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On the Le Verrier-Faddeev method's universality in domain of the matrix computations

Consider the matrix $\mathbf{A}$ being either real or complex and the identity matrix $\mathbf{E}$:

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} \delta_{1,1} & \cdots & \delta_{1,n} \\ \vdots & \ddots & \vdots \\ \delta_{n,1} & \cdots & \delta_{n,n} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

The Le Verrier–Faddeev algorithm is an extension of the Le Verrier algorithm based on the Cayley–Hamilton Theorem, and is shown below:

$$\begin{array}{lll} \mathbf{A}_1 = \mathbf{A} & \mathbf{B}_1 = \mathbf{A}_1 - q_1\mathbf{E} & q_1 = Sp(\mathbf{A}_1) \\ \mathbf{A}_2 = \mathbf{A}\mathbf{B}_1 & \mathbf{B}_2 = \mathbf{A}_2 - q_2\mathbf{E} & q_2 = \dfrac{Sp(\mathbf{A}_2)}{2} \\ \vdots & & \\ \mathbf{A}_{n-1} = \mathbf{A}\mathbf{B}_{n-2} & \mathbf{B}_{n-1} = \mathbf{A}_{n-1} - q_{n-1}\mathbf{E} & q_{n-1} = \dfrac{Sp(\mathbf{A})}{n-1} \\ \mathbf{A}_n = \mathbf{A}\mathbf{B}_{n-1} & \mathbf{B}_n = \mathbf{A}_n - q_n\mathbf{E} & q_n = \dfrac{Sp(\mathbf{A}_n)}{n} \end{array}$$

$$\mathbf{B}_n = \mathbf{0} \Rightarrow \mathbf{A}^{-1} = \frac{\mathbf{B}_{n-1}}{q_n} \quad \mathbf{A}^D = (-1)^{n-1}\mathbf{B}_{n-1} \quad \det\mathbf{A} = (-1)^{n-1}q_n$$

So this algorithm gives us: an associated matrix, an inverted matrix (if the given matrix is nonsingular), the coefficients of the characteristic (secular) equation and of course the matrix determinant. The characteristic equation

$$\phi(\lambda) = (-1)^n \left[\lambda^n - \sum_{k=1}^{n} q_k \lambda^{n-k}\right] = 0$$

gives us the eigenvalues of the matrix $\mathbf{A}$ and by:

$$\mathbf{Q}_k = \lambda_k^{n-1}\mathbf{E} + \sum_{j=1}^{n-1} \lambda_k^{n-1-j}\mathbf{B}_j$$

the eigenvector for k-th eigenvalue being an arbitrary column of the shown above nonzero matrix, i.e.

$$\mathbf{Q}_k = [\mathbf{q}_1, \ldots, \mathbf{q}_n]_k$$

If the roots of the characteristic polynomial are multiple then the eigenvectors for k-th eigenvalue are determined by:

$$\frac{d^i \mathbf{Q}_k}{d\lambda_k^i} = \frac{d^i}{d\lambda_k^i}\left[\lambda_k^{n-1}\mathbf{E} - \sum_{j=1}^{n-1} \lambda_k^{n-j-1}\mathbf{B}_j\right] \underset{0\leqslant i\leqslant n_k-1}{\forall} \ \underset{1\leqslant k\leqslant s}{\forall} \qquad \sum_{k=1}^{s} n_k = n$$

and we choose such nonzero columns of the left hand side of the above shown matrix equations those which fulfill the eigenproblem for the matrix $\mathbf{A}$, and if for a given eigenvalue λ_k a Jordan block appears then we can determine the appropriate root vectors too.

Remark. For a real nonsymmetric matrix having complex single eigenvalues, the matrix has n distinct complex eigenvectors for each such eigenvalue that the author has proved by experimental computations using his own software. By the author's knowledge, any other method cannot lead to such result.