In-depth analysis of a differential game modelling dynamic oligopoly with sticky prices

(Dogłębna analiza gry różniczkowej modelującej dynamiczny oligopol z lepkimi cenami)

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Konferencja Zastosowań Matematyki
12 września 2016

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What if prices do not adjust immediately (menu costs etc.)?

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- Firms consider dynamic optimization problems: firm i maximizes over q_i , $J^i_{0,x_0}(q_1, \ldots, q_N) =$
  $= \int_0^\infty e^{-\rho t} \left(p(t)q_i(t) - cq_i(t) - \frac{q_i(t)^2}{2} \right) dt,$
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  Two formulations:
  - open loop strategies: q_i are measurable functions of time;
  - feedback strategies: q_i are functions of price; in all above definitions $q_i(t)$ is replaced by $q_i(p(t))$.

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Literature

- R. Cellini, L. Lambertini, 2004, *Dynamic Oligopoly with Sticky Prices: Closed-Loop, Feedback and Open-Loop Solutions*, Journal of Dynamical and Control Systems **10**, 303-314.
- C. Fershtman, M. I. Kamien, 1987, *Dynamic Duopolistic Competition with Sticky Prices*, Econometrica **55**, 1151-1164.
- M. Simaan, T. Takayama, 1978, *Game Theory Applied to Dynamic Duopoly with Production Constraints*, Automatica **14**, 161-166.
- A. Wiszniewska-Matyszkiel, M. Bodnar, F. Mirota, 2014, *Dynamic Oligopoly with Sticky Prices: Off-Steady-State Analysis*, Dynamic Games and Applications **5**, 568-598, DOI 10.1007/s13235-014-0125-z.

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  - S. Aseev, V. Veliov, 2012, *Maximum principle for infinite-horizon optimal control with dominating discount*, Dynamics of Continuous, Discrete and Impulsive Systems **19**, 43-62.

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  - however, the same authors noticed that it is not Liapunov stable – it was proven to be a saddle point,
  - thus, there is no reason to assume that a solution from any other initial state will converge to it.
- Incomplete analysis in N-players feedback case.

Assev-Veliov infinite horizon Maximum Principle

- Consider a dynamic optimization problem
  - Maximize

$$J_{0,x_0}(u) = \int_{t=0}^{\infty} e^{-\rho t} g(t, x(t), u(t)) dt,$$

where the trajectory x is the trajectory corresponding to control u and it is defined by

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \text{for } t > 0, \\ x(0) = x_0, \end{cases}$$

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Theorem

*Under some unpleasant technical assumptions A1–A4 core relations (**CR**) of the Pontriagin maximum principle hold together with terminal condition (**TC**).*

(A1) The functions f and g and their partial derivatives with respect to x are continuous in (x, u) for every fixed t and uniformly bounded as functions of t over every bounded set of (x, u).

(A2) There exist numbers $\mu, r, \kappa, c_1 \geqslant 0$ and $\beta > 0$ such that for every $t \geqslant 0$

  (i) $\|x^*(t)\| \leqslant c_1 e^{\mu t}$ and
  (ii) for every control u for which the Lebesgue measure of $\{t : u(t) \neq u^*(t)\} \leqslant \beta$, the corresponding trajectory exists on $\mathbb{R}_+$ and $\|\frac{\partial g(t,y,u^*(y))}{\partial x}\| \leqslant \kappa(1 + \|y\|^r)$ for every $y \in \mathrm{conv}\{x(t), x^*(t)\}$, where conv denotes the convex hull.

(A3) There are numbers $\eta \in \mathbb{R}, \gamma > 0$ and $c_2 \geqslant 0$ such that for every $\zeta \in \mathbb{X}$ with $\|\zeta - x_0\| < \gamma$ state equation with initial condition replaced by $x(0) = \zeta$ has a solution x^ζ defined on $\mathbb{R}_+$, such that $x^\zeta(t) \in \mathbb{X}$, for all $t \geqslant 0$, and

$$\|x^\zeta(t) - x^*(t)\| \leqslant c_2\|\zeta - x_0\|e^{\eta t}.$$

(A4) $\rho > \eta + r\max\{\eta, \mu\}$ for r, η, μ from (A2) and (A3).

Assev-Veliov Maximum Principle — 2

- Hamiltonian:
  $H(x,t,u,\psi) = e^{-\rho t} g(t,x,u) + \langle \psi, f(t,x,u) \rangle$
- Let (x^*, u^*) be the optimal pair and A1–A4 hold,
- then there exists an absolutely continuous costate variable ψ^* such that

(i) (**CR**)

- for a.e. t, $u^*(t)$ maximizes the hamiltonian $H(x^*(t), t, u, \psi^*(t))$,
- $\dot{\psi}^*(t) = -\frac{\partial H(x^*(t),t,u^*(t),\psi^*(t))}{\partial x}$,

(ii) (**TC**)

- for every $t \geqslant 0$ the integral
  $$I^*(t) = \int_t^\infty e^{-\rho w} \left[Z_{(x^*,u^*)}(w) \right]^{-1} \frac{\partial g(w,x^*(w),u^*(w))}{\partial x} dw,$$
  - where $Z_{(x^*,u^*)}(t)$ is the normalised fundamental matrix of the following linear system $\dot{z}(t) = -\frac{\partial f(x^*(t),t,u^*(t))}{\partial x} z(t)$,

  converges absolutely, and

(iii) $I^*(t) = [Z_{(x^*,u^*)}(t)]^{-1} \psi^*(t)$.

Open loop

Application of the Pontriagin maximum principle to optimization of player i, given strategies of the remaining players q_j.

- Present value hamiltonian $H_i^{PV}(p,t,q_i,\lambda_i) = pq_i - cq_i - \frac{q_i^2}{2} + \lambda_i s(A - \sum_{j\neq i} q_j(t) - q_i)$
- for redefined costate variable $\lambda_i(t) := \psi(t)e^{\rho t}$.

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- for redefined costate variable $\lambda_i(t) := \psi(t)e^{\rho t}$.
- Costate variable – shadow price λ_i fulfils
  - $\dot{\lambda}_i(t) = \lambda_i \rho - \frac{\partial H_i^{PV}(p(t),t,q_i(t),\lambda_i(t))}{\partial p}$
  - with transversality condition $\lambda_i(t)e^{-\rho t} \to 0$,
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  - with transversality condition $\lambda_i(t)e^{-\rho t} \to 0$,
  - and $\lambda_i(t) > 0$ for every t.
- Optimal strategy $q_i(t) \in \mathrm{Argmax}_{q_i \in \mathbb{R}_+} H_i^{\mathrm{PV}}(p(t),t,q_i,\lambda_i(t))$.

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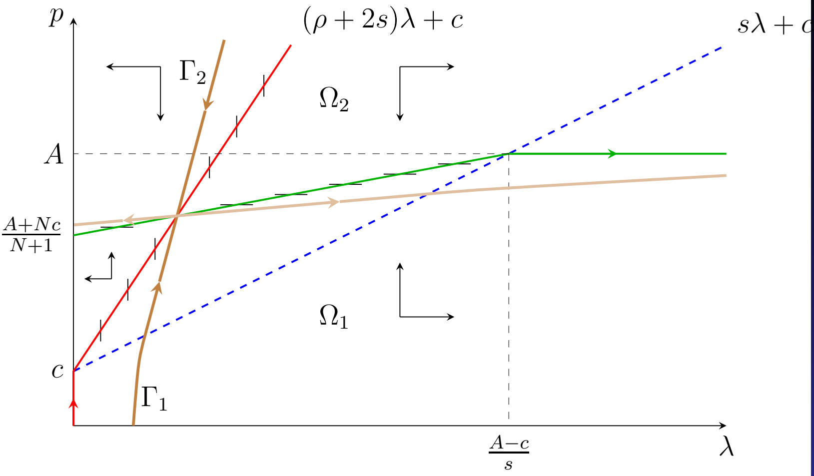
  and
- $$\dot{p} = \begin{cases} Ns^2\lambda - (N+1)sp + As + Ncs, & (\lambda, p) \in \Omega_2, \\ -sp + As, & (\lambda, p) \in \Omega_1. \end{cases}$$

Open loop – phase diagram of (λ, p)



Rysunek: Solid red line with vertical bars – λ-null-cline. Solid green line with horizontal bars – p-null-cline. Dark brown thick line with arrows denotes the stable saddle path. Dashed blue line is $p = s\lambda + c$ that divides the first quarter into region Ω_1 (below this line) and Ω_2 (above it).

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- Let us denote the intersection of the stable saddle path with line $p = s\lambda + c$ by $(\bar{\lambda}, \bar{p})$.

  If $p(t) < \bar{p}$ then $q_i(t) = 0$, otherwise $q_i(t) = p(t) - c - \lambda(t)s$.

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Feedback Nash equilibria – The Bellman equation

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- the Bellman equation $\rho V_i(p) = \sup_{q_i \geqslant 0} pq_i - cq_i - \frac{q_i^2}{2} + V_i'(p)s(A - \sum_{j \neq i} q_j(p) - q_i)$

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- V_i is the value function of player *i* given strategies of the remaining players;
- $q_i(p) \in \text{Argmax}_{q_i \geqslant 0} pq_i - cq_i - \frac{q_i^2}{2} + V_i'(p)s(A - \sum_{j \neq i} q_j(p) - q_i)$ defines optimal strategy of player *i* given strategies of the remaining players.

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  - Check the terminal condition to exclude one of solutions.
- The value function is

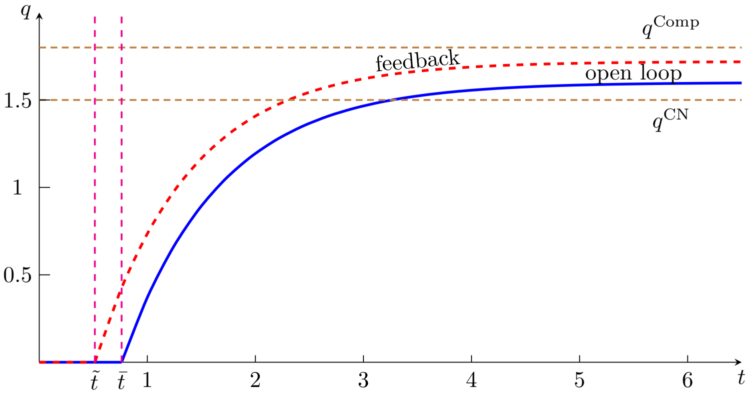
$$V_i(p) = \begin{cases} \frac{kp^2}{2} + hp + g & \text{for } p \geqslant \tilde{p} = \frac{c+sh}{1-sk}, \\ (A-p)^{-\frac{\rho}{s}}(A-\tilde{p})^{\frac{\rho}{s}}\left(\frac{k\tilde{p}^2}{2} + h\tilde{p} + g\right) & \text{otherwise.} \end{cases}$$

for unique k, h and g, with $k > 0$.

Symmetric feedback Nash equilibrium – results

- The game is linear-quadratic, so assume quadratic value function, identical for all players, and calculate the coefficients.
  - It does not work – for p below some $\tilde{p}$ equilibrium production turns out to be negative.
  - So change the candidate for equilibrium strategies below $\tilde{p}$ to 0 and the value function correspondingly.
  - Check the terminal condition to exclude one of solutions.
- The value function is
  $$V_i(p) = \begin{cases} \frac{kp^2}{2} + hp + g & \text{for } p \geqslant \tilde{p} = \frac{c+sh}{1-sk}, \\ (A-p)^{-\frac{\rho}{s}}(A-\tilde{p})^{\frac{\rho}{s}}\left(\frac{k\tilde{p}^2}{2} + h\tilde{p} + g\right) & \text{otherwise.} \end{cases}$$
  for unique k, h and g, with $k > 0$.
- Production at Nash equilibrium is
  $$q_i(p) = \begin{cases} p - c - s(kp+h) & \text{if } p \geqslant \tilde{p}, \\ 0 & \text{otherwise,} \end{cases}$$

Open and closed loop production



Rysunek: Open loop and feedback equilibria for the same initial price, for $A = 10$, $c = 1$, $\rho = 0.15$, $s = 0.2$, $N = 10$; static Cournot-Nash and competitive production levels for comparison.

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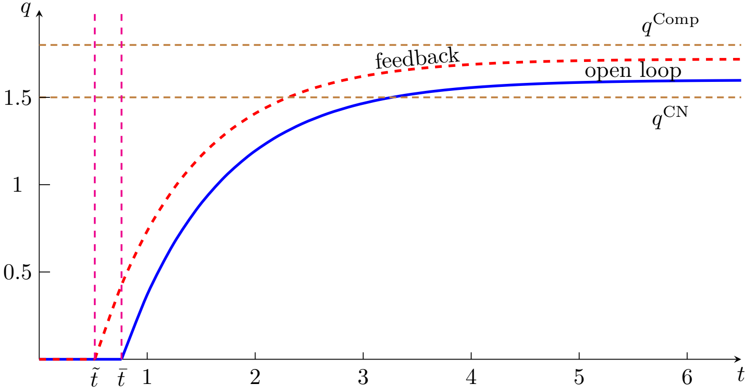

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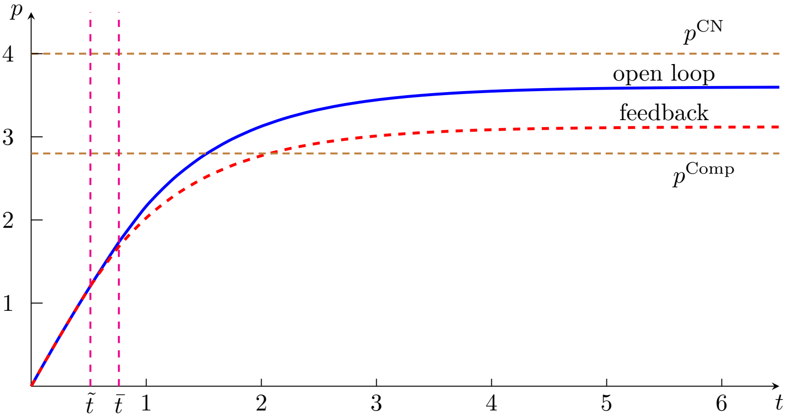
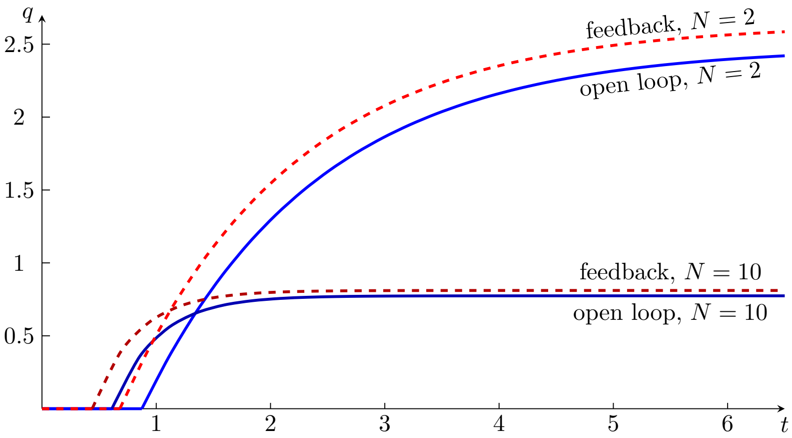
Correction by effect caused by dependence of other players' strategies on price in the feedback case!

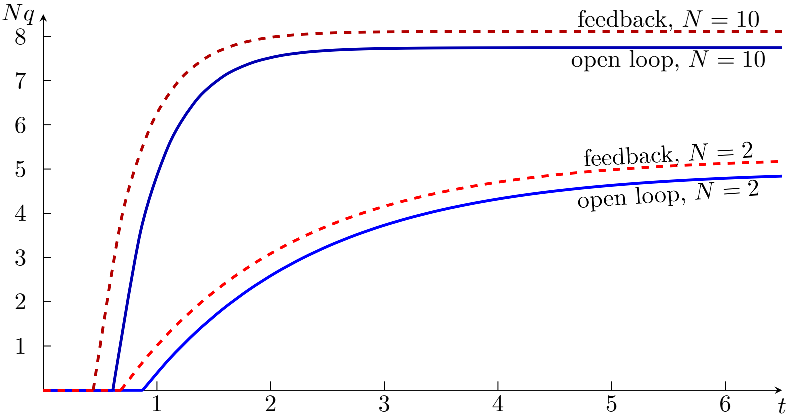
Open and closed loop price



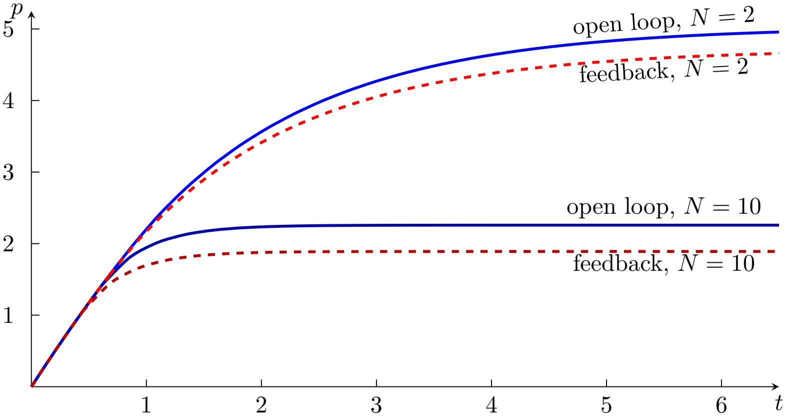
Open and closed loop Nash equilibria as the number of firms increases



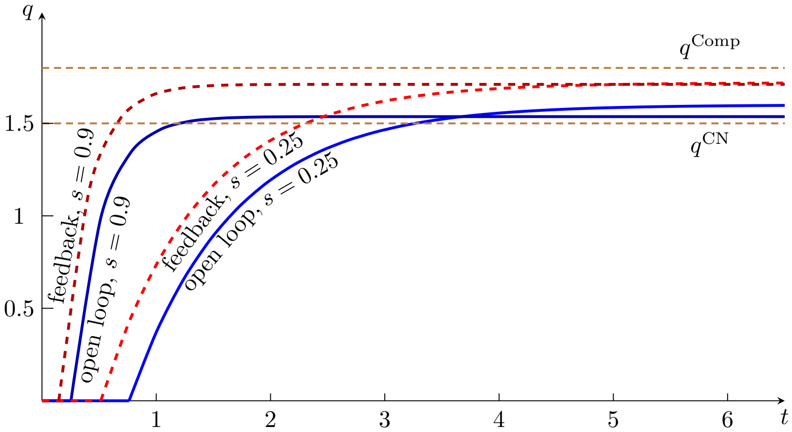
Aggregate production as the number of firms increases



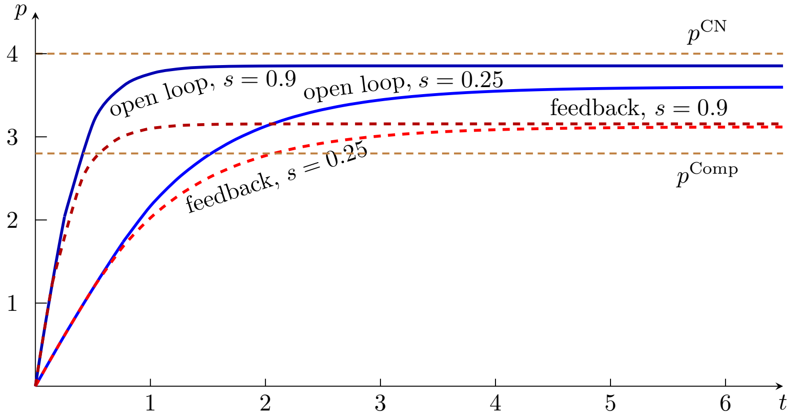
Price as the number of firms increases



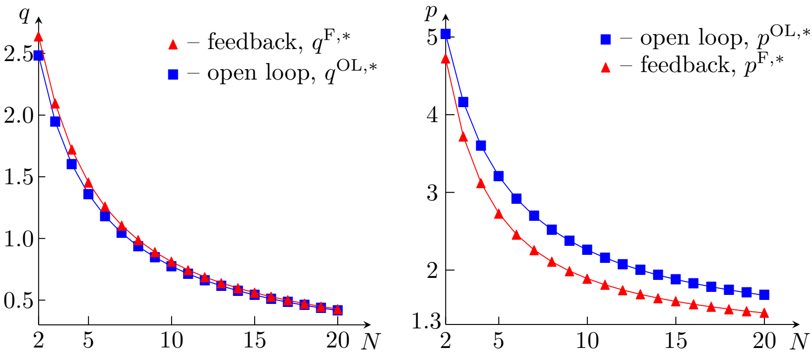
Open and closed loop Nash equilibria as the speed of adjustment increases



Price as the speed of adjustment increases

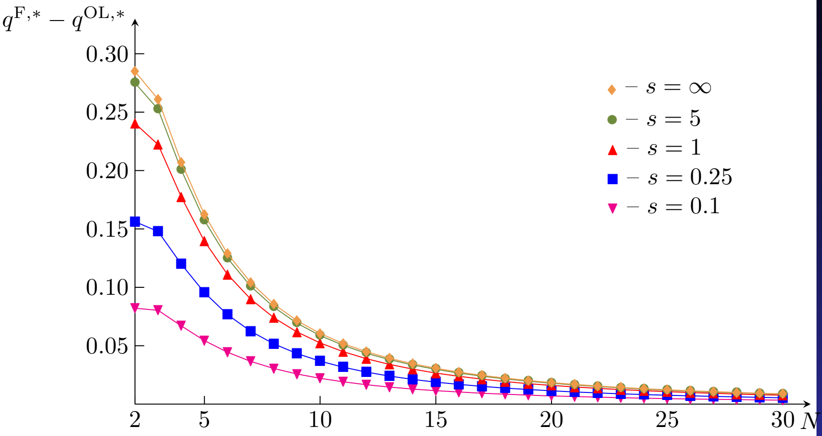


Steady state at Nash equilibria as a function of number of firms



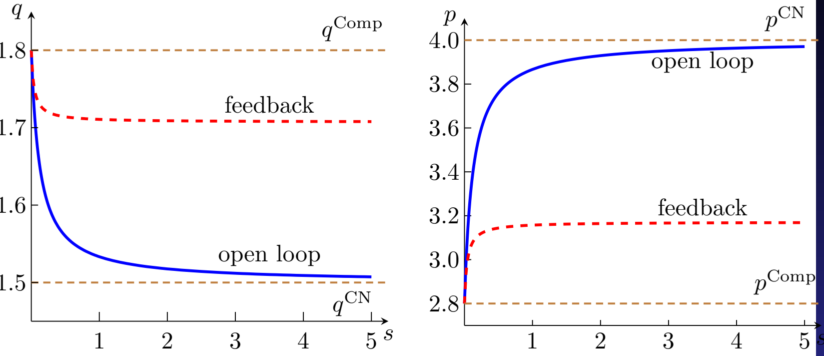
Rysunek: Dependence of the asymptotic (as $t \to +\infty$) of the production level (in the left-hand side panel) and the price level (in the right-hand side panel) in the Nash equilibrium on the number of firms *N*.

Difference between feedback and open loop Nash equilibrium steady state production



Rysunek: Dependence of the difference $q^{\text{feed},*} - q^{\text{ol},*}$ on the number of firms N for various values of the price stickiness.

Steady state at Nash equilibria as a function of speed of adjustment



Rysunek: Dependence of the asymptotic (as $t \to +\infty$) of the production level (in the left-hand side panel) and the price level (in the right-hand side panel) in the Nash equilibrium on the price stickiness s.

Conclusions

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- Feedback price is less than open loop price from the first moment at which feedback production is positive.

Conclusions continued

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- As speed of price adjustment $s \to 0$, both feedback and open loop equilibrium production and price tend to their static competitive equilibrium levels;
- while as $s \to \infty$, then open loop equilibrium production and price tend to their static Cournot oligopoly levels, while feedback – are between Cournot and competitive levels.

Thank you for your attention!