Large deviations for random evolutions with global balans conditions

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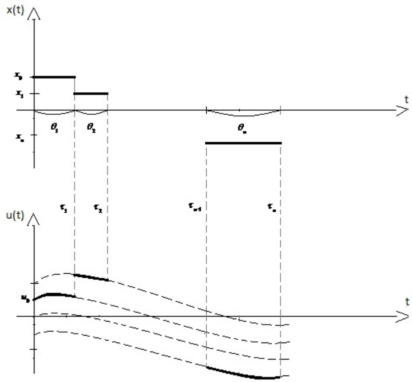
Two models of random evolutions

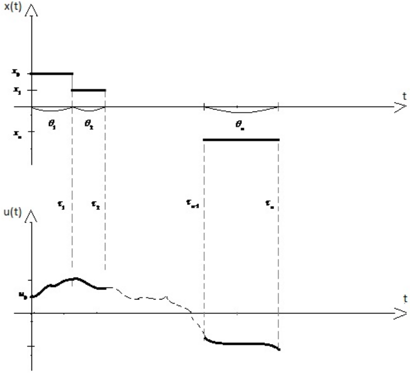
- 1) in case of accidental action on the evolution system jumps of some other trajectory of the same evolution



Random evolution with Markov switching

Two models of random evolutions

- 2) change random evolution or system which describes it continuing the movement from the position in which it was at the moment of impact on the trajectory of the new system evolution



Random evolution with Markov switching

Markov processes

- Switching process $x(t), t \geq 0,$ — jumping Markov process in dimensional phase space $(X, \mathbf{X})$
- Generator of Marov process

$$Q\varphi(x) = q(x) \int\limits_X P(x, dy)[\varphi(y) - \varphi(x)], \qquad (1)$$

where $P(x, B) = P\{x_{n+1} \in B | x_n = x\}$ – stochastic kernel, $B \in \mathbf{X}$,

- all operator are defined in Banach space of real-valued limited functions with supremum norm $\|\varphi(x)\| := \sup\limits_{x \in X} |\varphi(x)|$
- $\pi(B), B \in \mathbf{X}$ stationary distribution of Markov process

Random evolution with Markov switching

Continuous case. Averaging scheme.

- $du^{\varepsilon}(t) = C(u^{\varepsilon}(t); x(t/\varepsilon))dt, u^{\varepsilon}(0) = u_0 \in R^d,$
- $x_t^{\varepsilon} := x(t/\varepsilon)$– Markov process of switching.
- Markov process $u^{\varepsilon}(t), x_t^{\varepsilon}, t \geq 0, \varepsilon \longrightarrow 0,$ determined by generator $\mathbf{L}^{\varepsilon}\varphi(u;x) = [\varepsilon^{-1}Q + \mathbf{C}(x)]\varphi(u;x).$
- *There is* asymptotic representation $\mathbf{L}^{\varepsilon}\varphi^{\varepsilon}(u;x) = \hat{\mathbf{C}}\varphi(u) + \theta^{\varepsilon}(x)\varphi(u),$ $\sup_{x\in X}|\theta^{\varepsilon}(x)\varphi(u)| \longrightarrow 0, \varphi(u) \in C^2(R^d)$
- Weak convergence $u^{\varepsilon}(t) \Rightarrow \hat{u(t)}, \varepsilon \longrightarrow 0,$ takes place, where limited evolution satisfies equation $d\hat{u}(t)/dt = \hat{C}(\hat{u}(t)), \hat{u}(0) = u_0 \in R^d, \hat{C}(u) = \int_X \pi(dx)C(u;x)$
- Korolyuk, V. S., Limnios, N., (2005). Stochastic Systems in Merging Phase Space. World Scientific Publishing, 330 p.

Stochastic approximation procedure as controlled random evolution

Continuous stochastic approximation procedure

- Procedure of stochastic approximation $du^{\varepsilon}(t)/dt = a(t)C(u^{\varepsilon}(t), x(t/\varepsilon)), u^{\varepsilon}(0) = u$, where $a(t) > 0$ *control* function of properties $\int_0^{\infty} a(t)dt = \infty, \int_0^{\infty} a^2(t)dt < \infty$. Regression function $C(u;x), u \in R^d$, that there exists solution of equation $du_x(t)/dt = a(t)C(u_x(t), x), u_x(0) = u$.
- There is weak convergence $u^{\varepsilon}(t) \Rightarrow u^*, t \longrightarrow \infty$, where equilibrium point u^* satisfies equation $\hat{C}(u^*) = 0, \hat{C}(u) = \int_X \pi(dx)C(u;x)$
- Here controlled random evolution $u^{\varepsilon}(t)$ with MP x_t^{ε} has generator $\mathbf{L}^{\varepsilon}\varphi(u;x) = [\varepsilon^{-1}Q + a(t)\mathbf{C}(x)]\varphi(u;x)$.
- Chabanyuk, Ya.M., (2006). Continuous Procedure of Stochastic Approximation With Singular Perturbation Under Balance Conditions. Cybernetics and Systems Analysis, Vol. 42, No. 3, 133-139.

Stochastic optimization procedure as controlled random evolution

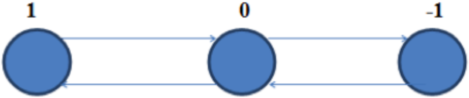
Stochastic optimization procedure as controlled random evolution

- Stochastic optimization procedure
  $du^{\varepsilon}(t)/dt = a(t)\,\bigtriangledown_{b(t)}\,C(u^{\varepsilon}(t), x(t/\varepsilon)), u^{\varepsilon}(0) = u\,,$
  where $\nabla_b C(u,\cdot) := \frac{C(u_i^+,\cdot)-C(u_i^-,\cdot)}{2b(t)}, (i=\overline{1,d}),$
  $u_i^{\pm} = u_i \pm b(t)e_i,\ e_i$ - vector with coordinates
  $\{0,...,1,0,...,0\}$.
- Weak convergence $u^{\varepsilon}(t) \Rightarrow u^*, t \longrightarrow \infty,$ takes place, where extremum point u^* satisfies equation
  $\hat{C}'(u^*) = 0, \hat{C}(u) = \int_X \pi(dx) C(u; x)$
- Khimka, U.T., Chabanyuk, Ya.M., (2009). Convergence of stochastic optimization procedure with markov switching. International Conference PDMU-2015, Abstracts. Skhidnytsia, April 27-30, Ukraine, 172.

Stochastic optimization procedure as controlled random evolution

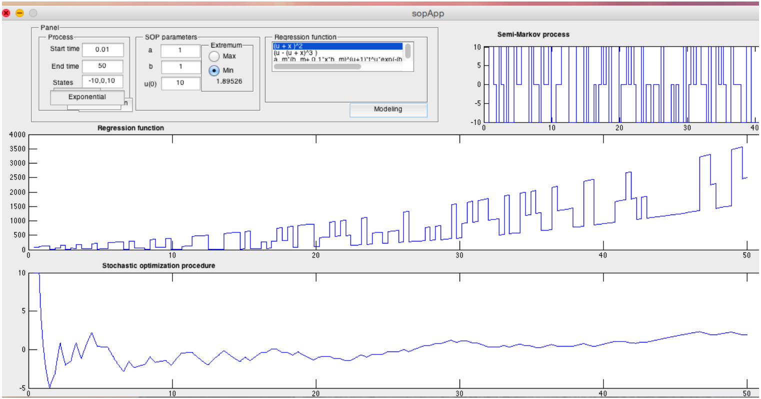
Example

- Let have some phenomenon given by function $C(u;x) = (u+x)^2$. On this phenomenon acts Markov process with exponentially distributed time spent in a state with three states
- Need to find a solution of equation $C'(u;x) = 0$, where $u \in R, x(t), t \geq t_0 > 0,$ – Markov process: $x \in \{-1, 0, 1\}$ with transition graph



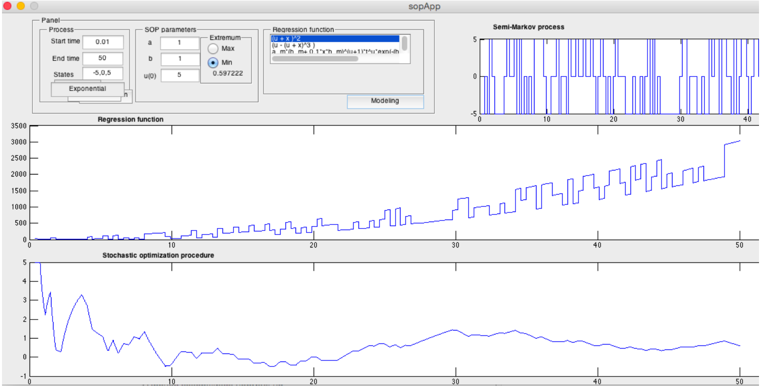
Stochastic optimization procedure as controlled random evolution

Example 1



Stochastic optimization procedure as controlled random evolution

Example 1



Stochastic optimization procedure as controlled random evolution

Example 2

- Model of software testing. Intensity error detection function: $\lambda(t) = \alpha\beta^{s+1}t^s e^{-\beta t}$.
- Regression function

$$\lambda(s,t) = \hat{\alpha}\hat{\beta}^{s+1}t^s \exp(-\hat{\beta}t). \qquad (2)$$

where $\hat{\alpha}$, $\hat{\beta}$ - point estimates for α та β.

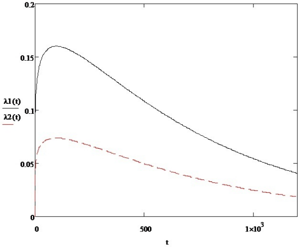
- PSO for function (2) with regard to parameter s
$\frac{ds(t)}{dt} = \frac{a(t)}{2b(t)}[\hat{\alpha}\hat{\beta}^{s(t)+b(t)+1}t^{s(t)+b(t)}e^{-\hat{\beta}t} -$
$-\hat{\alpha}\hat{\beta}^{s(t)-b(t)+1}t^{s(t)-b(t)}e^{-\hat{\beta}t}]$.
- Khimka, U.T., Chabanyuk, Ya.M., (2013). Difference procedure of stochastic optimization with impulse perturbation. Cybernetics and Systems Analysis, Vol. 49, No. 3, 145-162.

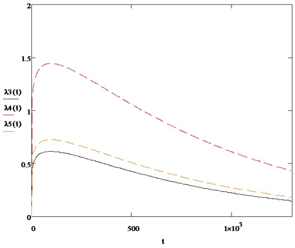
Stochastic optimization procedure as controlled random evolution

Example 2

Results of the application of PSO for testing model

Type	s (Model)	s (PSO)	μ(Model)	μ(PSO)
Trivial	0,154	0,252	56	64,902
Minor	0,144	0,201	479	573,506
Major	0,122	0,296	1204	1516,043
Critical	0,139	0,222	666	801,526
Bloker	0,153	0,387	122	138,966





Stochastic optimization procedure as controlled random evolution

Analysis methods

- At research of stochastic approximation procedure are used analysis methods of stochastic systems with use of a small parameter, martingale characterization of Markov processes, solutions of singular perturbation problem

Asymptotic normality

Problem formulation

- A continuous one-dimensional procedure of finding maximum point regression function $C(u, x)$ given by evolution equation

$$du^{\varepsilon}(t) = \frac{a}{t} \bigtriangledown_b C(u^{\varepsilon}(t), x(t/\varepsilon^2))dt, \tag{3}$$

- Normalized procedure given by ratio

$$v^{\varepsilon}(t) = \sqrt{t/\varepsilon^2} u^{\varepsilon}(t), t \geq t_0 > 0. \tag{4}$$

Asymptotic normality

Theorem

Theorem 1

Let the conditions of PSA convergence satisfies (3), Also additional conditions take place:

$$D1 : \rho^2 := 2\int_X \pi(dx)C_0(x)R_0C_0(x) > 0, C_0(x) := C'(0,x)$$

$$D2 : c := -\int_X \pi(dx)C_1(x) > 0, C_1(x) := C''(0,x)/2,$$

$$D3 : b := ac - 1/2 > 0.$$

Then for normalized PSA (4) there is weak convergence $v^\varepsilon(t) \to \zeta(t), \varepsilon \to 0$, in every finite interval $0 < t_0 \leq t \leq T, T > 0$, where $\zeta(t), t \geq 0$ – Ornstein-Uhlenbeck process with generator $\mathbf{L}\varphi(v) = -bv\varphi'(v) + \frac{\sigma^2}{2}\varphi''(v)$, and dispersion $\sigma^2 = a^2\rho^2$.

Random evolution in asymptotic small diffusion scheme

Asymptotic small diffusion scheme

The problem of large deviations for random processes is by one of the three major problems of the theory of convergence of random processes, including:
1) the problem of averaging or the law of large numbers;
2) the problem of the diffusion approximation or central limit theorem;
3) the problem of large deviations or exponential asymptotic behavior small probabilities.
Solution problems of large deviations realization requires four phases (see. [1], chapter 2):
1) calculate the limit exponential (nonlinear) operator, which determines the large deviation;
2) definition of exponential kompact;
3) determination of the principle of limiting the comparison to the operator; 4) constructions image variation functional action that defines large deviation.

Random evolution in asymptotic small diffusion scheme

Asymptotic small diffusion scheme

Consider random evolution with diffusion perturbation defined by stochastic differential equation[1,2]

$$\frac{du^{\varepsilon,\delta}(t)}{dt} = C(u^{\varepsilon,\delta}(t); x(t/\varepsilon^3)) + (\varepsilon^{-1} + \delta^{-1}) C_0(x(t/\varepsilon^3)), \tag{5}$$

where $u^{\varepsilon,\delta}(t) \in R^d, t \geq 0,$ – random evolution [1]; $C_0(x)$ – singular perturbation of regression function $C(u; x) \in C^2(R^d)$; $x(t)$ – Markov process in phase space of states $(X, \mathbf{X})$ with stationary distribution $\pi(B), B \in \mathbf{X}$ [1,2]. Limited evolution has a representation $du(t) = C(u(t))dt + \sqrt{\delta}\sigma dw(t), C(u) = \int_X \pi(dx) C(u; x).$ For σ relation $B = \sigma^T \sigma, B = 2 \int_X \pi(dx) C_0(x) \mathbf{R}_0 C_0(x).$

Random evolution in asymptotic small diffusion scheme

Asymptotic small diffusion scheme

Balance condition holds

$$\int_X \pi(dx) C_0(x) = 0.$$

The random evolution (5) is defined by the generator

$$\mathbf{L}^{\varepsilon,\delta}(x)\varphi(u,x) = [\varepsilon^{-3}\mathbf{Q} + (\varepsilon^{-1} + \delta^{-1})\mathbf{C}_0(x) + \mathbf{C}(x)]\varphi(u,x),$$

where

$$\mathbf{C}_0(x)\varphi(u) = C_0(x)\varphi'(u), \mathbf{C}(x)\varphi(u) = C(u;x)\varphi'(u).$$

Random evolution in asymptotic small diffusion scheme

Asymptotic small diffusion scheme

Theorem 3

Solution of singular perturbation problem [2] for generator $\mathbf{L}^{\varepsilon,\delta}(x)$ on perturbed test-function

$$\varphi^{\varepsilon,\delta}(u;x) = \varphi(u) + (\varepsilon+\delta)\varphi_1(u;x) + \varepsilon^3\varphi_2(u;x),$$

where $\frac{\varepsilon}{\delta} \to 1, \varepsilon \to 0$, has form

$$\mathbf{L}^{\varepsilon,\delta}(x)\varphi^{\varepsilon,\delta}(u;x) = \mathbf{L}^{\delta}\varphi(u) + \theta_L^{\varepsilon,\delta}(x)\varphi(u),$$

where limited generator $\mathbf{L}^{\delta}$ is determined from relation

$$\mathbf{L}^{\delta} = \Pi\mathbf{C}(x)\Pi + \delta\Pi\mathbf{C}_0(x)\mathbf{R}_0\mathbf{C}_0(x)\Pi$$

and residual term $\theta_L^{\varepsilon,\delta}(x)\varphi(u)$ is defined by the ratio

$$\sup_{x\in X}\theta_L^{\varepsilon,\delta}(x)\varphi(u) \to 0, \varepsilon \to 0.$$

Random evolution in asymptotic small diffusion scheme

Asymptotic small diffusion scheme

Theorem 4

The exponential generator

$$\mathbf{H}^{\delta}\varphi(u) := exp(-\frac{\varphi(u)}{\delta})\delta\mathbf{L}^{\delta}exp(\frac{\varphi(u)}{\delta})$$

[1,2] the limiting evolution

$$du(t) = C(u(t))dt + \delta^{1/2}\sigma dw(t)$$

with shift $C(u) = \int_X \pi(dx)C(u;x)$, converges to the limiting exponential generator

$$\mathbf{H}\varphi(u) = \frac{1}{2}\sigma^2[\varphi'(u)]^2 + C(u)\varphi'(u), \delta \to 0.$$

Finding the optimal portfolio in Markowitz model

Finding the optimal portfolio. Problem formulation

Markowitz model of optimal portfolio *

$$\begin{cases} r(u) = \sum_{i=1}^{d} r_i u_i \to max, \\ \tilde{C}(u,x) = \sum_{i=1}^{d} \sum_{j=1}^{d} S_{ij}(x) u_i u_j \to min, \ S_{ij}(x) = c_{ij} x_i x_j, \\ \sum_{i=1}^{d} a_{ij} u_i = b_j, \ j = \overline{1,m}, \\ \sum_{i=1}^{d} u_i = 1, \ u_i \geq 0, \end{cases} \tag{6}$$

where $r_i = r_i(t)$ — profitability of i th asset in t days, $x_i = x_i(t)$ — the risk of adverse change in market quotations i th asset, $c_{ij} = c_{ij}(t), i, j = \overline{1,n}$ — correlation coefficients of random variables of assets profitability (risk add j th kind of asset to i th portfolio).

* Markowitz Harry. The Optimization of a Quadratic Function Subject to Linear Constraints // Naval Research Logistics Quarterly. 1956 — Vol. 3. — P. 1–33.

* Sharp O., Aleksandr Г., Bailey G.. Investment , Trans. from eng. — M.: Infra-M, — 2001. — XII, — 1028 p.

Finding the optimal portfolio in Markowitz model

Modeling

Consider example when investor has six assets *. Need to find vector $(u_1, \ldots, u_6)^T$, which provides a minimum of functional $C(u,x) = -\mu r(u) + \widetilde{C}(u,x) \to min$,

$$r = (0.108,\ 0.136,\ 0.144,\ 0.151,\ 0.187,\ 0.191)^T, \qquad (7)$$

The matrix of assets correlation coefficients c has form

$$c = \begin{pmatrix} 1 & 0.32 & 0.4 & -0.3 & 0.31 & -0.18 \\ 0.32 & 1 & 0.45 & 0.43 & 0.29 & 0.3 \\ 0.4 & 0.45 & 1 & 0.4 & 0.46 & -0.29 \\ -0.3 & 0.43 & 0.4 & 1 & 0.41 & 0.4 \\ 0.31 & 0.29 & 0.46 & 0.41 & 1 & 0.25 \\ -0.18 & 0.3 & -0.29 & 0.4 & 0.25 & 1 \end{pmatrix} \qquad (8)$$

The value of the parameter μ reflects the attitude of investor to risk. Let put it equal to 1 (average or moderate attitude).
* MATLAB 7.

Finding the optimal portfolio in Markowitz model

Modeling

Each of the vector components $x = (x_1, \dots, x_6)$ can be in three states: $x_i = \{\Delta_i - \frac{1}{3};\ \Delta_i;\ \Delta_i + \frac{1}{3}\}$, $i = \overline{1,6}$, where Δ can be in three states:

$$\Delta = (0.36,\ 0.405,\ 0.45,\ 0.54,\ 0.60,\ 0.615)^T. \qquad (9)$$

The probability matrix of transition from state to state each component of the vector x has the form

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}. \qquad (10)$$

Finding the optimal portfolio in Markowitz model

Modeling

Thus, objective function has the form:

$$C(u,x) = -\sum_{i=1}^{d} r_i u_i + \sum_{i=1}^{d}\sum_{j=1}^{d} c_{ij} x_i x_j u_i u_j. \tag{11}$$

Discrete PSO has representation:

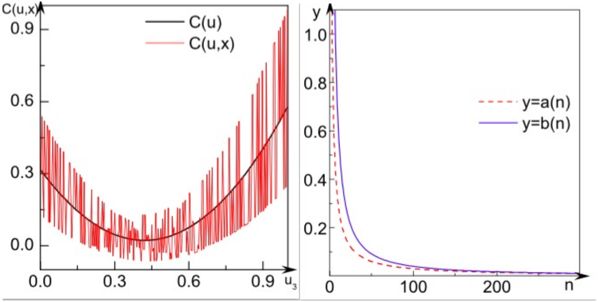
$$u_i^{(n+1)} = u_i^{(n)} + a^{(n)} \bigtriangledown_{b^{(n)}} C(u_i, x_i), \tag{12}$$

where $i = \overline{1, d},$ and n— number of iteration.
Averaged system takes the form:

$$\frac{\partial C}{\partial u_i} = -r_i + \sum_{j=1}^{d} c_{ij} x_{i0} x_{j0} u_j + c_{ii} x_{i0}^2 u_i, \tag{13}$$

Finding the optimal portfolio in Markowitz model

Result of modeling

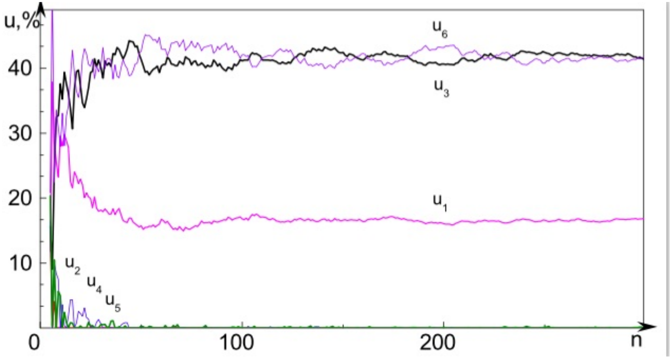


Here presented projection of the objective function on a plane $(C(u;x), u_3)$ at fixed $u_1 = u_2 = u_4 = u_5 = 0$ in the absence ($C(u)$) and presence of risk changes ($C(u,x)$). Control functions $a^{(n)}$, $b^{(n)}$ have form:

$$a^{(n)} = 3/n, \quad b^{(n)} = \frac{10}{n^{0.4}}.$$

Finding the optimal portfolio in Markowitz model

Result of modeling



$$u^* = (16.7,\ 0.0,\ 41.9,\ 0.0,\ 0.0,\ 41.4)^T.$$

Finding the optimal portfolio in Markowitz model

Result of modeling

n	u_1	u_2	u_3	u_4	u_5	u_6
u^0	17,45	0,00	42,16	0,00	1,29	39,10
1	16,70	0,00	41,90	0,00	0,00	41,40
2	18,50	0,00	43,50	0,00	0,10	37,90
3	18,20	0,00	42,00	0,00	0,00	39,70
4	18,10	0,00	42,50	0,00	0,00	39,40
5	18,00	0,00	40,00	0,00	0,00	42,00
6	19,30	0,00	39,80	0,00	0,10	40,70
7	20,50	0,00	43,10	0,00	0,00	36,30
8	20,30	0,00	41,00	0,10	0,00	38,60
9	19,20	0,00	39,40	0,00	0,00	41,30
10	17,80	0,00	42,30	0,00	0,10	39,80

Average value of income $\overline{r}(u) = 15,69\%$, variance $\sigma^2 = 1,7 \cdot 10^{-4}\%$, mean square deviation $\sigma = 0,1\%$.

Literature

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