Zadanie sterowania optymalnego opisane równaniem lepko-sprężystej belki i jego aproksymacja typu Galerkina

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Introduction

We study the optimal control problem for the transvers motions of a viscoelastic beam. The equation of motion in the y-direction of this beam is

$$\begin{aligned}&\frac{\partial^2 y}{\partial t^2}+\alpha\frac{\partial^4 y}{\partial x^4}-\left[\beta+\gamma\int\limits_0^l\left(\frac{\partial y(\xi,t)}{\partial \xi}\right)^2 d\xi\right]\frac{\partial^2 y}{\partial x^2}+\delta\frac{\partial^5 y}{\partial x^4\partial t}+\\&-\sigma\int_0^l\frac{\partial y(\xi,t)}{\partial \xi}\cdot\frac{\partial^2 y(\xi,t)}{\partial\xi\partial t}d\xi\cdot\frac{\partial^2 y}{\partial x^2}+\eta\frac{\partial y}{\partial t}=f(x,t).\end{aligned} \tag{1.1}$$

The parameters α, γ, δ, σ are positive phisical costants and $\beta,\ \eta\in\mathbb{R}$. The position $x\in(0,l)$ and the time $t\in(0,T)$ for $l,\ T<\infty$.

We consider, from the mechanical point of view, the boundary conditions corresponding to clamped ends, when

$$y(0,t) = y(l,t) = \frac{\partial y(0,t)}{\partial x} = \frac{\partial y(l,t)}{\partial x} = 0 \tag{1.2}$$

or the boundary hinged ends, when

$$y(0,t) = y(l,t) = \frac{\partial^2 y(0,t)}{\partial x^2} = \frac{\partial^2 y(l,t)}{\partial x^2} = 0. \tag{1.3}$$

We consider the initial-boundary value problem consisting of (1.1), the initial conditions

$$y(x,0) = y_0(x) \qquad \text{and} \qquad \frac{\partial y(x,0)}{\partial t} = y_1(x) \tag{1.4}$$

and the boundary conditions (1.2) or (1.3).

Let $\Omega = (0, l)$, where $l > 0$ is the natural length of the beam, $S = (0, T)$ and $Q = \Omega \times S$. We shall need the following space:

- Lebesgue spaces $L^2(\Omega)$, $L^2(Q)$

$$L^2(S; W) = \left\{ \omega \colon S \to W \mid \int_S \|\omega(t)\|_W^2 dt < \infty \right\}$$

and

$$L^\infty(S; W) = \left\{ \omega \colon S \to W \mid \operatorname{ess} \sup_{t \in S} \|\omega(t)\|_W < \infty \right\},$$

with the standard norms, where W is any Banach space.

- Sobolev spaces $H^2(\Omega)$, $H_0^2(\Omega)$, $H_0^1(\Omega)$ with the standard norms.

Let $V = H_0^2(\Omega)$ for clamped ends or $V = H^2(\Omega) \cap H_0^1(\Omega)$ (the closed subspace of $H^2(\Omega)$) for hinged ends and $H = L^2(\Omega)$. These spaces are equipped with standard norms. The embedding $V \subset H$ is continuos, dense and compact. Identifying H with its dual we have the evolution triple $V \subset H \subset V^*$. The duality pairing $\langle \varphi, \psi \rangle$ of V^* and V is identical with the inner product (φ, ψ) on H if $\varphi \in H$.

We define a weak solution of the equation (1.1) with the initial condition (1.4) and the boundary conditions (1.2) or (1.3) comme the solution of following equation

$$
\begin{gathered}
\langle \ddot{y}(t), \psi \rangle + \alpha(y_{xx}(t), \psi_{xx}) - \big(\beta + \gamma ||y_x(t)||_H^2\big)(y_{xx}(t), \psi) + \delta(\dot{y}_{xx}(t), \psi_{xx}) + \\
-\sigma(y_x(t), \dot{y}_x(t))(y_{xx}(t), \psi) + \eta(\dot{y}(t), \psi) = (f(t), \psi), \\
\forall_{\psi \in V} \ \text{ for a.e. } t \in S, \\
y(0) = y_0 \ \text{ and } \ \dot{y}(0) = y_1 \quad \text{for } y_0 \in V \text{ and } y_1 \in H,
\end{gathered}
\tag{1.5}
$$

where $(\varphi, \psi) = \int\limits_0^l \varphi(x)\psi(x)dx$ (the inner product on H).

In our first theorem we state the existence and uniqueness of weak solution (1.1)–(1.4).

Theorem 1

Suppose $f \in L^2(Q)$, $y_0 \in V$ *and* $y_1 \in H$. *Then, there exists a unique solution* y *of equations (1.5). This solution* $y \in L^\infty(S;V)$ *and* $\dot{y} \in L^\infty(S;H) \cap L^2(S;V)$.

Let us put in (1.5) $f = g + Bu$, where $g \in L^2(Q)$, $u \in U$ (the control space) and $B \in \mathcal{L}(U; L^2(Q))$. The equations (1.5) have a form

$$\begin{gathered}\langle \ddot{y}(t), \psi \rangle + \alpha(y_{xx}(t), \psi_{xx}) - \big(\beta + \gamma ||y_x(t)||_H^2\big)(y_{xx}(t), \psi) + \delta(\dot{y}_{xx}(t), \psi_{xx}) + \\ -\sigma(y_x(t), \dot{y}_x(t))(y_{xx}(t), \psi) + \eta(\dot{y}(t), \psi) = \big(g(t) + (Bu)(t), \psi\big), \\ \forall_{\psi \in V} \ \text{ for a.e. } t \in S,\end{gathered}$$

$$y(0) = y_0 \ \text{ and } \ \dot{y}(0) = y_1. \tag{1.6}$$

Now we define a nonlinear operator F from the separable Hilbert space U into a space
$X = \prod_{i=1}^{4} L^2(S;H)$ by

$$F(u) = (y, y_x, y_{xx}, \dot{y}),$$

where y is the unique solution of (1.6). The norm in the space X is given by the form

$$||F(u)||_X^2 = \int_0^T \left[||y(t)||_H^2 + ||y_x(t)||_H^2 + ||y_{xx}(t)||_H^2 + ||\dot{y}(t)||_H^2 \right] dt = \\ = ||y||_{L^2(S;V)}^2 + ||\dot{y}||_{L^2(S;H)}^2.$$

Lemma 2

Suppose $g \in L^2(Q)$, $y_0 \in V$, $y_1 \in H$ and the operator B is linear and bounded with the separable Hilbert space U into $L^2(Q)$. Then the operator F is locally Lipschitz continuous and a weakly continuous map.

Optimal control problem

The state of system of our control problem is described by an equation

$$\langle \ddot{y}(t),\psi\rangle + \alpha(y_{xx}(t),\psi_{xx}) - \big(\beta+\gamma||y_x(t)||^2_H\big)(y_{xx}(t),\psi) + \delta(\dot{y}_{xx}(t),\psi_{xx}) + \\ -\sigma(y_x(t),\dot{y}_x(t))(y_{xx}(t),\psi) + \eta(\dot{y}(t),\psi) = \big(g(t)+(Bu)(t),\psi\big), \\ \forall_{\psi\in V} \text{ for a.e. } t\in S,$$

$$y(0)=y_0 \text{ and } \dot{y}(0)=y_1. \tag{2.1}$$

The optimal control proplem (P) can be formulated as follows: find an optimal pair
$(u^0,y^0)\in U\times\mathcal{W}$ wchich minimizes a functional $J(u,y)$ where
$J: U\times\mathcal{W}\to\mathbb{R}$ and $y=y(u)$ is a unique solution of (2.1) for $u\in U$
and $\mathcal{W}=\{\omega\in L^2(Q) \mid \omega,\omega_x,\omega_{xx},\dot{\omega}\in L^2(Q)\}$ with a norm
$||(u,\omega||_{U\times\mathcal{W}} = ||u||_U + ||\omega||_{L^2(Q)} + ||\omega_x||_{L^2(Q)} + ||\omega_{xx}||_{L^2(Q)} + ||\dot{\omega}||_{L^2(Q)}$.

Theorem 3

Let the assumtions of Lemma 2 be satisfied, i.e. $g \in L^2(Q)$, $y_0 \in V$, $y_1 \in H$ *and the operator* B *is linear and bounded with the separable Hilbert space* U *into* $L^2(Q)$*. If the functional* J *is continous and convex on* $U \times \mathcal{W}$ *and the functional* $u \to J(u, y(u))$ *is coersive i.e.* $\lim_{||u|| \to \infty} J(u, y(u)) = \infty$*. Then, there exists at least one optimal pair* $(u^0, y^0) \in U \times \mathcal{W}$ *such that* $\inf_{u \in U} J(u, y(u)) = J(u^0, y^0)$ *where* $y^0 = y(u^0)$ *is the solution of (2.1) for* $u = u^0$*.*

In many enginering applications J may be quadratic functional in the form

$$J(u,y)=||u||_U^2+\lambda_1\int\limits_0^T\int\limits_0^l|y(t,x)-y_d|^2dxdt+\lambda_2\int\limits_0^T\int\limits_0^l|y_x(t,x)-y_d^1|^2dxdt+$$
$$+\lambda_3\int\limits_0^T\int\limits_0^l|y_{xx}(t,x)-y_d^2|^2dxdt+\lambda_4\int\limits_0^T\int\limits_0^l|\dot{y}(t,x)-y_d^3|^2dxdt$$

where $\lambda_i\geqslant 0$ for $i=1,\ldots,4$ and $\sum\limits_{i=1}^{4}\lambda_i=1$ and $y_d,\ y_d^1,\ y_d^2,\ y_d^3\in L^2(S;H)$ are desired functions. This functional represents the total anergy of the beams.

Approximation of the control problem

Here we recall some known results concerning the finite dimensional Galerkin approximation. They are basic for the convergence analysis of our optimal problem.

We consider a family $\{V_n\}_{n\in G}$ of finite dimensional subspaces of V which satisfies the following conditions:

$$\forall h_1, h_2 \in G \qquad (h_1 > h_2 \Longrightarrow V_{h_1} \subset V_{h_2}) \qquad \text{and} \qquad \overline{\bigcup_{h\in G} V_h} = V, \tag{3.1}$$

where the set $G \subset (0, 1]$ of parameters h has an accumulation point at 0. The approximation of space H is the same family $\{V_h\}_{h\in G}$ with an induced norm with H. The approximation of the spaces $L^2(S; V)$ and $L^2(S; H)$ is understood here as a family of space $\{L^2(S; V_h)\}_{h\in G}$ from respective norms.

As an approximate solutions of equation (2.1) we mean the family of functions $y_h \in L^2(S; V_h)$ which are the solutions of the following system

$$\langle \ddot{y}_h(t), \psi_h\rangle + \alpha(y_{hxx}(t), \psi_{hxx}) - \big(\beta + \gamma ||y_{hx}(t)||_H^2\big)(y_{hxx}(t), \psi_h) +$$
$$+ \delta(\dot{y}_{hxx}(t), \psi_{hxx}) - \sigma(y_{hx}(t), \dot{y}_{hx}(t))(y_{hxx}(t), \psi) + \eta(\dot{y}_h(t), \psi_h) =$$
$$= \big(g(t) + (Bu)(t), \psi_h\big), \qquad \forall_{\psi_h \in V_h} \text{ for a.e. } t \in S,$$

$$y_h(0) = y_{0h} \text{ and } \dot{y}_h(0) = y_{1h} \tag{3.2}$$

where y_{0h} and y_{1h} are the ortogonal projections y_0 and y_1 onto V_h whith the respective norms. From Theorem 1 conclude that for each $h \in G$ the equation (3.2) the unique solution $y_h \in L^2(S; V_h)$ and $\dot{y}_h \in L^2(S; V_h)$.

As an approximation of control space U we take a family $\{U_k\}_{k\in K}$ of finite dimensional subspaces of U which satisfies the following conditions:

$$\forall k_1, k_2 \in K \qquad (k_1 > k_2 \Longrightarrow U_{k_1} \subset U_{k_2}) \qquad \text{and} \qquad \overline{\bigcup_{k\in K} U_k} = U, \tag{3.3}$$

where the set $K \subset (0, 1]$ of parameters k has an accumulation point at 0.

Our approximated optimal control problem (P_{hk}) has the following form: find an optimal pair $(u_k^0, y_{hk}^0) \in U_k \times \mathcal{W}_h$ which minimizes the cost functional J i.e.

$$J(u_k^0, y_{hk}^0) = \inf_{u_k \in U_k} J(u_k, y_h(u_k))$$

where $y_{hk} = y_h(u_k)$ is the solution of the system

$$\begin{aligned} &\langle \ddot{y}_h(t), \psi_h \rangle + \alpha(y_{hxx}(t), \psi_{hxx}) - \big(\beta + \gamma ||y_{hx}(t)||_H^2\big)(y_{hxx}(t), \psi_h) + \\ &+ \delta(\dot{y}_{hxx}(t), \psi_{hxx}) - \sigma(y_{hx}(t), \dot{y}_{hx}(t))(y_{hxx}(t), \psi) + \eta(\dot{y}_h(t), \psi_h) = \\ &= \big(g(t) + (Bu_k)(t), \psi_h\big), \qquad \forall_{\psi_h \in V_h} \text{ for a.e. } t \in S, \\ &y_h(0) = y_{0h} \text{ and } \dot{y}_h(0) = y_{1h} \end{aligned} \tag{3.4}$$

and $\mathcal{W}_h = \{\omega_h \in L^2(S; V_h) \mid \omega_h, \omega_{hx}, \omega_{hxx}, \dot{\omega}_h \in L^2(S; V_h)\}$ with an induced norm of $\mathcal{W}$. The control problem (P_{hk}) is the lumped parameter system.

Theorem 4

Under the assumptions of Theorem 3 and the properties of Galerkin approximation (3.1) and (3.3), then the approximated control problem (P_{hk}) *has at least one solution* $u^0_{kh} \in U_k$.

Lemma 2 implies

Lemma 5

Let (u_k) be a sequence of elements in $U_k \subset U$ and (y_{hk}) be the corresponding sequence of solution of equation (3.4). If the assumptions of Lemma 2 and the properties of Galerkin approximation (3.1) and (3.3) be satisfied, then the following conditions hold:

(i) If $u_k \xrightarrow[k\to 0]{} \overline{u}$ *weakly in* U, *then*

$$y_{hk} \longrightarrow \overline{y} \text{ weakly in } L^2(S;H)$$

$$y_{hkx} \xrightarrow[h,k\to 0]{} \overline{y}_x \text{ weakly in } L^2(S;H),$$

$$y_{hkxx} \xrightarrow[h,k\to 0]{} \overline{y}_{xx} \text{ weakly in } L^2(S;H)$$

and

$$\dot{y}_{hk} \xrightarrow[h,k\to 0]{} \dot{\overline{y}} \text{ weakly in } L^2(S;H).$$

(ii) If $u_k \xrightarrow[k\to 0]{} \overline{u}$ *strongly in* U, *then*

$$y_{hk} \xrightarrow[h,k\to 0]{} \overline{y} \text{ strongly in } \mathcal{W}$$

where the function $\overline{y}$ *is unique solution of system (1.6) for* $u = \overline{u}$.

Let us now consider the convergence of approximation for prblem (P).

Theorem 6

If the assumptions of Theorem 4 be satisfied, then there exist weakly condensation points of a set of solutions of the optimization problem (P_{hk}) *in* $U \times \mathcal{W}$ *and each of these points is a solution of the optimization problem* (P).

This paper is a generalization of our paper:
A. Just, Z. Stempień *Pareto optimal control problem and its Galerkin approximation for a nonlinear one-dimensional extensible beam equation*, Opuscula Math. **36**(2) (2016), 239–252.

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial^4 y}{\partial x^4} - \left[\beta + \gamma \int_0^l \left(\frac{\partial y(\xi,t)}{\partial \xi}\right)^2 d\xi\right] \frac{\partial^2 y}{\partial x^2} + \delta \frac{\partial^5 y}{\partial x^4 \partial t} +$$
$$-\sigma \int_0^l \frac{\partial y(\xi,t)}{\partial \xi} \cdot \frac{\partial^2 y(\xi,t)}{\partial \xi \partial t} d\xi \cdot \frac{\partial^2 y}{\partial x^2} + \eta \frac{\partial y}{\partial t} = f(x,t).$$