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ON THE NUMERICAL SOLUTION OF AN ODE USING AN ORTHOGONAL WAVELET EXPANSION

We treat here a differential equation of an arbitrary order m

$$\sum_{r=0}^{m} a_r(z) y^{(r)}(z) = f(z) \tag{1}$$

for a given interval:

$$z \in [c, d] \tag{2}$$

which is split, for wavelet rank p, into $s = 2^p$ subintervals:

$$z_{\alpha_p} \in [c_{\alpha_p}, d_{\alpha_p}] = [c + 2^{-p}\alpha_p(d-c), c + 2^{-p}(\alpha_p + 1)(d-c)] \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{3}$$

$$z_{\alpha_p} = c + 2^{-p}[\alpha_p(d-c) + (z-c)] \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{4}$$

where we expand both the known function f(z) and the unknown function y(z) either by orthogonal polynomials and wavelets orthogonal polynomials defined in standard interval

$$x \in [a, b] \tag{5}$$

and $s = 2^p$ standard subintervals for wavelet rank p

$$x_{\alpha_p} \in [a_{\alpha_p}, b_{\alpha_p}] = [a + 2^{-p}\alpha_p(b-a), a + 2^{-p}(\alpha_p + 1)(b-a)] \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{6}$$

$$x_{\alpha_p} = a + 2^{-p}[\alpha_p(b-a) + (x-a)] \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{7}$$

$$P_p f(z_{\alpha_p}) = P_p f_{\alpha_p}(z) = \sum_{j=0}^{n} f_{\alpha_p,j}\phi_j(x) = \sum_{j=0}^{n} f_{\alpha_p,j}\phi_{j,p}(x_{\alpha_p}) \ \ (m \le n) \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{8}$$

$$f_{\alpha_p,j} = \frac{(f_{\alpha_p}, \phi_j)_{L^2[a,b]}}{||\phi_j||^2_{L^2[a,b]}} = \frac{(f, \phi_{j,p})_{L^2[a_{\alpha_p}, b_{\alpha_p}]}}{||\phi_{j,p}||^2_{L^2[a_{\alpha_p}, b_{\alpha_p}]}} \quad \forall_{0 \le j \le n} \ \forall_{0 \le \alpha_p \le 2^p - 1} \tag{9}$$

$$P_p y(z_{\alpha_p}) = P_p y_{\alpha_p}(z) = \sum_{j=0}^{n} y_{\alpha_p,j}\phi_j(x) = \sum_{j=0}^{n} y_{\alpha_p,j}\phi_{j,p}(x_{\alpha_p}) \ \ (m \le n) \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{10}$$

and where the appropriate coefficients are unknown Remark that:

$$\phi^{(r)}_{j,p}(x_{\alpha_p}) = \phi^{(r)}_j(x)\left(\frac{dx}{dx_{\alpha_p}}\right)^r = 2^{pr}\phi^{(r)}_j(x) \quad \forall_{0 \le j \le n} \ \forall_{0 \le r \le m} \ \forall_{0 \le \alpha_p \le 2^p - 1} \tag{11}$$

$$w_{\phi_p}(x_{\alpha_p}) = w_\phi(x) \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{12}$$

The scalar products and norms are defined below:

$$(f,\phi_j)_{L^2[a,b]} = \int_a^b f(x)\phi_j(x)w_\phi(x)dx \quad \underset{0\le j\le n<\infty}{\forall} \tag{13}$$

$$||\phi_j||^2_{L^2[a,b]} = \int_a^b \phi_j^2(x)w_\phi(x)dx \quad \underset{0\le j\le n<\infty}{\forall} \tag{14}$$

Here the interval [a,b] means an standard interval for the appropriate types of orthogonal polynomials i.e.

$$x\in[-1,1],\ \ x\in[0,\infty[,\ \ x\in]-\infty,\infty[\tag{15}$$

where: the left hand side interval stands for Jacobi, Gegenbauer, Legendre, Chebyshev polynomials of first and second kind, the central interval stands for Laguerre polynomials, the right hand side interval stands for Hermite polynomials. The weight functions are defined:

$$w_{C^\rho}(x) = (1-x^2)^{\rho-0.5}\ \ (\rho\ge 0) \tag{16}$$

for Gegenbauer polynomials and here 3 special cases occur:

$$w_{C^0}(x) = w_T(x) = (1-x^2)^{-0.5} \tag{17}$$

for Chebyshev polynomials of first kind

$$w_{C^{0.5}}(x) = w_P(x) = 1 \tag{18}$$

for Legendre polynomials

$$w_{C^1}(x) = w_U(x) = (1-x^2)^{0.5} \tag{19}$$

for Chebyshev polynomials of second kind and further:

$$w_L(x) = e^{-x} \tag{20}$$

for Laguerre polynomials

$$w_H(x) = e^{-x^2} \tag{21}$$

for Hermite polynomials.

Between the nonstandard interval [c,d] variable z and the standard interval [a,b] variable x exists the following relations:

$$z(x) = 0.5(d+c) + \frac{(d-c)}{b-a}x - 0.5\frac{(d-c)(b+a)}{(b-a)} \quad (-\infty < c \le z \le d < \infty) \tag{22}$$

when

$$x\in[-1,1]=[a,b] \tag{23}$$

and

$$z(x) = c + px \quad (-\infty \leq c < \infty) \wedge (p \neq 0) \tag{24}$$

when

$$(x \in [0, \infty[) \vee (x \in]-\infty, \infty[) \tag{25}$$

The reciprocal transformations are as follows:

$$x(z) = 0.5(a+b) + \frac{(b-a)}{d-c}x - 0.5\frac{(b-a)(d+c)}{(d-c)} \quad (-1 = a \leq x \leq b = 1) \tag{26}$$

when

$$z \in [c, d] \tag{27}$$

and

$$x(z) = \frac{z-c}{p} \quad (-\infty \leq c < \infty) \wedge (p \neq 0) \tag{28}$$

when

$$(z \in [c, \infty[) \vee z \in]-\infty, \infty[\tag{29}$$

Analoguous relations exist for the variables in the subintervals.

The coefficients of the differential equation

$$a_r(z_{\alpha_p}) = a_{r,\alpha_p}(z) \underset{0 \leq r \leq m}{\forall} \; \underset{0 \leq \alpha_p \leq 2^p - 1}{\forall} \tag{30}$$

must be squares integrable in general

Let consider first the case p=0 i.e. classical orthogonal expansion. Then let rewrite the differential equation (1):

$$\sum_{r=0}^{m} a_r(z) \sum_{j=0}^{n} y_j \phi_j^{(r)}(x) (\frac{dx}{dz})^r = f(z) \tag{31}$$

We impose first the initial conditions for $m \geq 1$:

$$y^{(r)}(c) = \sum_{j=r}^{n} y_j \phi_j^{(r)}(a) (\frac{dx}{dz})^r \underset{0 \leq r \leq m-1}{\forall} \tag{32}$$

Let now write them in a more covenient mode for further treatment:

$$\sum_{j=r}^{m-1} y_j \phi_j^{(r)}(a) (\frac{dx}{dz})^r = y^{(r)}(c) - \sum_{k=m}^{n} y_k \phi_k^{(r)}(a) (\frac{dx}{dz})^r \underset{0 \leq r \leq m-1}{\forall} \tag{33}$$

which can be written as the equations system

$$\sum_{j=r}^{m-1} B_{r,j} y_j = g_r \underset{0 \leq r \leq m-1}{\forall} \tag{34}$$

where:

$$B_{r,j} = \phi_j^{(r)}(a) (\frac{dx}{dz})^r \underset{0 \leq j, r \leq m-1}{\forall} \tag{35}$$

$$g_r = y^{(r)}(c) - \sum_{k=m}^{n} y_k \phi_k^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0\le r\le m-1} \tag{36}$$

The solution is:

$$y_j = \sum_{r=0}^{m-1} (B^{-1})_{j,r} g_r = \sum_{r=0}^{m-1} (B^{-1})_{j,r} [y^{(r)}(c) - \sum_{k=m}^{n} \phi_k^{(r)}(a)(\frac{dx}{dz})^r] \quad \forall_{0\le j\le m-1} \tag{37}$$

Analogously we impose the boundary conditions for $m \geq 2$, where 2 cases are distinct:

1) $[0.5m] = 0.5m$

$$y^{(r)}(c) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0\le r\le 0.5m-1} \tag{38}$$

$$y^{(r)}(d) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(b)(\frac{dx}{dz})^r \quad \forall_{0\le r\le 0.5m-1} \tag{39}$$

rearranged as:

$$\sum_{j=r}^{m-1} y_j \phi_j^{(r)}(a)(\frac{dx}{dz})^r = y^{(r)}(c) - \sum_{k=m}^{n} y_k \phi_k^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0\le r\le 0.5m-1} \tag{40}$$

$$\sum_{j=r}^{m-1} y_j \phi_j^{(r)}(b)(\frac{dx}{dz})^r = y^{(r)}(d) - \sum_{k=m}^{n} y_k \phi_k^{(r)}(b)(\frac{dx}{dz})^r \quad \forall_{0\le r\le 0.5m-1} \tag{41}$$

being an equations system:

$$\sum_{j=r}^{m-1} B_{r,j} y_j = g_r \quad \forall_{0\le r\le m-1} \tag{42}$$

where:

$$B_{r,j} = \phi_j^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0\le r\le 0.5m-1} \ \forall_{0\le j\le m-1} \tag{43}$$

$$B_{r,j} = \phi_j^{(r-0.5m)}(b)(\frac{dx}{dz})^{r-0.5m} \quad \forall_{0.5m\le r\le m-1} \ \forall_{0\le j\le m-1} \tag{44}$$

$$g_r = y^{(r)}(c) - \sum_{k=m}^{n} y_k \phi_k^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0\le r\le 0.5m-1} \tag{45}$$

$$g_r = y^{(r-0.5m)}(d) - \sum_{k=m}^{n} y_k \phi_k^{(r-0.5m)}(b)(\frac{dx}{dz})^{r-0.5m} \quad \forall_{0.5m\le r\le m-1} \tag{46}$$

The solution is:

$$y_j = \sum_{r=0}^{m-1} (B^{-1})_{j,r} g_r \quad \forall_{0\le j\le m-1} \tag{47}$$

$$y_j = \sum_{r=0}^{0.5m-1} (B^{-1})_{j,r}[y^{(r)}(c) - \sum_{k=m}^{n} \phi_k^{(r)}(a)(\frac{dx}{dz})^r] + \sum_{r=0.5m}^{m-1} (B^{-1})_{j,r}[y^{(r-0.5m)}(d) - \sum_{k=m}^{n} \phi_k^{(r-0.5m)}(b)(\frac{dx}{dz})^{r-0.5m}] \quad (48)$$

2) $[0.5m] \neq 0.5m$ and $([0.5(m+1)] = 0.5(m+1)) \wedge ([0.5(m-1)] = 0.5(m-1))$
subcase a)

$$y^{(r)}(c) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0 \leq r \leq 0.5(m+1)-1} \quad (49)$$

$$y^{(r)}(d) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(b)(\frac{dx}{dz})^r \quad \forall_{0 \leq r \leq 0.5(m-1)-1} \quad (50)$$

rearranged as:

$$\sum_{j=r}^{m-1} y_j \phi_j^{(r)}(a)(\frac{dx}{dz})^r = y^{(r)}(c) - \sum_{k=m}^{n} y_k \phi_k^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0 \leq r \leq 0.5(m+1)-1} \quad (51)$$

$$\sum_{j=r}^{m-1} y_j \phi_j^{(r)}(b)(\frac{dx}{dz})^r = y^{(r)}(d) - \sum_{k=m-1}^{n} y_k \phi_k^{(r)}(b)(\frac{dx}{dz})^r \quad \forall_{0 \leq r \leq 0.5(m-1)-1} \quad (52)$$

being an equations system:

$$\sum_{j=r}^{m-1} B_{r,j} y_j = g_r \quad \forall_{0 \leq r \leq m-1} \quad (53)$$

where:

$$B_{r,j} = \phi_j^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0 \leq r \leq 0.5(m+1)-1} \; \forall_{0 \leq j \leq m-1} \quad (54)$$

$$B_{r,j} = \phi_j^{(r-0.5(m+1))}(b)(\frac{dx}{dz})^{r-0.5(m+1)} \quad \forall_{0.5(m+1) \leq r \leq m-1} \; \forall_{0 \leq j \leq m-1} \quad (55)$$

$$g_r = y^{(r)}(c) - \sum_{k=m}^{n} y_k \phi_k^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0 \leq r \leq 0.5(m+1)-1} \quad (56)$$

$$g_r = y^{(r-0.5(m+1))}(d) - \sum_{k=m}^{n} y_k \phi_k^{(r-0.5(m+1))}(b)(\frac{dx}{dz})^{r-0.5(m+1)} \quad \forall_{0.5(m+1) \leq r \leq m-1} \quad (57)$$

The solution is:

$$y_j = \sum_{r=0}^{m-1} (B^{-1})_{j,r} g_r \quad \forall_{0 \leq j \leq m-1} \quad (58)$$

subcase b)

$$y^{(r)}(c) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0 \leq r \leq 0.5(m-1)-1} \quad (59)$$

$$y^{(r)}(d) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(b)(\frac{dx}{dz})^r \quad \forall_{0 \le r \le 0.5(m+1)-1} \tag{60}$$

rearranged as:

$$\sum_{j=r}^{m-1} y_j \phi_j^{(r)}(a)(\frac{dx}{dz})^r = y^{(r)}(c) - \sum_{k=m}^{n} y_k \phi_k^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0 \le r \le 0.5(m-1)-1} \tag{61}$$

$$\sum_{j=r}^{m-1} y_j \phi_j^{(r)}(b)(\frac{dx}{dz})^r = y^{(r)}(d) - \sum_{k=m-1}^{n} y_k \phi_k^{(r)}(b)(\frac{dx}{dz})^r \quad \forall_{0 \le r \le 0.5(m+1)-1} \tag{62}$$

being an equations system:

$$\sum_{j=r}^{m-1} B_{r,j} y_j = g_r \quad \forall_{0 \le r \le m-1} \tag{63}$$

where:

$$B_{r,j} = \phi_j^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0 \le r \le 0.5(m-1)-1} \; \forall_{0 \le j \le m-1} \tag{64}$$

$$B_{r,j} = \phi_j^{(r-0.5(m+1))}(b)(\frac{dx}{dz})^{r-0.5(m-1)} \quad \forall_{0.5(m-1) \le r \le m-1} \; \forall_{0 \le j \le m-1} \tag{65}$$

$$g_r = y^{(r)}(c) - \sum_{k=m}^{n} y_k \phi_k^{(r)}(a)(\frac{dx}{dz})^r \quad \forall_{0 \le r \le 0.5(m-1)-1} \tag{66}$$

$$g_r = y^{(r-0.5(m-1))}(d) - \sum_{k=m}^{n} y_k \phi_k^{(r-0.5(m+1))}(b)(\frac{dx}{dz})^{r-0.5(m-1)} \quad \forall_{0.5(m-1) \le r \le m-1} \tag{67}$$

The solution is:

$$y_j = \sum_{r=0}^{m-1} (B^{-1})_{j,r} g_r \quad \forall_{0 \le j \le m-1} \tag{68}$$

Now we make the following functional:

$$J = || \sum_{k=0}^{n} y_k \sum_{r=0}^{m} a_r \phi_k^{(r)} (\frac{dx}{dz})^r - f||^2_{L^2[a,b]} = MIN \tag{69}$$

$$J = \int_a^b [\sum_{k=0}^{n} y_k \sum_{r=0}^{m} a_r(z) \phi_k^{(r)}(x)(\frac{dx}{dz})^r - f(z)]^2 w_\phi(x) dx = MIN \tag{70}$$

in which we consider the boundary or initial conditions as shown above, then

$$J = || \sum_{k=0}^{m-1} [\sum_{s=0}^{m-1} (B^{-1})_{k,l} g_l] \sum_{r=0}^{m} a_r \phi_k^{(r)} (\frac{dx}{dz})^r + \sum_{k=m}^{n} y_k \sum_{r=0}^{m} a_r \phi_k^{(r)} (\frac{dx}{dz})^r - f||^2_{L^2[a,b]} = MIN \tag{71}$$

$$\frac{\partial J}{\partial y_k} = 0 \quad \underset{m \le k \le n}{\forall} \tag{72}$$

the resulting equations system leads us to compute the coefficients

$$\sum_{k=m}^{n} A_{l,k} y_k = h_l \quad \underset{m \le l \le n}{\forall} \tag{73}$$

$$y_k = \sum_{l=m}^{n} (A^{-1})_{k,l} h_l \quad \underset{m \le l \le n}{\forall} \tag{74}$$

The matrix $\mathbf{A}$ is splitted as follows in purpose to show explicitly its elements:

$$A_{l,k} = A^{(0)}_{l,k} + A^{(1)}_{l,k} + A^{(2)}_{l,k} + A^{(3)}_{l,k} \quad \underset{m \le k, l \le n}{\forall} \tag{75}$$

where:

$$A^{(0)}_{l,k} = \sum_{r=0}^{m}\sum_{q=0}^{m} (\frac{dx}{dz})^{r+q} (a_r \phi^{(r)}_l, a_q \phi^{(q)}_k)_{L^2[a,b]} \underset{m \le k, l \le n}{\forall} \tag{76}$$

being available for all cases of limits problems.

For the initial values problem

$$A^{(1)}_{l,k} = \sum_{r=0}^{m}\sum_{q=0}^{m}\sum_{j=0}^{m-1}\sum_{s=0}^{m-1}\sum_{v=0}^{m-1}\sum_{w=0}^{m-1} (B^{-1})_{j,s} \phi^{(s)}_l(a) (B^{-1})_{v,w} \phi^{(w)}_k(a) (\frac{dx}{dz})^{r+q+s+w} (a_r \phi^{(r)}_j, a_q \phi^{(q)}_v)_{L^2[a,b]} \quad \underset{m \le k, l \le n}{\forall} \tag{77}$$

$$A^{(2)}_{l,k} = -\sum_{r=0}^{m}\sum_{q=0}^{m}\sum_{j=0}^{m-1}\sum_{s=0}^{m-1} (B^{-1})_{j,s} \phi^{(s)}_l(a) (\frac{dx}{dz})^{r+q+s} (a_r \phi^{(r)}_j, a_q \phi^{(q)}_l)_{L^2[a,b]} \quad \underset{m \le k, l \le n}{\forall} \tag{78}$$

$$A^{(3)}_{l,k} = -\sum_{r=0}^{m}\sum_{q=0}^{m}\sum_{v=0}^{m-1}\sum_{w=0}^{m-1} (B^{-1})_{v,w} \phi^{(w)}_k(a) (\frac{dx}{dz})^{r+q+w} (a_r \phi^{(r)}_l, a_q \phi^{(q)}_v)_{L^2[a,b]} \quad \underset{m \le k, l \le n}{\forall} \tag{79}$$

For the boundary value problem:

$$A^{(1)}_{l,k} = \sum_{r=0}^{m}\sum_{q=0}^{m}\sum_{j=0}^{m-1}\sum_{v=0}^{m-1} G_{l,j} G_{k,v} (\frac{dx}{dz})^{r+q} (a_r \phi^{(r)}_l, a_q \phi^{(q)}_k)_{L^2[a,b]} \underset{m \le k, l \le n}{\forall} \tag{80}$$

$$A^{(2)}_{l,k} = -\sum_{r=0}^{m}\sum_{q=0}^{m}\sum_{j=0}^{m-1} G_{l,j} (\frac{dx}{dz})^{r+q} (a_r \phi^{(r)}_j, a_q \phi^{(q)}_k)_{L^2[a,b]} \quad \underset{m, \le l, k \le n}{\forall} \tag{81}$$

$$A^{(3)}_{l,k} = -\sum_{r=0}^{m}\sum_{q=0}^{m}\sum_{v=0}^{m-1} G_{k,v} (\frac{dx}{dz})^{r+q} (a_r \phi^{(r)}_l, a_q \phi^{(q)}_v)_{L^2[a,b]} \quad \underset{m, \le l, k \le n}{\forall} \tag{82}$$

where for:

1) $[0.5m] = 0.5m$

$$G_{l,j} = \sum_{s=0}^{0.5m-1} (B^{-1})_{j,s}\phi_l^{(s)}(a)(\frac{dx}{dz})^s + \sum_{s=0.5m}^{m-1} (B^{-1})_{j,s}\phi_l^{(s-0.5m)}(b)(\frac{dx}{dz})^{s-0.5m} \underset{0\leq j\leq m-1}{\forall} \underset{m\leq l\leq n}{\forall} \tag{83}$$

2) $[0.5m] \neq 0.5m$ and $([0.5(m+1)] = 0.5(m+1)) \wedge ([0.5(m-1)] = 0.5(m-1))$
subcase a)

$$G_{l,j} = \sum_{s=0}^{0.5(m+1)-1} (B^{-1})_{j,s}\phi_l^{(s)}(a)(\frac{dx}{dz})^s + \sum_{s=0.5(m+1)}^{m-1} (B^{-1})_{j,s}\phi_l^{(s-0.5(m+1))}(b)(\frac{dx}{dz})^{s-0.5(m+1)} \underset{0\leq j\leq m-1}{\forall} \underset{m\leq l\leq n}{\forall} \tag{84}$$

subcase b)

$$G_{l,j} = \sum_{s=0}^{0.5(m-1)-1} (B^{-1})_{j,s}\phi_l^{(s)}(a)(\frac{dx}{dz})^s + \sum_{s=0.5(m-1)}^{m-1} (B^{-1})_{j,s}\phi_l^{(s-0.5(m-1))}(b)(\frac{dx}{dz})^{s-0.5(m-1)} \underset{0\leq j\leq m-1}{\forall} \underset{m\leq l\leq n}{\forall} \tag{85}$$

The vector **h** is splitted too for the same purpose:

$$h_l = h_l^{(0)} + h_l^{(1)} + h_l^{(2)} + h_l^{(3)} \underset{m\leq l\leq n}{\forall} \tag{86}$$

where:

$$h_l^{(0)} = \sum_{r=0}^{m} (\frac{dx}{dz})^r (f, a_r\phi_l^{(r)})_{L^2[a,b]} \underset{m\leq l\leq n}{\forall} \tag{87}$$

being available for all limit problems
For the initial values problem:

$$h_l^{(1)} = -\sum_{r=0}^{m}\sum_{j=0}^{m-1}\sum_{s=0}^{m-1} (B^{-1})_{j,s}\phi_l^{(s)}(a)(\frac{dx}{dz})^{r+s}(f, a_r\phi_j^{(r)})_{L^2[a,b]} \underset{m\leq l\leq n}{\forall} \tag{88}$$

$$h_l^{(2)} = -\sum_{r=0}^{m}\sum_{q=0}^{m}\sum_{v=0}^{m-1}\sum_{w=0}^{m-1} (B^{-1})_{v,w}y^{(w)}(c)(a_r\phi_l^{(r)}, a_q\phi(q)_v)_{L^2[a,b]}(\frac{dx}{dz})^{r+q} \underset{m\leq l\leq n}{\forall} \tag{89}$$

$$h_l^{(3)} = \sum_{r=0}^{m}\sum_{q=0}^{m}\sum_{j=0}^{m-1}\sum_{s=0}^{m-1} (B^{-1})_{j,s}\phi_l^{(s)}(a)(a_r\phi_j^{(r)}, a_q\phi_v^{(q)})_{L^2[a.b]}(\frac{dx}{dz})^{r+q+s} \underset{m\leq l\leq n}{\forall} \tag{90}$$

For the boundary values problem:

$$h_l^{(1)} = -\sum_{r=0}^{m}\sum_{j=0}^{m-1} (G^{-1})_{l,j}(f, a_r\phi_j^{(r)})_{L^2[a,b]}(\frac{dx}{dz})^r \underset{m\leq l\leq n}{\forall} \tag{91}$$

$$h_l^{(2)} = -\sum_{r=0}^{m}\sum_{q=0}^{m}\sum_{v=0}^{m-1} H_v(a_r\phi_l^{(r)}, a_q\phi_v^{(q)})_{L^2[a,b]}(\frac{dx}{dz})^{r+q} \underset{m\leq l\leq n}{\forall} \tag{92}$$

$$h_l^{(3)} = \sum_{r=0}^{m}\sum_{q=0}^{m}\sum_{v=0}^{m-1}\sum_{j=0}^{m-1} H_v G_{l,j}(a_r\phi_j^{(r)}, a_q\phi_v^{(q)})_{L^2[a,b]}(\frac{dx}{dz})^{r+q} \quad \forall_{m\leq l\leq n} \tag{93}$$

where for:

1) $[0.5m] = 0.5m$

$$H_v = \sum_{w=0}^{0.5m-1}(B^{-1})_{v,w}y^{(w)}(c) + \sum_{w=0.5m}^{m-1}(B^{-1})_{v,w}y^{(w-0.5m)}(d) \quad \forall_{0\leq v\leq m-1} \tag{94}$$

2) $[0.5m] \neq 0.5m$

subcase a)

$$H_v = \sum_{w=0}^{0.5(m+1)-1}(B^{-1})_{v,w}y^{(w)}(c) + \sum_{w=0.5(m+1)}^{m-1}(B^{-1})_{v,w}y^{(w-0.5(m+1))}(d) \quad \forall_{0\leq v\leq m-1} \tag{95}$$

subcase b)

$$H_v = \sum_{w=0}^{0.5(m-1)-1}(B^{-1})_{v,w}y^{(w)}(c) + \sum_{w=0.5(m-1)}^{m-1}(B^{-1})_{v,w}y^{(w-0.5(m-1))}(d) \quad \forall_{0\leq v\leq m-1} \tag{96}$$

For the case of using the Hermite polynomials up to ordrer n we replace a by å and b by $\mathring{b}$:

$$(a^\circ = -[|x_0^{(n+1)}| + 0.5]) \wedge (b^\circ = [x_n^{(n+1)} + 0.5]) \tag{97}$$

$$(H_{n+1}(x_0^{(n+1)}) = 0) \wedge (H_{n+1}(x_n^{(n+1)}) = 0) \tag{98}$$

For the case of using the Laguerre polynomials up to ordrer n we replace a by 0 and b by $\mathring{b}$:

$$(a = 0) \wedge (b^\circ = [x_n^{(n+1)} + 0.5]) \tag{99}$$

$$L_{n+1}(x_n^{(n+1)}) = 0 \tag{100}$$

$$P_0 y(z) = Py(z) = \sum_{j=0}^{n} y_j\phi_j(x) \tag{101}$$

The approximated solution Py(z) tends to y(z) for $n \to \infty$

Consider now a p wavelet orthogonal expansion, then we rewrite the differential equation (1) in 2 equivalent forms:

$$\sum_{r=0}^{m} a_r(z_{\alpha_p})\sum_{j=0}^{n} y_{j,\alpha_p}\phi_j^{(r)}(x)(\frac{dx}{dz_{\alpha_p}})^r = f(z_{\alpha_p}) \quad \forall_{0\leq\alpha_p\leq 2^p-1} \tag{102}$$

$$\sum_{r=0}^{m} a_{r,\alpha_p}(z)\sum_{j=0}^{n} y_{j,\alpha_p}\phi_j^{(r)}(x)(\frac{dx}{dz})^r = f_{\alpha_p}(z) \quad \forall_{0\leq\alpha_p\leq 2^p-1} \tag{103}$$

let compute the derivatives of the Py(z) solution at the given points as follows:

$$Py^{(r)}(c_{\alpha_p}) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(a_{\alpha_p})(\frac{dx}{dz})^r \quad \underset{0\leq r\leq n}{\forall} \ \underset{1\leq \alpha_p\leq 2^p-1}{\forall} \tag{104}$$

$$Py^{(r)}(d_{\alpha_p}) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(b_{\alpha_p})(\frac{dx}{dz})^r \quad \underset{0\leq r\leq n}{\forall} \ \underset{0\leq \alpha_p\leq 2^p-2}{\forall} \tag{105}$$

We impose first the initial conditions for $m \geq 1$:

$$y^{(r)}(c) = \sum_{j=r}^{n} y_j \phi_j^{(r)}(a)(\frac{dx}{dz})^r \quad \underset{0\leq r\leq m-1}{\forall} \tag{106}$$

and later

$$Py^{(r)}(c) = \sum_{j=r}^{n} y_j \phi_j^{(r)}(a_{\alpha p})(\frac{dx}{dz})^r \quad \underset{0\leq r\leq m-1}{\forall} \tag{107}$$

Analogously we impose the boundary conditions for $m \geq 2$, where 2 cases are distinct:

1) $[0.5m] = 0.5m$

$$Py^{(r)}(c_{\alpha_p}) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(a_{\alpha_p})(\frac{dx}{dz})^r \quad \underset{0\leq r\leq 0.5m-1}{\forall} \ \underset{0\leq \alpha_p\leq 2^p-1}{\forall} \tag{108}$$

$$Py^{(r)}(d_{\alpha_p}) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(b_{\alpha_p})(\frac{dx}{dz})^r \quad \underset{0\leq r\leq 0.5m-1}{\forall} \ \underset{0\leq \alpha_p\leq 2^p-1}{\forall} \tag{109}$$

2) $[0.5m] \neq 0.5m$

subcase a)

$$Py^{(r)}(c_{\alpha_p}) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(a_{\alpha_p})(\frac{dx}{dz})^r \quad \underset{0\leq r\leq 0.5(m+1)-1}{\forall} \ \underset{0\leq \alpha_p\leq 2^p-1}{\forall} \tag{110}$$

$$Py^{(r)}(d_{\alpha_p}) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(b_{\alpha_p})(\frac{dx}{dz})^r \quad \underset{0\leq r\leq 0.5(m-1)-1}{\forall} \ \underset{0\leq \alpha_p\leq 2^p-1}{\forall} \tag{111}$$

subcase b)

$$Py^{(r)}(c_{\alpha_p}) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(a_{\alpha_p})(\frac{dx}{dz})^r \quad \underset{0\leq r\leq 0.5(m-1)-1}{\forall} \ \underset{0\leq \alpha_p\leq 2^p-1}{\forall} \tag{112}$$

$$Py^{(r)}(d_{\alpha_p}) = \sum_{j=0}^{n} y_j \phi_j^{(r)}(b_{\alpha_p})(\frac{dx}{dz})^r \quad \underset{0\leq r\leq 0.5(m+1)-1}{\forall} \ \underset{0\leq \alpha_p\leq 2^p-1}{\forall} \tag{113}$$

Now we make the following functionals:

$$J_{p,\alpha_p} = ||\sum_{k=0}^{n} y_{k,\alpha_p} \sum_{r=0}^{m} a_{r,\alpha_p} \phi_k^{(r)} (\frac{dx}{dz_{\alpha_p}})^r - f||^2_{L^2[a,b]} = MIN \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{114}$$

$$J_{p,\alpha_p} = \int_a^b [\sum_{k=0}^{n} y_{k,\alpha_p} \sum_{r=0}^{m} a_r(z) \phi_k^{(r)}(x) (\frac{dx}{dz_{\alpha_p}})^r - f(z_{\alpha_p})]^2 w_\phi(x) dx = MIN \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{115}$$

in which we consider the boundary or initial conditions as shown above, then

$$\frac{\partial J_{p,\alpha_p}}{\partial y_{k,\alpha_p}} = 0 \quad \forall_{m \le k \le n} \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{116}$$

the resulting equations systems leads us to compute the coefficients

$$\sum_{k=m}^{n} A_{l,k,\alpha_p} y_{k,\alpha_p} = h_{l,\alpha_p} \quad \forall_{m \le l \le n} \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{117}$$

The results are:

$$P_p y_{\alpha_p}(z) = P_p y(z_{\alpha_p}) = \sum_{j=0}^{n} y_{j,\alpha_p} \phi_j(x) \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{118}$$

$$P_p y(z) = \sum_{\alpha_p=0}^{2^p-1} P_p y(z_{\alpha_p}) \chi_{\alpha_p}(z) \tag{119}$$

where:

$$(\chi_{\alpha_p}(z) = 1) \Leftrightarrow (z = z_{\alpha_p}) \tag{120}$$

and finally:

$$P_{\sum_{k=0}^{p} k} y(z) = (p+1)^{-1} \sum_{k=0}^{p} P_k y(z) \tag{121}$$

When:

$$a_{r,\alpha_p}(z) = a_r \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{122}$$

then for a given p

$$A_{l,k,\alpha_p} = A_{l,k} \quad \forall_{m \le l,k \le n} \quad \forall_{0 \le \alpha_p \le 2^p - 1} \tag{123}$$

It is not necessary to impose any conditions to the solution of (1) for an stiff one, but only when the equation (1) is inhomogeneous. An example of a stiff ODE will be given:

$$y^{(1)}(z) + 100 y(z) = sinz \quad y(0) = 0 \tag{124}$$

having the analytical solution:

$$y(z) = \frac{sinz - 0.01(cosz - e^{-100z})}{1.0001} \tag{125}$$

We sample this solution in the interval $z \in [0, 4]$ with rate Δz=0.1 and for the same interval we perform the computations of the approximated solutions both with and without the given initial condition. We use here up to the p=2 rank wavelets (when the initial condition is given) and apply the LS method shown above, and p=0 and p=2 rank wavelet (when the initial condition is discarded) both by LS and Galerkin methods. n=9 order of orthogonal expansion