar{c}_2^2}\right) & 2 \\ 2\omega^2\gamma_1\gamma_2 & \left(2\omega^2 - \dfrac{s}{\bar{c}_2^2}\right) \end{vmatrix} = \left(2\omega^2 - \frac{s}{\bar{c}_2^2}\right)^2 - 4\omega^2\gamma_1\gamma_2$$

**<u>The Lamb boundary value problem</u> :** $\hat{\bar{\sigma}}_{33} = -\hat{\bar{p}}_3,\ \hat{\bar{\sigma}}_{23} = 0, \hat{\bar{\sigma}}_{13} = 0,\ \hat{\bar{q}}_3 = 0$

**<u>The Daniłowska boundary value problem</u> :** $\hat{\bar{\sigma}}_{33} = 0,\ \hat{\bar{\sigma}}_{23} = 0, \hat{\bar{\sigma}}_{13} = 0,\ \hat{\bar{q}}_3 = -\hat{\bar{p}}_0$

**Solution to Lamb-Daniłowska boundary value problem:**

$$\hat{\bar{\varepsilon}}_{23} = -l_2\gamma_1 C_1 e^{-\gamma_1 x_3} - l_2\gamma_3 C_3 e^{-\gamma_3 x_3} - \frac{l_2}{2\gamma_2} B_3 e^{-\gamma_2 x_3} - \frac{\gamma_2}{2l_2} B_2 e^{-\gamma_2 x_3}\ ,\ \ \hat{\bar{\varepsilon}}_{13} = -l_1\gamma_1 C_1 e^{-\gamma_1 x_3} - l_1\gamma_3 C_3 e^{-\gamma_3 x_3} - \frac{l_1}{2\gamma_2} B_3 e^{-\gamma_2 x_3} - \frac{\gamma_2}{2l_1} B_1 e^{-\gamma_2 x_3}\ ,$$

$$\hat{\hat{\varepsilon}}_{12} = -l_1 l_2 C_1 e^{-\gamma_1 x_3} - l_1 l_2 C_3 e^{-\gamma_3 x_3} + \frac{l_1}{2l_2} B_2 e^{-\gamma_2 x_3} - \frac{l_2}{2l_1} B_1 e^{-\gamma_2 x_3}$$

$$\hat{\hat{\varepsilon}}_{11} = -l_1^2 C_1 e^{-\gamma_1 x_3} + l_1^2 C_3 e^{-\gamma_3 x_3} + B_1 e^{-\gamma_2 x_3}, \quad \hat{\hat{\varepsilon}}_{22} = l_2^2 C_1 e^{-\gamma_1 x_3} + l_2^2 C_3 e^{-\gamma_3 x_3} + B_2 e^{-\gamma_2 x_3} \quad , \quad \hat{\hat{\varepsilon}}_{33} = \gamma_1^2 C_1 e^{-\gamma_1 x_3} + \gamma_3^2 C_3 e^{-\gamma_3 x_3} + B_3 e^{-\gamma_2 x_3}$$

$$\bar{\beta}\hat{\hat{\theta}} = -\left(\frac{s}{\bar{c}_1^2} - \bar{l}_1^2\right)\rho\bar{c}_1^2 C_1 e^{-\gamma_1 x_3} - \left(\frac{s}{\bar{c}_1^2} - \bar{l}_3^2\right)\rho\bar{c}_1^2 C_3 e^{-\gamma_3 x_3}$$

$$\hat{\hat{\upsilon}}_1 = s l_1\left(C_1 e^{-\gamma_1 x_3} + C_3 e^{-\gamma_3 x_3}\right) + \frac{s}{l_1} B_1 e^{-\gamma_2 x_3}, \quad \hat{\hat{\upsilon}}_2 = s l_2\left(C_1 e^{-\gamma_1 x_3} + C_3 e^{-\gamma_3 x_3}\right) + \frac{s}{l_2} B_2 e^{-\gamma_2 x_3} \quad , \quad \hat{\hat{\upsilon}}_3 = -s\left(\gamma_1 C_1 e^{-\gamma_1 x_3} + \gamma_3 C_3 e^{-\gamma_3 x_3}\right) - \frac{s}{\gamma_2} B_3 e^{-\gamma_2 x_3}$$

We introduce the solution vector $X = [\upsilon_1, \upsilon_2, \upsilon_3, \varepsilon, \theta, \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}]^T$, then

$$\hat{\hat{\bar{X}}}_k = \sum_{n=1}^{3} \hat{\hat{\bar{F}}}_{kn}(s, \xi_1, \xi_2) e^{-\gamma_n x_3} \qquad k = 1,2,\ldots,11; \text{ where } \gamma_n = \sqrt{\bar{l}_n^2 + \omega^2}, \quad \omega^2 = (-i\xi_1)^2 + (-i\xi_2)^2$$

Apply the inverse Fourier transform and de Hoop transformation we get

$$\bar{X}_k = \frac{1}{4\pi^2}\sum_{n=1}^{3} \bar{l}_n^2(s)\int_{R^2} \bar{\varphi}_{nk}(s)\hat{\bar{\Phi}}_{kn}(s,p,q) e^{-\bar{l}_n\left(x_3\sqrt{p^2+q^2+1} - irp\right)} dpdq$$

$$\eta_1 = \xi_1 \frac{x_1}{r} + \xi_2 \frac{x_2}{r} \quad \eta_2 = -\xi_1 \frac{x_2}{r} + \xi_2 \frac{x_1}{r} \qquad \text{where} \quad r = \sqrt{x_1^2 + x_2^2}, \quad \eta_1 = \bar{l}_n(s)p, \quad \eta_2 = \bar{l}_n(s)q$$

Modified Cagniar’s method - replacement of the integral with respect to real axis by the integral over suitable chosen integral curve

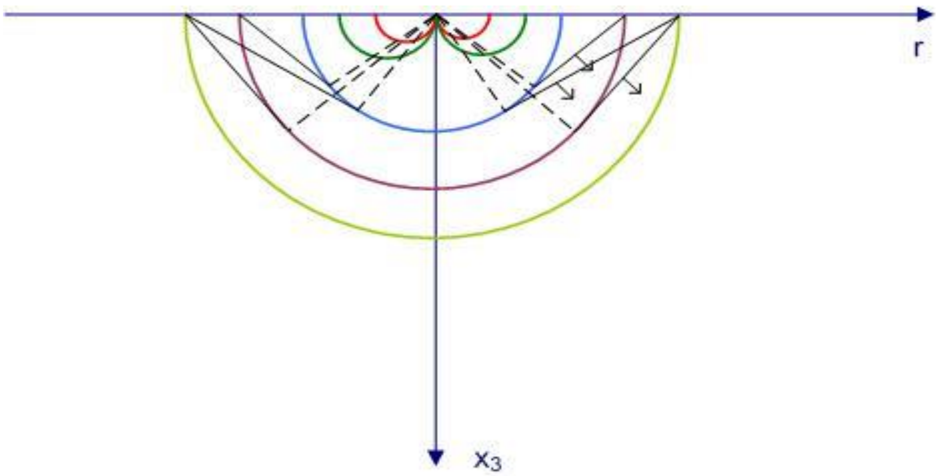
$$x_3\sqrt{p^2+q^2+1} - irp = \tau \quad , \quad p^{\pm}(\tau, x_3, r, q) = \frac{ir}{R^2}\tau \pm \frac{x_3}{R^2}\sqrt{\tau^2 - R^2\left(q^2+1\right)}, \quad \tau > R\sqrt{q^2+1}, \quad R = \sqrt{r^2 + x_3^2}$$

$$\overline{X}_k = \frac{1}{4\pi^2}\sum_{n=1}^{3} \bar{l}_n^2(s)\overline{\varphi}_{kn}(s)\int_{-\infty}^{\infty} dq \int_{R\sqrt{q^2+1}}^{\infty} \phi_{kn}(\tau, q) e^{-\bar{l}_n \tau} d\tau\,, \quad \phi_{kn}(\tau, q) = \left[\hat{\overline{\Phi}}_{nk}(p, q)\frac{\partial p}{\partial \tau}\right]_{p=p^-}^{p=p^+}$$

$$\overline{X}_k = \frac{1}{4\pi^2}\sum_{n=1}^{3}\int_0^{\infty} G_{kn}(s) e^{-\bar{l}_n(s)\tau} d\tau \left[H(\tau - R)\sqrt{\left(\frac{\tau}{R}\right)^2 - 1}\int_{-1}^{1}\tilde{\Phi}_{kn}(\tau, \eta) d\eta\right], \qquad G_{kn}(s) = -\bar{l}_n^2(s)\overline{\varphi}_{kn}, \;\; q = \eta\sqrt{\left(\frac{\tau}{R}\right)^2 - 1}$$

Appling Efros theorem we get $X_k(t, r, x_3) = \frac{1}{4\pi^2}\sum_{n=1}^{3}\int_R^{\infty} g_{kn}(t, \tau) R_{kn}(\tau, r, x_3) d\tau\,, \quad \int_{-1}^{1}\tilde{\Phi}_{kn}(\tau, \eta) d\eta = R_{kn}$

**Lamb –Daniłowska problem - the geometry of waves in nonlocal model**




Geometry of waves in nonlocal model: two Rayleigh’s (red, green), three cone waves ( with arrows), mechanical transversal wave (blue), coupled thermal wave (vaiolet), coupled logitudinal wave (light green).